II. 4-manifolds

A quick overview

<u>Theorem</u> (Markov, 1958)

$\nexists$ algorithm s.t.

<u>Input:</u> Triangulation of closed 4-manifolds
M_1, M_2

<u>Output:</u> Tells you whether or not $M_1 \underset{\text{homeo.}}{\cong} M_2$.

<u>Proof idea:</u> 1. Every f.p. group $\Gamma = \langle a_1, \dots, a_m \mid R_1, \dots, R_n \rangle$
is $\Gamma = \pi_1(M)$, M closed, orientable 4-manifold

<u>Pf:</u> $\pi_1\left(\underset{\substack{j=1 \\ \uparrow \\ \text{connect sum}}}{\overset{m}{\#}} (S^1 \times S^3)\right) \cong F_m$ by van Kampen

Pick loops $\alpha_1, \dots, \alpha_n$ representing R_i
embedded, disjoint

Let $N := \#_{j=1}^{m} (S^1 \times S^3) - \bigcup \text{nbhd}(\alpha_i)$

cpt manifold w/ bdry $\partial N \cong \overbrace{(S^1 \times S^2) \cup \cdots \cup (S^1 \times S^2)}^{n}$

For each relator R_i, glue $S^2 \times D^2$ to N, via word α_i $\left[\partial(S^2 \times D^2) \cong S^2 \times S^1\right]$

2. Markov: $\nexists$ algorithm

<u>Input</u>: 2 f.p. groups

<u>Output</u>: Tells you if isomorphic $\square$

So classifying <u>all</u> 4-manifolds is a little hopeless. But of the 4-manifolds that arise in "nature"

<u>Two Big Classes:</u>

1. $\pi_1 M = 0$
2. Algebraic surfaces := smooth 2-dim complex proj. variety.

Two main invariants:

$\pi_1 M$ and $(H_2(M; \mathbb{Z}), Q_M)$

$\uparrow$ intersection form.

B. The intersection form

__Recall:__ closed, oriented $2n$-manifold M.

$\exists$ bilinear form $Q_M : H^n(M;\mathbb{Z})/\text{torsion} \times H^n(M;\mathbb{Z})/\text{torsion} \to \mathbb{Z}$

$$(\alpha, \beta) \longmapsto \langle \alpha \cup \beta, [M] \rangle$$

cup product

$$= \int \alpha \wedge \beta .$$

viewing as forms

__Dually:__ $H_n(M;\mathbb{Z})/\text{torsion} \times H_n(M;\mathbb{Z})/\text{torsion} \to \mathbb{Z}$

$$(\alpha, b) \longmapsto \langle PD(a) \cup PD(b), [M] \rangle$$

Poincaré dual

Will also call Q_M.

__Remarks:__

1. Q_M is non-degenerate, $\forall \alpha \neq 0$, $\exists \beta$ s.t. $Q_M(\alpha,\beta) \neq 0$
2. Since $a \cup b = (-1)^n\, b \cup a$

   n odd $\Rightarrow$ Q_M is skew symmetric (symplectic)

   n even $\Rightarrow$ Q_M is symmetric

So, e.g. when n is even Q_M is represented by a symmetric matrix wrt. any basis.

Q_M as an intersection number

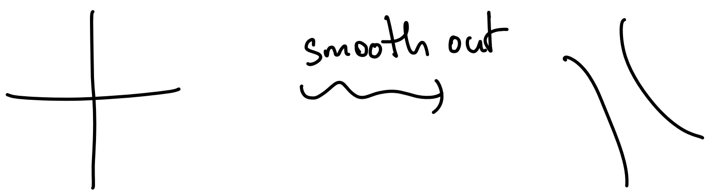
Prop: M any (oriented) 4-manifold, $\xi \in H_2(M;\mathbb{Z})$

$\Rightarrow \exists g \geq 0, \ \exists i : \Sigma_g \underset{\text{embedding}}{\hookrightarrow} M$ s.t. $i_*([\Sigma_g]) = \xi$.

