Lecture 3 problems

1. Do problems 3,4,5 from Lecture 2 problems.

2. (5 points determine a conic) In the lecture I noted that the parameter space of *all* (not necessarily smooth) degree d hypersurfaces in $\mathbb{P}^n$ is a projective space. The point of this exercise is to exhibit a standard kind of argument in algebraic geometry that uses topology of parameter spaces.

   Let $\mathcal{C}$ be the parameter space of all conics [1] in $\mathbb{P}^2$.

   (a) An arbitrary conic is of the form $C = Z(F)$ where

   $$F(x, y, z) = a_1x^2 + a_2y^2 + a_3z^2 + a_4xy + a_5yz + a_6xz$$

   with each $a_i \in \mathbb{C}$. Deduce that $\mathcal{C} \cong \mathbb{P}^5$.

   (b) Prove that for any point $p \in \mathbb{P}^2$, the set of conics $C \in \mathcal{C}$ containing p is a hyperplane in $\mathcal{C} \cong \mathbb{P}^5$.

   (c) Prove that for any 5 points $\{p_1, \ldots, p_5\}$ of $\mathbb{P}^2$, there exists a conic $C \in \mathcal{C}$ containing $\{p_i\}$.

   (d) Prove that for a general conic (i.e. any conic $C \in \mathcal{C} - Y$ for a fixed subvariety $Y \subset \mathcal{C}$, if $\{p_i\}$ are in general position (i.e. all distinct, and no 3 on a line) then $\{p_i\}$ determine C uniquely.

   (e) For any $d \geq 2$, determine the maximal $N = N(d)$ with the property that any N points in $\mathbb{P}^2$ lie on a degree d curve in $\mathbb{P}^2$.

3. (The incidence variety is a covering map) With terminology as in the lecture, let $X_{3,3}$ be the parameter space of smooth cubic surfaces, let $\mathrm{Gr}_{\mathbb{C}}(2, 4)$ be the Grassmannian of (projective) lines in $\mathbb{P}^3$, let M be the incidence variety

   $$M := \{(S, L) : L \in X_{3,3}, L \subset S\} \subset X_{3,3} \times \mathrm{Gr}_{\mathbb{C}}(2, 4)$$

   and let $\pi : M \to X_{3,3}$ be defined by $\pi(S, L) := S$. The goal of this exercise is to prove, as claimed in my lecture, that π is a covering map. We will do this by proving that the derivative map $D\pi$ is invertible, showing that π is a local diffeomorphism.

   (a) Consider an arbitrary point $(S, L) \in M$. By changing coordinates we can assume that $L = Z(x_2, x_3) \subset \mathbb{P}^3$. In affine coordinates around $L \subset \mathrm{Gr}_{\mathbb{C}}(2, 4)$ we consider the 2-dimensional complex subspaces of $\mathbb{C}^4$ spanned by the rows of

   $$\begin{pmatrix} 1 & 0 & a_2 & a_3 \\ 0 & 1 & b_2 & b_3 \end{pmatrix}$$

   which gives a neighborhood of all the lines in $\mathrm{Gr}_{\mathbb{C}}(2, 4)$ around the line L, which itself corresponds to the point in $\mathrm{Gr}_{\mathbb{C}}(2, 4)$ where $(a_2, a_3, b_2, b_3) = (0, 0, 0, 0)$.

[1] Recall that a *conic* is the zero set of a homogeneous, degree 2 polynomial $F \in \mathbb{C}[x, y, z]$.

Now on $X_{3,3}$ we have coordinates C_I where $F = \sum c_I x^I$, in the usual notation. Let c denote the vector of all the c_I's, and let $a = (a_2, a_3)$ and $b = (b_2, b_3)$. Verify that

$$(a, b, c) \in \mathrm{Gr}_{\mathbb{C}}(2, 4) \times X_{3,3} \Leftrightarrow F(s \cdot (1, 0, a_2, a_3) + t \cdot (1, 0, b_2, b_3)) = 0 \forall s, t \quad (1)$$

that is, F vanishes on the line in $\mathbb{P}^3$ corresponding to 2-plane in $\mathbb{C}^4$ with coordinate in $\mathrm{Gr}_{\mathbb{C}}(2, 4)$ given by (a_2, a_3, b_2, b_3).

(b) Expand out (1) into a sum $\sum_i s^i t^{3-i} F_i(a, b, c)$ for polynomials F_i. Deduce that the incidence variety M is a subvariety of $X_{3,3} \times \mathrm{Gr}_{\mathbb{C}}(2, 4)$.

(c) Let $d_1 = a_2, d_2 = a_3, d_3 = b_2, d_4 = b_3$. Prove that the Jacobian matrix $(\partial F_i / \partial d_j)$ is invertible at $(d_1, d_2, d_3, d_4) = (0, 0, 0, 0)$. Deduce that $D\pi$ is a local diffeomorphism.

(d) Deduce that π is a covering map.