Mod(M^4) Minicourse: Exercises, 1

4-manifold topology

1. Work out the details of Markov's proof that (triangulable) 4-manifolds are not classifiable up to homeomorphism.

2. Prove that there are exactly two homeomorphism types of oriented S^2-bundles M over S^2, one of course being $S^2 \times S^2$. [Hint: Write the base S^2 as a union of two disks glued along their boundary $\alpha \cong S^1$. Show that M is determined by a homotopy class of maps $\alpha \to \mathrm{SO}(3)$, and deduce the result.]

3. Work through the details of both proofs given in lecture that, in a 4-manifold M, each element $c \in H_2(M;\mathbb{Z})$ can be represented by an embedded surface.

4. (Intersection form basics) Let M be a closed, oriented $2n$-manifold.

   (a) Prove that the intersection forms on H^n and H_n given in lecture give (isometrically) isomorphic lattices.

   (b) Prove that the intersection form Q_M on M is unimodular.

   (c) Let $\bar{M}$ denote M with the opposite orientation. Verify that
   $$Q_{\bar{M}} \cong -Q_M.$$

   (d) Let M_1 and M_2 be closed, oriented $2n$-manifolds, and let M be their connect sum. Prove that
   $$Q_M \cong Q_{M_1} \oplus Q_{M_2}.$$

5. (Intersection form examples) Compute the intersection form Q_M on $H_2(M;\mathbb{Z})/\text{torsion}$ for the following 4-manifolds.

   (a) The 4-torus T^4.

   (b) The product $S_g \times S_h$ of two surfaces of genus g (resp. h).

   (c) The nontrivial S^2-bundle over S^2.

6. (Monodromy and intersection form - a simple example) Consider two disjoint, essential simple closed curves a, b on the genus 3 surface S_3 that are not homotopic but that are homologous; equivalently, the union $a \cup b$ bounds a subsurface. Let $f \in \mathrm{Diff}^+(S_3)$ be the product of twists $f := T_a \circ T_b^{-1}$. Let
   $$\rho : \pi_1(S_2) \to \mathbb{Z} \to \mathrm{Diff}^+(S_3)$$
   be the homomorphism that takes a nonseparating, simple closed curve $\mu \subset S_2$ to f and that kills the other 3 homology classes in a standard basis for $H_1(S_2;\mathbb{Z})$ containing $[\mu]$. Let M be the S_3-bundle over S_2 determined by ρ.

   (a) Prove that $H^2(M;\mathbb{Z}) \cong H^2(S_2 \times S_3;\mathbb{Z})$ as abelian groups. [Note: This is easier if you know Leray's Theorem (see Hatcher) or spectral sequences.]

(b) Prove that the lattice $(H_2(M;\mathbb{Z}), Q_M)$ is not isomorphic to the lattice $H_2(S_2 \times S_3;\mathbb{Z})$ endowed with its intersection form. In particular M is not homotopy equivalent to $S_2 \times S_3$.

For much deeper and more interesting stuff in the direction of properties of the monodromy of a surface bundle versus the cup product structure on its cohomology, see papers by Nick Salter and Lei Chen.

7. (The Kummer manifold, I) Let A be a complex 2-torus $A := \mathbb{C}^2/\Lambda$ where $\Lambda \cong \mathbb{Z}^4$ is the subgroup given by the $\mathbb{Z}$-span of four vectors in $\mathbb{C}^2$ that are linearly independent over $\mathbb{R}$. For example $\Lambda := \mathbb{Z}[i]^2$. Note that A is an abelian group under addition modulo Λ.

   (a) Verify that the involution $\iota : A \to A$ defined by
$$\iota(z, w) := (-z, -w)$$
   has precisely sixteen fixed points, corresponding to the sixteen 2-torsion points of A.

   (b) Show that the quotient $A/\mathbb{Z}/2\mathbb{Z}$ by the group generated by ι is not a manifold.

   (c) Let $\tilde{M}$ be the (complex) blowup of A at its sixteen 2-torsion points. Verify that ι induces an action of $\mathbb{Z}/2\mathbb{Z}$ by diffeomorphisms on $\tilde{M}$.

   (d) Prove that this $\mathbb{Z}/2\mathbb{Z}$ action on $\tilde{M}$ is still not free, but that the quotient
$$M := \tilde{M}/(\mathbb{Z}/2\mathbb{Z})$$
   is a smooth manifold.

   (e) Prove that $H_2(M;\mathbb{Q}) \cong \mathbb{Q}^{22}$. [Hint: Recall transfer.]

   (f) Let $\{e_1, \ldots, e_{16}\}$ denote the exceptional divisors in $\tilde{M}$, and let $\pi : \tilde{M} \to M$ denote the quotient map. Prove for each $1 \le i \le 16$ that $\pi(e_i)$ is an embedded 2-sphere in M, and that it has self-intersection number -2.

8. (The Kummer manifold, II) The purpose of this problem is to prove that the Kummer manifold M (notation from Problem 7) is a *K3 surface*; that is, it has the following two properties:

   (a) Prove that $\pi_1(M) = 0$.

   (b) Prove that M has a nowhere vanishing holomorphic 2-form. Note that $dz_1 \wedge dz_2$ is a globally defined, nowhere vanishing, holomorphic 2-form on A.

Quadratic forms

1. Prove that the lattice $(1) \oplus 9(-1)$ is (isometrically) isomorphic to the lattice $E_8(-1) \oplus (1) \oplus (-1)$. You are allowed to use the classification of indefinite lattices.

2. (Constructing Γ_{4m}). The purpose of this exercise is to construct a sequence of interesting positive-definite lattices. Fix $m \ge 1$, and let $\{e_1, \ldots, e_{4m}\}$ be the standard basis of $\mathbb{R}^{4m}$, equipped with the standard inner product. Let $\Gamma_{4m} \subset \mathbb{R}^{4m}$ be the $\mathbb{Z}$-span of the set of vectors $\{e_i + e_j\}$ and the vector $\frac{1}{2}(e_1 + \cdots + e_{4m})$.

(a) Prove that Γ_{4m}, equipped with the restriction to Γ_{4m} of the standard inner product on $\mathbb{R}^{4m}$, is a unimodular lattice. [Hint: First consider the index of the $\mathbb{Z}$-span of $\{e_i + e_j\}$ in the $\mathbb{Z}$-span of $\{e_i\}$.

(b) Prove that Γ_{4m} is an even lattice when m is even, and an odd lattice if m is odd.

(c) Prove that for even $m \geq 2$, the signature of Γ_{4m} is $4m$.

(d) Prove that Γ_{4m} is *indecomposable*: it cannot be written as an orthogonal direct sum of two sublattices.

(e) Prove that Γ_8 has exactly 240 elements v of minimal length; that is with $v^2 = -2$.

3. (a) Prove that Γ_{16} is not (isometrically) isomorphic to $\Gamma_8 \oplus \Gamma_8$. [Hint: prove that the set of vectors $v \in \Gamma_{16}$ with $v^2 = 2$ are exactly $\{\pm e_i \pm e_j : i \neq j\}$, and that these do not generate Γ_16. Prove that this set (in Γ_8) does generate Γ_8.]

   (b) Let U be the rank 2 hyperbolic lattice. Prove that $\Gamma_{16} \oplus U$ is (isometrically) isomorphic to $\Gamma_8 \oplus \Gamma_8 \oplus U$.