Last time:

(Ω, Q_Ω) integral, non-degen. unimodular

- rank
- signature
- parity

Example:

① $\Omega \cong \mathbb{Z}^d$, $Q = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$

$= d(1)$

$= (1)^{\oplus d}$

rank $= d$

pos def.

odd

$(Q = d(-1)$ negative definite$)$

② $Q = p(1) \oplus q(-1)$, $p, q \geq 1$

rank $p+q$

sig $p-q$

odd

indef.

③ The hyperbolic lattice $U = \left(\mathbb{Z}^2, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)$, $\{e_1, e_2\}$

rank 2

<u>even</u>

$\{e_1, e_2\}$ basis

$e_i^2 = 0$

$e_1 e_2 = e_2 e_1 = 1$

<u>Note:</u> Checking evenness, it is enough to check on basis vectors, e.g.

$$(e_1+e_2)(e_1+e_2) = e_1^2 + \cancelto{0}{2e_1e_2} + e_2^2 \mod 2$$

diagonalizable over $\mathbb{R}$: $\underbrace{e_1-e_2, e_1+e_2}$ → (not a $\mathbb{Z}$-basis!)

$$\begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix} \rightsquigarrow \text{type } (1,1)$$

sig. 0

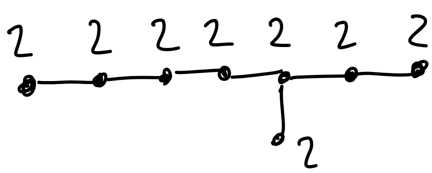
④ Let $\{e_1, \dots, e_8\}$ = std. basis for $\mathbb{R}^8$ with std. inner prod.

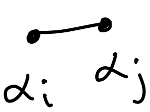
Let $\Omega = \mathbb{Z}\text{-span}\{e_2-e_3, e_3-e_4, \dots, e_7-e_8, e_7+e_8, \frac{1}{2}(e_1+e_8-e_2-\cdots-e_7)\}$ } call these $\alpha_1, \dots, \alpha_8$

Q = restriction of std. inner prod. to Ω.

<u>Check:</u>
- $Q > 0$
- Q integral
- Q even.

$$A_Q = \begin{pmatrix} 2 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & & & & \\ & & -1 & 2 & -1 & & & \\ & & & -1 & 2 & -1 & & -1 \\ & & & & -1 & 2 & -1 & \\ & & & & & -1 & 2 & \\ & & & & -1 & & & 2 \end{pmatrix}$$





$\rightsquigarrow \alpha_i \cdot \alpha_j = -1$

else if $i \neq j$ $\alpha_i \cdot \alpha_j = 0$.

- Q is unimodular

This (Ω, Q) denoted by $E8$
$(\Omega, -Q)$ " " $E8(-1)$

<u>Example</u>: Let $S = Z(F) \subseteq \mathbb{P}^3$
<u>smooth</u> quartic surface $F \in \mathbb{C}[x_0, \dots, x_3]_{(4)}$
(homog. deg 4 polynomials)

e.g. $S = \{[x_0 : \cdots : x_3] : \sum_i x_i^4 = 0\}$

<u>Non obvious facts:</u>

1. $\pi_1(S) = 0$
2. $H_S = (\mathbb{Z}^{22}, Q_S) = E8(-1)^{\oplus 2} U^{\oplus 3}$

$$\left(\begin{array}{ll} \text{type:} & (3,19) \\ \text{sig.} & -16 \\ \text{rank:} & 22 \end{array}\right)$$

<u>Hasse – Minkowski Theorem:</u>

Let (Ω, Q) be a unimodular lattice
("Q is an integral unimodular quad. form")

Suppose Q is <u>indefinite</u>.
Then (Ω, Q) is isometrically isomorphic to one of:

Q odd: $a(1) \oplus b(-1)$ $\quad a, b \geq 1.$

$$\mathbb{Z}^{a+b}, \quad \begin{pmatrix} \overbrace{1 \ \ddots \ 1}^{a} & \\ & \overbrace{-1 \ \ddots \ -1}^{b} \end{pmatrix}$$

Q even: $a E_8(\pm 1) \oplus b U$ $\quad b \geq 1, \quad a \geq 0$

- Husimoller – Milnor
- Serre's Course in Arithmetic

Remarks:

1. Pos. def. + even case: finitely many for each fixed rank, but $\# \to \infty$.

rank	#	
8	1	E_8
16	2	$2E_8, \Gamma_{16}$
24	24	
40	$\gtrsim 10^{51}$	

see HW

2. Have relations: $(1) \oplus 9(-1) \cong E_8(-1) \oplus (1) \oplus (-1)$

Pf: You!

D. Simply-connected 4-manifolds

M^4 closed, $\pi_1 M = 0$ (ugh PD+UCT)

$$H_i(M,\mathbb{Z}) = \begin{cases} \mathbb{Z} & i=0,4 \\ 0 & i=1,3 \\ \mathbb{Z}^d & \text{some } d \end{cases}$$

$$\mathcal{H}_M := (H_2(M;\mathbb{Z}), Q_M)$$

Freedman's Theorem

Thm (Freedman, 1982):

Existence: $\forall$ sym. unimodular integral bilinear form B $\exists$ closed (topological) 4-manifold M s.t.

$$\pi_1 M = 0 \quad \text{and} \quad Q_M \cong B.$$

Uniqueness:

1. B even $\Rightarrow$ M is unique
2. B odd $\Rightarrow$ $\exists$ exactly two homeo. classes, at most one of which is smoothable.

Corollary 1. (4-dim Poincaré conj.)

M^4 closed, orientable. Then $M \underset{\substack{\text{htpy}\\ \text{equiv.}}}{\simeq} S^4 \Longleftrightarrow M \underset{\text{homeo.}}{\cong} S^4$.

Pf: $Q_{S^4} = 0$. □

Example:

1. $Q_{S^2 \times S^2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ even, indefinite, type (1,1).

   So if $\pi_1 M = 0$ and $Q_M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

   $\Longrightarrow M \underset{\cong}{\simeq} S^2 \times S^2$.

2. $M = a\,\mathbb{CP}^2 \# b\,\overline{\mathbb{CP}}^2$, $Q_M = a(1) + b(-1)$.

   get all odd indef. forms.

Corollary 2: $\exists$ unsmoothable even simply conn. 4-manifolds.

Proof:

a. Rochlin (1952): If $\pi_1 M^4 = 0$ and Q_M even and if M is smooth, then $16 \mid \sigma(Q_M)$

signature.

Note: $\sigma(Q_M)$ is always divisible by 8

b. Freedman $\Rightarrow$ $\exists!$ top. closed M^4, $\pi_1 M^4 = 0$ s.t.

$$Q_M = E_8. \qquad \sigma(Q_{E_8}) = 8$$

"E_8 manifold"

<u>Thm (Donaldson, 1983):</u> M^4 closed, $\pi_1 M^4 = 0$, M^4 smooth

① $Q_M > 0$ or < 0 $\Rightarrow$ $Q_n = n(1)$ or $n(-1)$

② If $Q_M = aE_8(\pm 1) \oplus bU$ and $a > 0$

$$\Rightarrow b \geq 3.$$