Proof #1:

(1) $\exists$ immersion $i : \Sigma_g \xrightarrow{e} M$ (Hatcher) with $i_*[\Sigma_g] = \xi$

$\Rightarrow$ finitely many self intersections



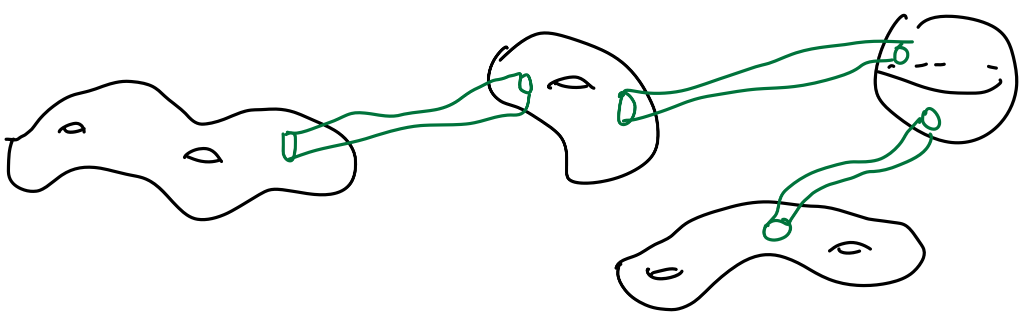
(see figure 3.2, pg 113 in Scorpan)

local picture

$z_1 z_2 = 0$ $\qquad\qquad$ $z_1 z_2 = \varepsilon$

$\overset{\text{get}}{\rightsquigarrow}$ embedded, possibly disconnected surface representing ξ.

(3) Tubing



$\square$

Proof #2: $\exists$ bijection

$$\{\text{elements of } H^2(M;\mathbb{Z})\} \longleftrightarrow [M, K(\mathbb{Z},2)]$$

$\mathbb{CP}^\infty$

htpy classes of maps

$$M \to \mathbb{CP}^\infty$$

$$H^*(\mathbb{CP}^\infty) = \mathbb{Z}[\alpha]$$

deg. 2

So $\xi \rightsquigarrow F_\xi : M^4 \to \mathbb{CP}^\infty$

$\rightsquigarrow F_\xi : M^4 \to \mathbb{CP}^2$ (4-skeleton)

$\rightsquigarrow F_\xi \pitchfork \mathbb{CP}^1$.

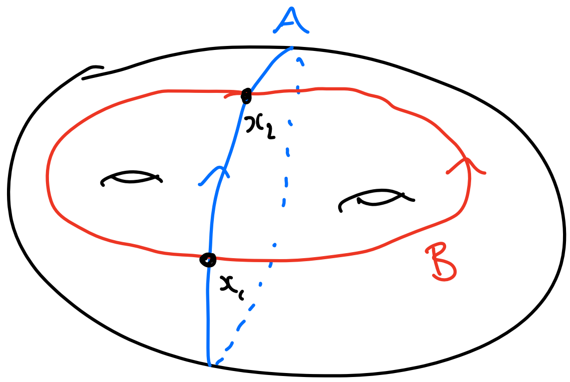
$\Rightarrow F_\xi^{-1}(\mathbb{CP}^1)$ is the required 2-manifold. $\square$

Def: $A^n, B^n \subset M^{2n}$ closed, $A \pitchfork B$ closed

Alg. int. # is: $$[A]\cdot[B] = \sum_{x\in A\cap B} \varepsilon(x)$$

$\varepsilon(x)$ is ± 1 (orientation)

e.g.



