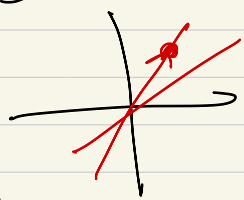
Lecture 1 — 7/18/22

Note: See webpage for problem sheet for this lecture

0. Recalling $\mathbb{P}^n$ basics

$\leftarrow \mathbb{CP}^n$

$$\mathbb{P}^n := \mathbb{CP}^n = \{\mathbb{C}\text{-lines in } \mathbb{C}^{n+1}\}$$
$$= \mathbb{C}^{n+1} - \{0\} \big/ \mathbb{C}^*$$



$$\lambda \cdot (z_0, \dots, z_n) = (\lambda z_0, \dots, \lambda z_n)$$

- $\mathbb{P}^n$ is an n-dim closed complex manifold (2n-dim, oriented real manifold)
- $\mathbb{P}^1 = \mathbb{CP}^1 \cong S^2$

- $V \subset \mathbb{C}^{n+1}$ a hyperplane.

$\rightsquigarrow$ hyperplane $\mathcal{H} \subset \mathbb{P}^n$ via

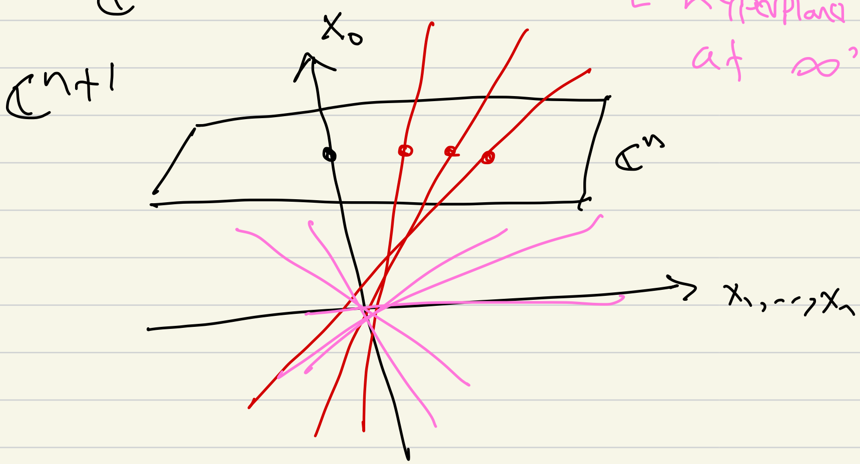
$$\pi : \mathbb{C}^{n+1} - \{0\} \longrightarrow \mathbb{P}^n$$

$$\mathcal{H} := \pi(V) \cong \mathbb{P}^{n-1}$$

- $\mathbb{P}^n = \{[x_0 : \cdots : x_n] : x_0 \neq 0\} \cup \{[0 : x_1 : \cdots : x_n]\}$

$= \{[1 : x_1 : \cdots : x_n] : x_i \in \mathbb{C}\} \cup \underbrace{\mathbb{P}^{n-1}}_{\text{“hyperplane at } \infty\text{”}}$

$= \mathbb{C}^n$



$\mathbb{C}P^n$

- $H_i(\mathbb{P}^n;\mathbb{Z}) = \begin{cases} \mathbb{Z} & i=0,2,\dots,2n \\ 0 & i \text{ odd} \end{cases}$

$$\mathbb{P}^1 \subset \mathbb{P}^2 \subset \cdots \subset \mathbb{P}^{n-1} \subset \mathbb{P}^n$$

the hyperplane class

"line" in $\mathbb{P}^n$

$H_{2k}(\mathbb{P}^n;\mathbb{Z}) \cong \mathbb{Z}$ generated by $[\mathbb{P}^k]$

the image of any $(k+1)$-dim linear subspace of $\mathbb{C}^{n+1}$ under $\pi: \mathbb{C}^{n+1} - \{0\} \longrightarrow \mathbb{P}^n$

- dual view: $\exists$ nice 2-form ω_{FS} - Fubini-Study ~~ω~~ on $\mathbb{P}^n$

$$[\omega_{FS}] := PD([H]) \in H^2(\mathbb{P}^n;\mathbb{Z})$$

hyperplane class

"The Kahler class of $\mathbb{P}^n$"

