Mod(M^4) Minicourse: Exercises, 2

Topology of K3 surfaces

1. If you haven't done so already, do the two problems on the Kummer manifold from Exercises, 1. The Kummer manifold equipped with a complex structure (there are many) is called a *Kummer surface*.

2. Consider the *Fermat quartic surface*

$$M := \{[x_0 : x_1 : x_2 : x_3] \in \mathbb{P}^3 : x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0\}.$$

(a) Prove that there is a 4-sheeted branched cover of M over $\mathbb{P}^2$ with branch locus a smooth quartic curve in $\mathbb{P}^2$.

(b) Use (a) to prove that $\pi_1(M) = 0$.

(c) It is a classical theorem that every smooth quartic curve $C \subset \mathbb{P}^2$ has 28 *bitangents*: lines in $\mathbb{P}^2$ tangent to C in two points (each has multiplicity 2, as demanded by Bezout (equivalently, intersection theory). Lift these to M, together with C itself, and look at the span of all of the homology classes of these lifts in $H_2(M;\mathbb{Z})$. What is the rank of this span?

(d) Building on part (c), find a basis for $H_2(M;\mathbb{Q})$.

(e) If you are really ambitious, find a basis for $H_2(M;\mathbb{Z})$ and verify by hand that

$$Q_M \cong E_8(-1)^{\oplus 2} \oplus U^{\oplus 3}.$$

3. Find an explicit, nowhere-vanishing holomorphic 2-form θ on the Fermat quartic surface M. Along with the fact that $\pi_1(M) = 0$ this shows that M (and hence any smooth quartic surface in $\mathbb{P}^3$) is a K3 surface.

Hint: Restrict to a chart U_i where $x_i = 0$, so e.g. on U_0 there are coordinates $(1, y_1, y_2, y_3), y_i \in \mathbb{C}$ where $y_i := x_i/x_0$, so that $U_0 \cap M$ is the 0 locus of

$$F(y_1, y_2, y_3) := 1 + y_1^4 + y_2^4 + y_3^4.$$

Now consider the holomorphic 2-form on U_1 given by

$$\theta := \frac{dy_j \wedge dy_k}{\partial F/\partial y_\ell}$$

for distinct $j, k, l \neq 0$. How do these forms depend on the choices of j, k, ℓ? Show that there are such forms defined on each U_i that agree on the overlaps, giving the required θ.

4. Consider the *double plane* M over a smooth sextic curve: that is, M is branched cover $M \to \mathbb{P}^2$ branched over a smooth sextic curve $C := V(F) \subset \mathbb{P}^2$ (here $F \in \mathbb{C}[x_0, x_1, x_2]$ is homogeneous of degree 6).

(a) Construct a nowhere-vanishing holomorphic 2-form θ on M. Hint: as above, construct θ on affine charts. For example, when $x_0 \neq 0$ let

$$\theta := \frac{dx_1 \wedge dx_2}{\sqrt{F(1, x_1, x_2)}}.$$

(b) Prove that $\pi_1(M) = 0$, so that M is a K3 surface.

Mapping class groups of 4-manifolds

5. Verify that the action of complex conjugation on $H_2(\mathbb{P}^2; \mathbb{Z})$ is $-\mathrm{Id}$.

6. Let M be any oriented 4-manifold. Let $S \subset M$ be an embedded 2-sphere in M with $S \cdot S = -2$. Let T_S denote the Dehn twist about S.

   (a) Check the details of the construction of T_S given in the lecture.

   (b) Verify that T_S has order 2 in the smooth mapping class group of M.

   (c) Let $v := [S] \in H_2(M; \mathbb{Z})$. Prove that T_S induces the reflection in $H_2(M; \mathbb{R})$ fixing the hyperplane $v^\perp$ and taking v to $-v$.

7. Let M be the blowup of $\mathbb{P}^2$ at two points, with hyperplane class h and exceptional divisors e_1, e_2.

   (a) Prove that the homology class $[e_1] - [e_2] \in H_2(M; \mathbb{Z})$ has self-intersection -2, and that it can be represented by a 2-sphere $S \subset \mathbb{P}^2$.

   (b) Given part (a), there is a Dehn twist T_S. Work out the action of T_S on $H_2(M; \mathbb{Z})$ in the basis $\{h, e_1, e_2\}$.

8. (A cool representation) The purpose of this problem is to describe a pretty representation that helped Looijenga and I guess the existence of a subgroup $\mathrm{SL}(4, \mathbb{Z}) < \mathrm{Mod}(M)$, where M is a K3 surface. Let V be the standard representation of $\mathrm{SL}(4, \mathbb{R})$ on $\mathbb{R}^4$. Fix an isomorphism $\wedge^4 V \cong \mathbb{R}$.

   (a) Prove that

   $$Q : \wedge^2 V \times \wedge^2 V \to \mathbb{R}$$

   defined by

   $$Q(a \wedge b, c \wedge d) := a \wedge b \wedge c \wedge d \in \wedge^4 V \cong \mathbb{R}$$

   is a nondegenerate, symmetric bilinear form on the 6-dimensional vector space $\wedge^2 V$.

   (b) Prove that Q has signature $(3, 3)$.

   (c) Note $\wedge^2 V$ is a 6-dimensional representation of $\mathrm{SL}(4, \mathbb{R})$ via

   $$T(u \wedge v) := T(u) \wedge T(v).$$

   Prove that this action preserves the bilinear form Q, thus giving a faithful representation

   $$\mathrm{SL}(4, \mathbb{R}) \to \mathrm{O}(3, 3)(\mathbb{R}) \to \mathrm{O}(3, 19)(\mathbb{R}) \tag{0.1}$$

   where the second map is the obvious inclusion.

9. (The Kummer subgroup of the mapping class group of a K3 surface, after Farb-Looijenga) The representation (0.1) in the previous problem has a topological incarnation, and indeed allows us to guess the existence of the following action. Let M be the Kummer manifold, constructed by taking say the blowup at the sixteen 2-torsion points of the 2-dimensional complex torus $A := \mathbb{C}^2/\mathbb{Z}[i]^2 \cong T^4$, and modding out by the involution $(z, w) \mapsto (-z, -w)$. Now $\mathrm{SL}(4, \mathbb{Z})$ acts on $H_2(A; \mathbb{Z})$ in the standard linear way.

   (a) Take an element $T \in \mathrm{SL}(4, \mathbb{Z})$ with the very special property that, via its action on $\mathbb{R}^4$ also thought of as the 2-dimensional complex vector space $\mathbb{C}^2$, it takes complex lines to complex lines. Prove that T induces a diffeomorphism of the Kummer manifold M.

   (b) Now prove that *any* $T \in \mathrm{SL}(4, \mathbb{Z})$ induces a diffeomorphism on M. [This is not so easy - you should use the fact that $\mathrm{GL}(2, \mathbb{C})$ lies in $\mathrm{SL}(4, \mathbb{R})$ and there is a path from any point of $\mathrm{SL}(4, \mathbb{R})$ to a point of $\mathrm{GL}(2, \mathbb{C})$. You need to do a kind of interpolation, the problem being that a general element of $\mathrm{SL}(4, \mathbb{R})$ does not induce a diffeomorphism of the blowup of A.

   (c) Prove using the Freedman-Quinn Theorem that the above construction gives a faithful representation

$$\mathrm{SL}(4, \mathbb{Z}) \to \mathrm{Mod}(M)$$

   that agrees (when tensored with $\mathbb{R}$) with the representation (0.1).