Lecture 2 problems

1. (Degree-genus formula) Give two other proofs of the degree-genus formula, as follows:

   (a) Let C be a smooth degree $d \geq 2$ curve in $\mathbb{P}^2$. Pick a line $L \subset \mathbb{P}^2$ and a point $p \in \mathbb{P}^2$ not on C or L. There is a map $\pi : C \to L \cong \mathbb{P}^1$ given by $\pi(m) :=$ the intersection of the unique line through p and m and the line L. Prove that for generic choices of L and p, the map π is a branched cover with ramification points of order 2, making C a 2-sheeted branched cover of $\mathbb{P}^1$, branched over some number r of points of $\mathbb{P}^1$. Compute r, and then use the Riemann-Hurwtiz formula to compute the genus of C.

   (b) I proved in my lecture that any two smooth degree d hypersurfaces in $\mathbb{P}^n$ are diffeomorphic, so it suffices to construct, for each $d \geq 1$, a single smooth, degree d curve $C \subset \mathbb{P}^2$ of genus $(d-1)(d-2)/2$. To do this:

      i. Start with d lines in general position, and show that they are the zero set of polynomial $\mathbb{C}[x, y, z]_{(d)}$.

      ii. Use the fact that the space of smooth degree d curves in $\mathbb{P}^2$ is dense in the space of all degree d curves in $\mathbb{P}^2$ to show that the zero set of some (hence any) perturbed polynomial is given by the original union of lines with their intersection points smoothed to little tubes.

      iii. Now try to figure out the genus. Try examples with small d first.

2. (Computing the canonical class of $\mathbb{P}^2$) Theorem 14.10 of Milnor-Stasheff's "Characteristic Classes" gives a computation of the total Chern class of the cotangent bundle of $\mathbb{P}^n$, and hence of the tangent bundle (it's the negative of the one for the cotangent bundle). This proof is somewhat formal. The goal of this exercise is to give a proof in the classical style (which actually proves more) that the first Chern class $c_1(\mathbb{P}^2)$ of the (cotangent bundle of) $\mathbb{P}^2$ equals $-3\mathrm{PD}([H])$, where $\mathrm{PD}([H])$ is the Poincaré dual of the hyperplane class $[H] \in H_2(\mathbb{P}^2; \mathbb{Z})$. The proof easily generalizes to prove that $c_1(\mathbb{P}^n) = -(n+1)\mathrm{PD}([H])$.

   The only fact we will need is the following. The first Chern class $c_1(T^*M)$ of the cotangent bundle of an algebraic surface M is equal to the following: let θ be a meromorphic 2-form on M. Then $c_1(T^*M)$ is the the homology class of the set of zeros (counted with order of vanishing) minus the homology class of the set of poles (counted with order of the pole).

   As a toy case, let $M = \mathbb{P}^1 = \mathbb{C}_z \cup \{\infty\}$. Let $\theta = dz$ on $\mathbb{C}_z$. Then the local coordinate at ∞ is $w = 1/z$, and so

   $$\theta = d(1/w) = \frac{-1}{w^2} dw$$

   so that θ has a pole of order 2 at ∞. Since θ has no zeros, it follows that the homology class discussed above is $-2[\{\infty\}] \in H_0(\mathbb{P}^1)$; here $[\infty]$ is a 0-cycle, and -2 is the coefficient.

   Now to use the above to compute $c_1(T^*\mathbb{P}^2)$.

(a) Choose two generic meromorphic 1-forms θ_1, θ_2 on $\mathbb{P}^2$. There are (at least) two ways to do this: you can work as above by just considering say dz and dw on $\mathbb{C}^2_{z,w}$, and extend to the $\mathbb{P}^1$ at infinity, generalizing the above. Another way is to specify two holomorphic 1-forms on $\mathbb{C}^3_{z_0,z_1,z_2} - \{0\}$ that are invariant under scaling by any $\lambda \in \mathbb{C}^*$; for example on can take dz_1/z_1 and dz_2/z_2.

(b) Compute the zero set and pole set of $\theta_1 \wedge \theta_2$, and the order of vanishing (resp. order of the pole).

(c) Use the above-mentioned theorem to prove that $c_1(T^*M) = -3[H]$, where $[H \in H_2(\mathbb{P}^2; \mathbb{Z})$ is the hyperplane class.

3. (Lines on the Fermat cubic) Let $S \subseteq \mathbb{P}^3$ be the Fermat cubic

$$S := Z(x_0^3 + x_1^3 + x_2^3 + x_3^3).$$

(a) Prove directly that S has 27 lines, and find their equations. How many of the lines lie in the set of real points $S(\mathbb{R})$?

(b) Prove that, given any line $L \subset S$, there are exactly 10 other lines in S that intersect L.

(c) Prove that, given any two disjoint lines $L_1, L_2 \subset S$, there are exactly 5 other lines in S meeting both L_1 and L_2.

(d) Using ideas from my lecture, prove that all of the above statements are true for any smooth cubic surface.

4. (27 real lines) Consider the projective variety (over $\mathbb{C}$)

$$S := Z(\sum_{i=0}^{4} x_i, \sum_{i=o}^{4} x_i^3) \subset \mathbb{P}^4.$$

This is a surface in the hyperplane $Z(\sum_{i=0}^4 x_i) \subset \mathbb{P}^4)$. Prove that all 27 lines on S are real; that is, each one is defined by linear equations with real coefficients.

5. (Lines in higher dimensions and degrees) Fix the field $\mathbb{C}$.

(a) Prove that the space of all degree 5 hypersurfaces in $\mathbb{P}^4$ can be identified with $\mathbb{P}^{125}$.

(b) A comment (not a problem): Let $U \subset \mathbb{P}^{125}$ be the space of smooth degree 5 hypersurfaces in $\mathbb{P}^4$. Let $\mathrm{Gr}(2,5)$ denote the Grassmannian of 2-planes in $\mathbb{C}^5$. It turns out that the set

$$\{(X, L) \in U \times \mathrm{Gr}(2,5) : \ L \text{ is a line in } X\}$$

is a smooth algebraic variety, and that it has the same dimension as U.

(c) Although one might guess from (b) that a smooth hypersurface of degree 5 in $\mathbb{P}^4$ contains only finitely many lines, the *Fermat hypersurface* $Z(x_0^5 + x_1^5 + \cdots + x_4^5) \subset \mathbb{P}^4$ contains infinitely many lines. Prove this.