- $PD([H]) \in H^2(\mathbb{P}^n;\mathbb{Z}) = \mathrm{Hom}(H_2(\mathbb{P}^n;\mathbb{Z}),\mathbb{Z})$
  is given by "Take intersection # with H":

$$H_2(\mathbb{P}^n;\mathbb{Z}) \longrightarrow \mathbb{Z}$$
$$[X] \longmapsto \underbrace{[X]\cdot[H]}_{\text{alg int. \#}}$$

$$[X] \longmapsto \int_X \omega_{FS}$$

$$PD(\underbrace{[H_1 \cap \cdots \cap H_r]}_{\mathbb{P}^{n-r}}) = [\omega_{FS}^r]$$
$$:= \underbrace{\omega_{FS} \wedge \cdots \wedge \omega_{FS}}_{r \text{ times}}$$

$$H^*(\mathbb{P}^n;\mathbb{Z}) = \mathbb{Z}[\omega_{FS}] \big/ (\omega_{FS}^{n+1})$$

Projective Varieties

Let $F \in \mathbb{C}[X_0, \dots, X_n]_{(d)}$:= homog. of degree d

$\deg(X_0^{a_0} \cdots X_n^{a_n}) := \sum a_i$

$\forall \lambda \in \mathbb{C}^* \quad F(\lambda X_0, \dots, \lambda X_n) = \lambda^d F(X_0, \dots, X_n)$

$Z(F) := \{ [X_0 : \dots : X_n] : F(X_0, \dots, X_n) = 0 \} \subseteq \mathbb{P}^n$

is well-defined

__Def:__ A __proj. variety__ is

$X := Z(F_1, \dots, F_m) = Z(F_1) \cap \dots \cap Z(F_m)$
$\subseteq \mathbb{P}^n$

$F_i \in \mathbb{C}[X_0, \dots, X_n]_{(d_i)}$

- X is compact
- X is __hypersurface of degree d__ in $\mathbb{P}^n$ if $X = Z(F)$, $F \in \mathbb{C}[X_0, \dots, X_n]_{(d)}$

If X is a hypersurface

- X is <u>smooth</u> (proj. var.) if it is ~~also~~ a submanifold of $\mathbb{P}^n$

$\overset{\Longleftrightarrow}{\text{IFT}}$ $\forall\, [a_0 : \cdots : a_n] \in X$

$$\left[\partial F/\partial x_0 (a_0, \ldots, a_n) \;\ldots\; \partial F/\partial x_n (a_0, \ldots, a_n) \right] \neq [0 \;\cdots\; 0]$$

<u>Remark:</u> In the space of all degree d hypersurfaces, the condition of being smooth is generic

<u>Ex:</u> $F(x_0, \ldots, x_n) := x_0^d + \cdots + x_n^d$

$\mathbb{P}^n \supset Z(F) =$ Fermat hypersurface is smooth ($(n-1)$-dim closed, complex manifold)

$d \geq 1$

$$[\partial F/\partial x_0 \;\ldots\; \partial F/\partial x_n] = [d x_0^{d-1} \; d x_1^{d-1} \cdots d x_n^{d-1}] \neq [0 \;\cdots\; 0]$$

Theme 1: The main property distinguishing the topology of a smooth proj. variety (r-dimensional) among all smooth 2r-dim., closed, oriented manifolds is:

Call it X

~~a~~ X comes with

$$i : X \hookrightarrow \mathbb{P}^n$$

$$i^* \omega^k \in H^{2k}(X; \mathbb{Z})\ !$$

Equivalently, $[X \cap H] \in H_{2\dim(X)-2}(X; \mathbb{Z})$

$[X \cap H \cap H'] \in H_{2\dim X - 4}$

$\vdots$

Theme #2: Solutions of polynomial equations are governed by the topology of the zero-sets.

Ex: Recall FTA: $f(z) = a_n z^n + \cdots + a_1 z + a_0$, $a_i \in \mathbb{C}$

Then f has n zeros counted with multiplicity

$f(x) = x^2 - a$, $a \in \mathbb{C}$

e.g. $f(z) = z^3(z-1)^2$

5 zeros

I. Complex curves in $\mathbb{P}^2$

Q: $7x^3y - (4\pi + i)xy + 2y^5 = 0$

and $x^7 + y^3 - xy^2 = 0$

how many common solutions $(x, y) \in \mathbb{C}^2$?

