Lecture 3

Hypersurfaces

degree d Hypersurface X in $\mathbb{P}^n$ is $X = Z(F)$

$$F \in \mathbb{C}[X_0, \dots, X_n]_{(d)}$$

Let $V_{d,n} := \mathbb{C}[X_0, \dots, X_n]_{(d)}$

$$\dim_{\mathbb{C}}(V_{d,n}) = \binom{d+n}{d}$$

Consider $\{\deg d \text{ hypersurfaces } Z(f) \subseteq \mathbb{P}^n\}$

$$= \mathbb{P}(V_{d,n})$$

$$\cong \mathbb{P}^{\binom{n+d}{d} - 1}$$

Parameter space of deg d hyp. in $\mathbb{P}^n$

$PGL_{n+1}\mathbb{C}$

$\mathbb{P}^n \circlearrowleft$

Q: $Z(X_0^d + \dots + X_n^d)$

vs.

$Z(X_0 X_1 X_2^{d-2} + (\pi + i) X_1^{d-5} X_2^5 - 17 X_n^d)$

$\Sigma_{d,n} := \{\text{Singular degree } d \text{ hyp. in } \mathbb{P}^n\} \in \mathbb{P}^n$

$= \{ Z(F) \in \mathbb{P}(V_{d,n}) : \exists [a_0 : \cdots : a_n] \in \cancel{Z(F)} \}$

s.t. $\partial F / \partial x_i (a_0, \ldots, a_n) = 0$

$\forall i$

Sidebar (Euler's identity):

$\forall F \in \mathbb{C}[x_0, \ldots, x_n]_{(d)}$

$\Longrightarrow d \cdot F = \sum \partial F/\partial x_i \cdot x_i$

Pf: you! $\square$

Cor: If $\partial F / \partial x_i (a_0, \ldots, a_n) = 0 \quad \forall i$

$\Longrightarrow (a_0, \ldots, a_n) \in Z(F)$

Amazing Classical Result:

Given $N, d_1, \ldots, d_r \geq 1$ $\exists$ polynomial

$\mathrm{Res} \in \mathbb{Z}\left[a_1, \ldots, a_{\binom{N+1+d_1}{d_1} + \cdots (\ )}\right]$

s.t.: $\forall F_1, \ldots, F_r$, $F_j \in \mathbb{C}[X_0, \ldots, X_N]_{(d_j)}$

$F_1(X_0, \ldots, X_N) = \sum a_I X^I$

$F_1, \dots, F_r$ have a common zero in $\mathbb{P}^N$ $\iff$ $\mathrm{Res}(\{a_I\}) = 0$

(Ex: $F(x) = x^2 + bx + c$ $\qquad b^2 - 4c = 0$)

Cor: $\Sigma_{d,n} := \{ Z(F) \in \mathbb{P}(V_{d,n}) : \exists [a_0 : \dots : a_n] \in \mathbb{P}^n$ s.t. $\partial F / \partial x_j (a_0, \dots, a_n) = 0 \}$

is a hypersurface $Z(\mathrm{Res}) \subset \mathbb{P}^{\binom{n+d}{d} - 1}$

(incredibly singular)

Open in all but a few tiny complexity (small d,n): $\quad H^*(\Sigma_{d,n}; \mathbb{Q}) = ?$

$H^*(X_{d,n}) = ?$

Let $\quad X_{d,n} = \mathbb{P}(V_{d,n}) - \Sigma_{d,n}$

$= \{$ smooth deg d hyper. in $\mathbb{P}^n \}$

Note: $X_{d,n}$ is a smooth manifold, open and dense in $\mathbb{P}(V_{d,n})$

$F \in \mathbb{C}[x,y]_{(d)}$ $\quad Z(F) \subset \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$

Application:

Fix $d \geq 1$, $n \geq 2$

<u>Thm</u>: Any 2 smooth, degree d hypersurfaces in $\mathbb{P}^n$ are diffeomorphic.

