# Mod($M^4$) Minicourse: Exercises, 3

First: feel free to go back to problems from the first two days, and discuss/solve them.

### Complex manifolds

1. Prove that every complex manifold has a natural orientation.
2. (Complex manifolds) Prove that the following are complex manifolds.
   (a) $\mathbb{CP}^2$.
   (b) Complex tori $\mathbb{C}^2/\Lambda$, where $\Lambda \cong \mathbb{Z}^4$ is the $\mathbb{Z}$-span of four vectors in $\mathbb{C}^2$ that are linearly independent over $\mathbb{R}$.
   (c) Hopf manifolds: The product $S^3 \times S^1$. [Hint: fix a real number $\lambda > 0$, and consider the action $(z, w) \mapsto (\lambda z, \lambda w)$ on $\mathbb{C}^2 - \{(0, 0)\}$.] One of the first theorems in Hodge theory is that the odd Betti numbers of smooth projective varieties are even. Deduce that the Hopf manifolds are not even homotopy equivalent to a smooth projective variety.
   (d) Smooth projective varieties $X$: the common zero set $X = Z(F_1, \ldots, F_m) \subset \mathbb{P}^n$ of a finite set $\{F_i\}$ of degree $d_i \geq 1$ homogeneous polynomials $F_i \in \mathbb{C}[x_0, \ldots, x_n]$.
   (e) Products of complex manifolds.
3. (Blowup basics) Let $M$ be a closed, complex surface and let $p \in M$. Let $\mathrm{Bl}_p(M)$ denote the blow-up of $M$ at $p$, and let $e \subset \mathrm{Bl}_p(M)$ denote the exceptional divisor.
   (a) Prove in a way analogous to the "$e^2 = -1$ handout" (on the minicourse webpage) that $e^2 = -1$ (recall that $e^2$ is shorthand here for the algebraic intersection number of $e$ with itself).
   (b) Give a different proof that $e^2 = -1$ by finding an explicit smooth perturbation $e'$ of $e$ so that $e \cap e'$ is transverse. Then compute the signed intersection number.
   (c) Prove that $\mathrm{Bl}_p(M)$ is diffeomorphic to the connect sum of $M$ and $\overline{\mathbb{CP}}^2$.
   (d) Prove that $\mathrm{Bl}_p(M)$ has a canonical complex structure, depending only on $p$.
4. Prove that a smooth *quadric surface* in $\mathbb{P}^3$ (that is, the zero set in $\mathbb{P}^3$ of a homogeneous, degree 2 polynomial $F \in \mathbb{C}[x, y, z]$) is diffeomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$.

### Topology of smooth cubic surfaces

1. (Cubic surfaces are blowups) Let $M$ be a smooth cubic surface. Prove that $M$ is diffeomorphic to the blowup of $\mathbb{P}^2$ at 6 points as follows.
   (a) There are two lines $L_1, L_2 \subset M$ that are disjoint. [Hint: Any two lines will span a plane $P \subset \mathbb{P}^3$. Consider $P \cap M$.]
   (b) Prove that there are precisely 5 lines $\ell \subset M$ such that $\ell \cap L_1 \neq \emptyset$ and $\ell \cap L_2 \neq \emptyset$.

(c) Define a rational map $\pi : L_1 \times L_2 \dashrightarrow M$ via

$$\pi(x_1, x_2) := \overline{x_1 x_2} \cap M$$

and note that this map is defined as long as the line $\overline{x_1 x_2}$ is not contained in $M$. Use part (b) to prove that $\pi$ extends to a map $\pi'$ from the blowup of $L_1 \times L_1$ at 5 points to $M$.

(d) Define $\psi : M - (L_1 \cup L_2) \to L_1 \times L_2$ by

$$\psi(m) := (\mathrm{span}\{m, L_1\} \cap L_2, \mathrm{span}\{m, L_2\} \cap L_1).$$

Prove that $\psi$ extends to a morphism $\psi'$ from $M$ to the blowup of $L_1 \times L_2$ at 5 points.

(e) Prove that $\pi'$ and $\psi'$ are inverses of each other, so in particular $\psi'$ is a diffeomorphism of $M$ with the blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ at 5 points.

(f) Prove that the blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ at 2 points is isomorphic (as a smooth projective variety) - hence diffeomorphic - to the blowup of $\mathbb{P}^2$ at one point.

(g) Deduce fro the above that $M$ is diffeomorphic to the blowup of $\mathbb{P}^2$ at 6 points. Check that these points must be in general position: they are distinct; no 3 are colinear; and no 6 lie on a conic.

2. (Seeing the 27 lines in the blowup model) Let $M$ be a smooth cubic surface, which is diffeomorphic to the blowup of $\mathbb{P}^2$ at 6 distinct points $\{p_1, \ldots, p_6\}$ in *general position*: no 3 on a line, and not all 6 on a conic. Let $\pi : M \to \mathbb{P}^2$ be the blowdown map.

   (a) Let $\hat{L}_{ij}, i \neq j$ be the *strict transform* of the unique line $L_{ij}$ through $p_i$ and $p_j$, meaning that $L_{ij}$ is the Zariski closure in $M$ of $\pi^{-1}(L - \{p_i\})$. This means that $\hat{L}_{ij}$ is the union of $\pi^{-1}(L - \{p_i\})$ together with 6 points. Prove that $\{\hat{L}_{ij} : i \neq j\}$ is a set of 15 lines in $M$.

   (b) Note that the 6 exceptional divisors in $M$ give 6 more lines in $M$.

   (c) For $1 \leq i \leq 6$, let $C_i \subset M$ be the strict transform of the conic containing all the $p_j$ except for $p_i$. Prove that this gives the remaining 6 lines in $M$.

3. For a smooth cubic surface $S$, the group $H_2(S; \mathbb{Z})$ has a basis given by 6 mutually disjoint lines in $S$ together with the lift of the hyperplane class in $\mathbb{P}^2$ via the blowup of $\mathbb{P}^2$ at 6 points. Write down explicitly the classes in $H_2(M; \mathbb{Z})$ of the other 21 lines in $S$ in terms of this given basis for $H_2(S; \mathbb{Z})$.