A: homogenize

$$f(x,y,z) = 7x^3yz - (4\pi+i)xyz^3 + 2y^5$$

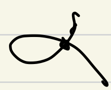
$$g(x,y,z) = x^7 + y^3z^4 - xy^2z^4 = 0$$

$z = 1 \rightsquigarrow$ original polys

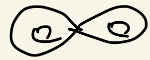
$f|_{\text{this}} =$

$$Z(f),\ Z(g) \subseteq \mathbb{P}^2 = \underbrace{\{[x:y:1]\}}_{\mathbb{C}^2} \cup \{[x:y:0]\}$$

- $Z(f)$, $Z(g)$ are complex curves



When $Z(f)$, $Z(g)$ smooth, you get 2 1-dim complex submanifolds of $\mathbb{P}^2$



Q1: $|Z(f) \cap Z(g)| = ?$

Q2: $Z(f) \cong \Sigma_g$ What is g?

If f smooth

A2: $g = \dfrac{(d-1)(d-2)}{2}$ depends only on $\deg(f)$

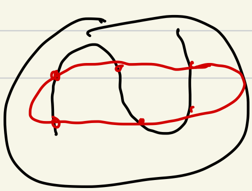
- If f and g have a common factor, say

$$f(x,y,z) = \alpha(x,y,z)\, \beta(x,y,z)$$
$$g = \alpha \cdot \gamma$$

$$\Longrightarrow Z(f) \cap Z(g) \supset Z(\alpha)$$

∞ many solutions.

Bezout's Thm: Suppose $f, g \in \mathbb{C}[x,y,z]$ are homogeneous of degrees d_1, d_2. Assume f and g have no common factor. Then $Z(f) \cap Z(g)$ is finite, and has $d_1 d_2$ elements counted with mult.



For generic f, g, $|Z(f) \cap Z(g)| = d_1 d_2$

Proof:

① $Z(f) \subset \mathbb{P}^2 \rightsquigarrow$

$$[Z(f)] \in H_2(\mathbb{P}^2;\mathbb{Z}) = \langle [\overset{\cong \mathbb{P}^1}{H}] \rangle$$

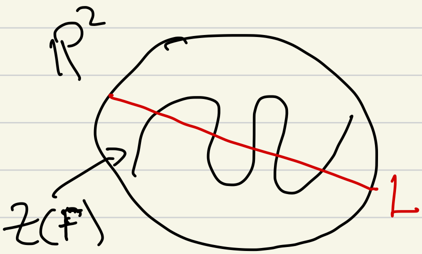
line in $\mathbb{P}^2$
e.g.
$\{[0:y:z]\}$

so

$$[Z(f)] = r \cdot [H]$$

What is r?

For a generic line $L \subset \mathbb{P}^2$



$|Z(f) \cap L| = \deg f$ by FTA

"linear bezout" = FTA

$$f(x,y,z) = x^3 + 2xy^2 + yz^2$$

line $\{[0:y:z]\} \cap Z(f) = \{ \underbrace{[0:y:z]}_{\mathbb{P}^1 = \mathbb{C} \cup \text{pt. at } \infty} : yz^2 = 0 \}$

$A^p \pitchfork B^q$

$\Longrightarrow$ signs all $= +1$

Since for 2 complex submanifolds $A^p, B^q \subset M^{p+q}$

On the other hand, basic intersection theory

$$\Longrightarrow \underbrace{[L] \bullet [Z(f)]}_{\text{alg int } \#} = |L \cap Z(f)|$$

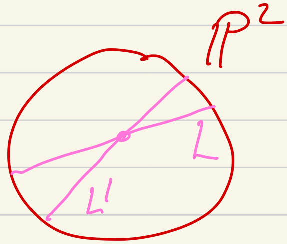
$= \deg(f)$ by above

$$\Longrightarrow \quad [L] \bullet (r[L]) = \deg(f)$$

$$\Longrightarrow \quad r \underbrace{[L]\cdot[L]}_{=1} = \deg(f)$$

$$\Longrightarrow \quad r = \deg(f)$$

i.e. $$[Z(f)] = (\deg f)[L]$$

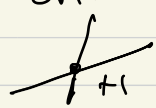


(2) $$[Z(f)] \bullet [Z(g)] = \sum_{z \in Z(f) \cap Z(g)} \pm 1$$

Assume $Z(f) \pitchfork Z(g)$

$= \sum 1$ by orientations

$= |Z(f) \cap Z(g)|$



Thus $|Z(f) \cap Z(g)| = [Z(f)] \bullet [Z(g)]$

$\xRightarrow{\text{by } (1)}$

$$= \left(\deg(f)[L]\right) \cdot \left(\deg(g)[L]\right)$$

$$= \deg(f)\deg(g) \cdot [L]\cdot[L] \to 1$$

Next time: degree-genus formula