$\varepsilon(x_1) = -\varepsilon(x_2)$, so

$[A]\cdot[B] = 0.$

<u>Fundamental Theorem of Intersection Theory:</u>

$$Q_M([A],[B]) = [A]\cdot[B]$$

Now, let $\dim(M) = 4$, M closed oriented

$$H_2(M;\mathbb{Z})/\text{torsion} \cong H^2(M;\mathbb{Z})/\text{torsion} \cong \mathbb{Z}^d, \quad d \geq 0.$$

Q_M symmetric ($\frac{4}{2}$ is even)

<u>Def:</u> A <u>lattice</u> Ω is a free abelian group $\mathbb{Z}^d$ equipped with a non-degenerate symm. bilinear form Q_Ω.
(note: has assoc. quadratic form)

- basis for $\Omega \rightsquigarrow A$ symm. matrix $Q_\Omega(u,v) = uAv^T$
- $\Omega \cong \Gamma \iff \exists F : \Omega \xrightarrow{\cong} \Gamma$ of free ab. groups s.t. $F^* Q_\Gamma \cong Q_\Omega$.
- $\operatorname{rank}(\Omega) := d$.
- Ω is <u>unimodular</u> if $\det(A) = 1$.

Prop: M^4 closed, oriented.

Let $H_M := \left(H_2(M,\mathbb{Z})/\text{torsion},\ Q_M\right)$ is a unimodular lattice.

Pf: exercise. $\square$

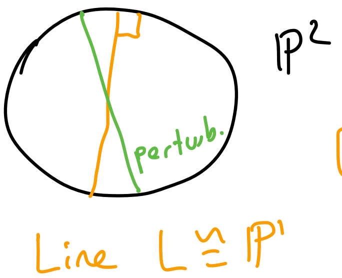
First Examples:

① $M = S^4$, $H_M = 0$

② $M = \mathbb{C}P^2 = \mathbb{P}^2$

$$H_M = (\mathbb{Z}, (1))$$

Pf:

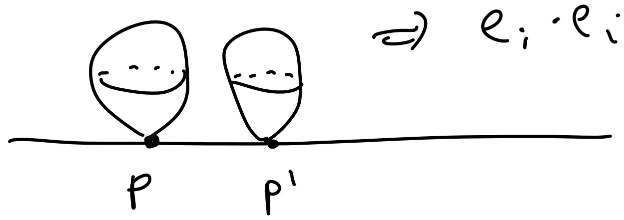


$[L]\cdot[L] = 1 \Rightarrow Q_M = (1)$.

③ $M = S^2 \times S^2$

$$H_M = \left(\mathbb{Z}^2, \begin{array}{c} \begin{array}{cc} e_1 & e_2 \end{array} \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{array}\right)$$

$e_1 = [S^2 \times \{p\}]$
$e_2 = [\{p\} \times S^2]$



$\Rightarrow e_i \cdot e_i = 0$.

④ $M = M_1 \# M_2$

$$\Rightarrow Q_M = Q_{M_1} \underset{\perp}{\oplus} Q_{M_2} = \left(\begin{array}{c|c} Q_{M_1} & 0 \\ \hline 0 & Q_{M_2} \end{array}\right).$$

C. Integral Quadratic Forms

Isomorphism Invariants of (Ω, Q_Ω):

1. $\mathrm{rank}(\Omega)$
2. Signature $\sigma(Q_\Omega)$ of Q_Ω.

diagonalize matrix A of Q_Ω over $\mathbb{R}$

$$\begin{pmatrix} \lambda_1 & & & & 0 \\ & \ddots & & & \\ & & \lambda_p & & \\ & & & \lambda_{p+1} & \\ 0 & & & & \ddots \ \lambda_{p+q} \end{pmatrix}$$

$p+q = \mathrm{rank}(\Omega)$

$\lambda_1, \dots, \lambda_p > 0$

$\lambda_{p+1}, \dots, \lambda_{p+q} < 0$

Say "Ω has type (p,q)"

$\sigma(\Omega) = p - q$

3. Definiteness (detectable from σ)

$Q_\Omega > 0$ means pos. def. $Q(v,v) > 0 \ \forall v \neq 0$

$Q_\Omega < 0$ means neg. def. $Q(v,v) < 0 \ \forall v \neq 0$.

Q_Ω indefinite otherwise

4. Parity

Q_Ω is $\begin{cases} \underline{\text{even}} & \text{if } Q_\Omega(v,v) \in 2\mathbb{Z} \ \forall v \in \Omega \\ \text{odd} & \text{otherwise} \end{cases}$

<u>exercise:</u> enough to check on a basis of Ω.