Remark: deg - genus formula + Classif. of 2-dim closed real manifolds $\Longrightarrow$ Thm for $d \geq 1$, $n = 2$

<u>Proof</u>: Let

$$M \longrightarrow \mathcal{U}_{d,n} = \{(M,p) : M \in X_{d,n}, p \in M\} \subseteq X_{d,n} \times \mathbb{P}^n$$

$\pi^{-1}(M) = \{m \in M\}$

$$\downarrow \pi \qquad \pi(M,p) := M$$

$$M \in X_{d,n} = \mathbb{P}(V_{d,n}) - \Sigma_{d,n}$$

$\mathcal{U}_{d,n}$ = "The universal smooth deg d hyp. in $\mathbb{P}^n$"

$\Sigma_g \to \mathcal{M}_{g,*} \to \mathcal{M}_g$

1. π is a proper submersion.

2. <u>Ehresman Thm</u>: Any smooth proper submersion is a fiber bundle!

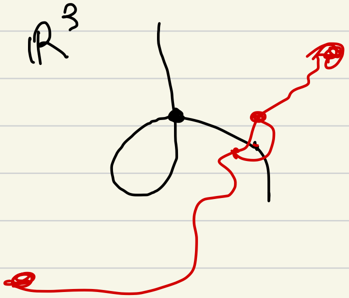
$$\cong \mathbb{P}^{\binom{n+d}{d}-1} \qquad \text{hypersurface}$$

3. $X_{d,n} = \mathbb{P}(V_{d,n}) - \Sigma_{d,n}$

is Path-Connected

<u>Pf:</u> $\Sigma_{d,n}$ has complex codimension 1

$\Longrightarrow$ real codim 2.

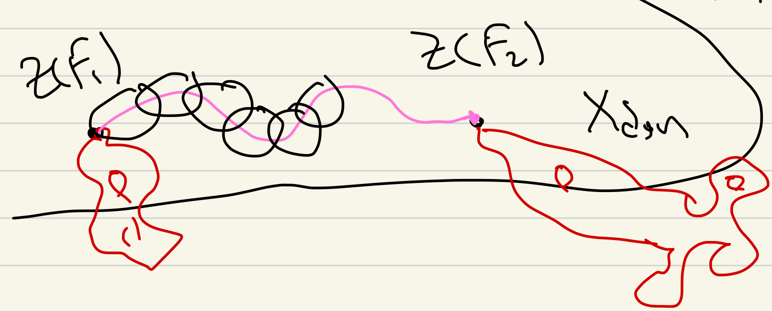


$\{$sing points of $\Sigma_{d,n}\}$ has complex codim 2 in $\mathbb{P}(V_{d,n})$

 $\Longrightarrow$ done.

$1+2+3 \Longrightarrow$ all fibers are diffeomorphic

$$M \longrightarrow U_{d,n}$$
$$\downarrow \pi$$
$$M \in X_{d,n}$$



$\square$

27 lines on a smooth cubic surface

Let $S = Z(F) \subseteq \mathbb{P}^3$

smooth cubic surface $\quad F \in [X_0, X_1, X_2, X_3]_{(3)}$

(an alg. surface)

So S is a closed complex 2-mfld (4-dim real mfld)

A <u>line</u> $L \subset \mathbb{P}^3$ is a linear copy of $\mathbb{P}^1$

- $L = Z(F_1, F_2) \cong \mathbb{P}^1$ $\quad \mathbb{C}^4 - \{0\} \to \mathbb{P}^3$, $\quad \mathbb{C}^2 - \{0\} \mapsto L$

  F_1, F_2 linear, $F_1 \neq \lambda F_2$

  i.e. $F_1 = a_0 X_0 + a_1 X_1 + a_2 X_2 + a_3 X_3$

<u>Example:</u> The <u>Fermat cubic surface</u>

is $S := Z(X_0^3 + X_1^3 + X_2^3 + X_3^3) \subseteq \mathbb{P}^3$

<u>smooth</u>: $[\partial F/\partial X_0 \cdots \partial F/\partial X_3] = [3X_0^2 \cdots 3X_3^2]$

$\neq 0$ at any $[a_0 : \cdots : a_3] \in \mathbb{P}^3$

Theorem (Cayley–Salmon): Every smooth cubic surface S contains exactly 27 distinct lines $L_1, \dots, L_{27}$

$$S \supset L_i \quad i = 1, \dots, 27$$

How to find the lines? (Still unknown)

Ex: $13 X_0^3 - 6 X_0 X_1 X_2 + \frac{75}{8} X_1^2 X_3 + 12 X_1 X_2 X_3 = 0$

Proof:

Step 1: The Fermat Cubic

$$X_0^3 + X_1^3 + X_2^3 + X_3^3 = 0$$

has 27 lines.

