Outline of lectures

I. The 2-manifold story, recap.

II. 4-manifolds

III. Mapping class groups of 4-manifolds

IV. Case Studies

$$\mathrm{Mod}(M) := \pi_0(\mathrm{Homeo}^+(M)) = \mathrm{Homeo}^+(M)/\mathrm{Homeo}^0(M) = \mathrm{Homeo}^+(M)/\text{isotopy}$$

When $\dim(M)=2$ $\rightarrow$ $= \boxed{\mathrm{Homeo}^+(M)/\text{htpy}}$

False for $\dim(M)=4$

I. 2-manifold story

(Primer on MCG, F-M)

Topological/smooth classification: Σ_g, $g \geq 0$.

Topological	metric	$\widetilde{\Sigma_g}$	holo 1-forms	{complex structures}
$\Sigma_0 = S^2$	$k \equiv 1$	$\mathbb{C}P^1$	$w \equiv 0$	unique
$\Sigma_1 = T^2$	$K \equiv 0$	$\mathbb{C}$	dz nowhere 0	∞ many
Σ_g, $g \geq 2$	$k < 0$	Δ	any w has $2g-2$ zeros	∞ many

What are the self-homeos of M?

$$\mathrm{Mod}(S^2) = 0$$
$$\mathrm{Mod}(T^2) = SL_2\mathbb{Z}$$

<u>Thm (Dehn, 1922)</u> Let $g \geq 0$. Then $\mathrm{Mod}(\Sigma_g)$ is generated by finitely many Dehn twists.

Now what?

① Normal forms

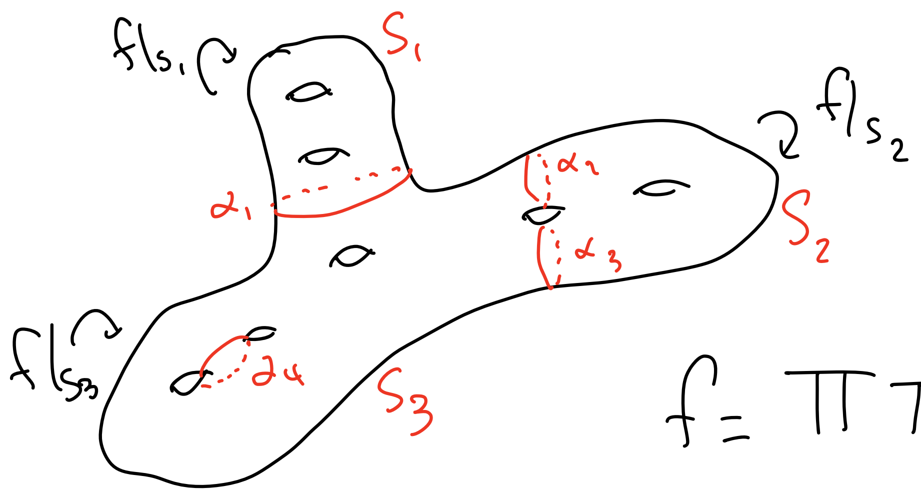
A. Nielsen–Thurston Trichotomy: $g \geq 2$

$\forall \varphi \in \mathrm{Mod}(\Sigma_g)$, $\exists F \in \mathrm{Homeo}^+(\Sigma_g)$ with $[F] = \varphi$ s.t.

1. F finite order (elliptic)
2. $\exists$ s.c.c. $\{\alpha_1, \dots, \alpha_n\}$ s.t. F permutes them. (reducible)
3. $\exists$ measured foliations $(\mathcal{F}_u, \mu_u)$, $(\mathcal{F}_s, \mu_s)$ that are F invariant s.t. F acts by mult. by λ, resp. λ^{-1} on them for some $\lambda > 1$. (pseudo Anosov, pA)

B. Thurston normal form (Ivanov's book, BLM in DMS)

$\forall \varphi \in \mathrm{Mod}(\Sigma_g)$, $\varphi \neq \mathrm{Id}$, $\exists f \in \mathrm{Homeo}^+(\Sigma_g)$ and $\exists$ s.c.c. $\{\alpha_i\}$ s.t. $[f] = \varphi^d$ for some d



powers of Dehn twists

$$f = \prod T_{\alpha_j}^{n_j} \circ f|_{S_1} \circ f|_{S_2} \circ f|_{S_3}$$

each $f|_{S_i} = \mathrm{Id}$ or a pA

This is all canonical! (canonical reduction system, crs) is the $\{\alpha_i\}$ up to isotopy

② Theorem (Preserved structures):

Let $g \geq 2$ $\forall \varphi \in \mathrm{Mod}(\Sigma_g)$ $\exists f \in \mathrm{Homeo}^+(\Sigma_g)$, $[f] = \varphi$ preserving.

1. hyperbolic metric $\Longleftrightarrow$ φ is finite order.

(different in dimension 4)

2. complex structure $\Longleftrightarrow$ " " " "

3. A proper, compact submanifold $N \subset \Sigma_g$ $\Longleftrightarrow$ φ is reducible

③ Realization Problems

$$\begin{tikzcd} \mathrm{Diff}^+ \Sigma_g \arrow[d, "\pi"'] \\ \mathrm{Mod}(\Sigma_g) \arrow[u, bend right, "\sigma"'] \end{tikzcd}$$

Q: Is there a section σ of π?

A: (Morita) no!

Q: What about over $G < \mathrm{Mod}(\Sigma_g)$?

Thm (Kirkhoff, Nielsen Realization): Yes when G finite

What about $[\mathrm{Mod}\, \Sigma_g : G] < \infty$?

Morita? (Lei Chen, N. Salter)

④ Best representatives

A pA F has minimal topological entropy in its htpy class

(5) Relationship with bundles (families)

$$\begin{array}{ccc} \Sigma_g \to & E & \\ & \downarrow & \\ & B & \end{array} \quad \overset{\rightsquigarrow}{\text{monodromy rep.}} \quad \rho : \pi_1(B) \longrightarrow \mathrm{Mod}(\Sigma_g)$$

Theorem (Eells-Earle): $g \geq 2$, $\exists$ bij

$$\left\{ \begin{array}{c} \Sigma_g \to E \\ \downarrow \\ B \end{array} \right\} \Big/ \cong \quad \longleftrightarrow \quad \mathrm{Hom}(\pi_1 B, \mathrm{Mod}(\Sigma_g)) \big/ \text{conj.}$$

(6) Relationship with moduli spaces. $(g \geq 2)$

$\mathcal{M}_g =$ "moduli space of genus g surfaces"

$:= \{\text{hyp. metrics on } \Sigma_g\} / \text{isometry}$

$= \{\text{Riem. metrics on } \Sigma_g\} / \text{diff}$

$= \{\text{complex structures of } \Sigma_g\} / \text{biholo.}$

$= \{\text{smooth genus } g \text{ complex curves}\} / \cong$

$= \{\text{singular flat structures} \ldots$

Fact: $\mathrm{Mod}(\Sigma_g) \cong \pi_1^{orb}(\mathcal{M}_g)$

$$\begin{array}{c} \mathrm{Mod}(\Sigma_g) \curvearrowright \mathrm{Teich}\,\Sigma_g \subset \mathbb{C}^{3g-3} \\ \downarrow \\ \mathcal{M}_g = \mathrm{Teich}\,\Sigma_g / \mathrm{Mod}\,\Sigma_g \end{array}$$