Ex:
$$L_1 = Z(X_0 + X_1, X_2 + X_3)$$
$$L_2 = Z(\omega^j X_0 + X_1, X_2 + X_3)$$
$j = 0, 1, 2 \qquad \omega^3 = 1$

Step 2 (The parameter space):

$$X_{3,3} := \left\{ \begin{matrix} \text{smooth cubic} \\ \text{surfaces} \end{matrix} \right\} = \mathbb{P}^{\binom{3+3}{3}-1} - \Sigma_{3,3}$$
$$= \mathbb{P}^{19} - \Sigma_{3,3}$$

So $X_{3,3}$ is path-connected

Step 3 (The incidence variety):

<u>Recall</u>: $\{\text{lines } \mathbb{P}^1 \hookrightarrow \mathbb{P}^3\} = \left\{ \begin{matrix} \text{2-dim} \\ \mathbb{C}\text{-linear subspaces} \\ \text{of } \mathbb{C}^4 \end{matrix} \right\}$
$$= Gr_{\mathbb{C}}(2,4)$$

is a smooth proj. var.

The <u>incidence variety</u> of lines on a smooth cubic surf. is:

$$\begin{array}{ll} M := \{(S,L) : S \in X_{3,3},\ L \overset{\text{line}}{\subset} S\} \\ \quad \downarrow \pi \qquad\qquad\qquad \subseteq X_{3,3} \times Gr(2,4) \\ S \in X_{3,3} \qquad\quad \pi((S,L)) := S \end{array}$$

$$\Longrightarrow \pi^{-1}(S) = \{\text{lines on } L\}$$

Key: Can compute $D\pi$ in coords (see Problem Set 3)

$\rightsquigarrow$ $D\pi$ is an $\cong$

$\rightsquigarrow$ π local diffeo

$\rightsquigarrow$ π is a covering space

Step 4 (Endgame – "Method of continuity")

$$\begin{array}{c} M \\ \downarrow \pi \\ X_{3,3} \end{array}$$

$\pi^{-1}(S) = \{\text{lines in } S\}$

- π covering (step 3) and $X_{3,3}$ is path-conn. (step 2)

$\Longrightarrow$ $|\pi^{-1}(S)|$ doesn't depend on S.

- Step 1 $\Longrightarrow$ $|\pi^{-1}(S)| = 27$ $\square$

Cor (not obvious): S = any smooth cubic surf.

$\Longrightarrow$ $S \underset{\text{diffeo}}{\cong} Bl_{\{p_1,\dots,p_6\}}(\mathbb{CP}^2)$

$\cong \mathbb{CP}^2 \# 6\, \overline{\mathbb{CP}^2}$

connect sum

$S \dashrightarrow L_1 \times L_2 = \mathbb{P}^1 \times \mathbb{P}^1$

Cor: $H_2(S;\mathbb{Z}) = \mathbb{Z}^7$ by mayer-viet.

$$Q_S = \begin{bmatrix} 1 & & & & & & \\ & -1 & & & & & \\ & & -1 & & & & \\ & & & -1 & & & \\ & & & & -1 & & \\ & & & & & -1 & \\ & & & & & & -1 \end{bmatrix}$$

$\# \quad \mathrm{Mod}(S) \longrightarrow \mathrm{Aut}(H_2(S;\mathbb{Z}), Q_S)$

$[f] \longmapsto f_*$

$= O(1,6)(\mathbb{Z})$

$< O(1,6)(\mathbb{R})$

$\cong \mathrm{Isom}(\mathbb{H}^6)$

$\mathbb{H}^6 / O(1,6)(\mathbb{Z})$