The work of G. Margulis

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1 General introduction

This is far from a complete survey; the results covered were selected according to the personal taste of the authors. A much more comprehensive account of Margulis's work up to 2008 is given in [Ji08]. The recent book [FKS22], besides giving a taste of the enormous impact of Margulis on the field he essentially created, also covers much more of Margulis's output than we do here. The reader should view this text as a "survey of surveys", as we try to present some glimpses of the work, and then refer the reader to more detailed surveys as appropriate.

2 Dynamical systems on homogeneous spaces

This section presents an exposition of the contributions of Margulis in the area of homogeneous dynamics, that is, dynamical and ergodic properties of actions on homogeneous spaces of Lie groups. Given a Lie group G and a closed subgroup $\Gamma \subset G$, one can consider the left action of any subgroup $F \subset G$ on G/Γ:

$$x \mapsto gx, \ x \in G/\Gamma, \ g \in F.$$

When F is a one-parameter subgroup, the action thus obtained is called a homogeneous (one-parameter) flow. Classical examples are given by geodesic and horocycle flows on surfaces of constant negative curvature, extensively studied in the 1930–1950s using geometric and representation-theoretic methods.

We remark that geodesic flows on surfaces of constant negative curvature are prototypical examples of Anosov flows, and orbits of horocycle flows are stable and unstable leaves relative to the geodesic actions.

Margulis' famous PhD Thesis "On some properties of Anosov flows" (or rather of U-systems, as they were called by Anosov back then), written in 1969 and published in 2004 [Mar04], made a foundational contribution to the theory. However, we will not be covering it here as we are limiting the scope to homogenous dynamics. Feel free to rewrite this further.

In what follows, G will be a semisimple Lie group with finite center, and Γ a lattice in G (this means that $\Gamma \subset G$ is a discrete subgroup, and the quotient G/Γ has finite Haar measure). A lattice Γ is *uniform* if G/Γ is compact.

The space of lattices in $\mathbb{R}^n$. Here we describe a family of homogeneous spaces particularly important for number-theoretic applications. Let $G = \mathrm{SL}(n, \mathbb{R})$, and let $\mathcal{L}_n$ denote the space of unimodular lattices in $\mathbb{R}^n$. (By definition, a lattice Δ is unimodular iff the volume of $\mathbb{R}^n/\Delta$ is equal to 1.) G acts on $\mathcal{L}_n$ as follows: if $g \in G$ and $\Delta \in \mathcal{L}_n$ is the $\mathbb{Z}$-span of the vectors $v_1, \ldots v_n$, then gv is the $\mathbb{Z}$-span of $gv_1, \ldots, gv_n$. This action is clearly transitive. The stabilizer of the standard lattice $\mathbb{Z}^n$ is $\Gamma = \mathrm{SL}(n, \mathbb{Z})$. This gives an identification of $\mathcal{L}_n$ with G/Γ.

One can consider a Haar measure on G (both left and right invariant) and the corresponding left-invariant measure on $\mathcal{L}_n$, which, as is well known, happens to be finite; that is, Γ is a lattice in G. An important feature of the quotient topology on G/Γ is that $\mathcal{L}_n$ is not compact (in other words, Γ is non-uniform). More precisely, Mahler's Compactness Criterion [Reference] says that a subset Q of $\mathcal{L}_n$ is bounded iff there exists $\epsilon > 0$ such that for any $\Lambda \in Q$ one has $\inf_{\mathbf{x} \in \Lambda \smallsetminus \{0\}} \|\mathbf{x}\| \geq \epsilon$. In other words, for $\epsilon > 0$ let $\mathcal{L}_n(\epsilon) \subset \mathcal{L}_n$ denote the set of lattices whose shortest non-zero vector has length at least ϵ. Then for any $\epsilon > 0$ the set $\mathcal{L}_n(\epsilon)$ is compact.

2.1 Unipotent Flows and Quantitative Non-Divergence.

Let $U = \{u_t\}_{t \in \mathbb{R}}$ be a unipotent one-parameter subgroup of G. (Recall that in $\mathrm{SL}(n, \mathbb{R})$ a matrix is unipotent if all its eigenvalues are 1. In a general Lie group an element is unipotent if its Adjoint (acting on the Lie algebra) is a unipotent matrix.) Examples of unipotent one-parameter subgroups:

$$\left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in \mathbb{R} \right\} \subset \mathrm{SL}(2, \mathbb{R}) \tag{2.1}$$

(the action of this subgroup on $\mathrm{SL}(2, \mathbb{R})/\Gamma$ induces the horocycle flow on the unit tangent bundle to the quotient of the hyperbolic plane by Γ), and

$$\left\{ \begin{pmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} : t \in \mathbb{R} \right\} \subset \mathrm{SL}(3, \mathbb{R}).$$

Consider the left action of a unipotent one-parameter subgroup of $\mathrm{SL}(n, \mathbb{R})$ on $\mathcal{L}_n$. When $n = 2$, every uniponent subgroup is conjugate to (2.1), and it is easy to show, see §2.1.1, that every orbit spends relatively little time outside $\mathcal{L}_2(\epsilon)$. In [Mar71, Mar75] Margulis proved the following result: for any one-parameter group $\{u_t\}$ of unipotent linear transformations, the orbit of every point in $\mathcal{L}_n$ under the semi-group $\{u_t : t \geq 0\}$ does not diverge to infinity. This was conjectured by Piateski-Shapiro and was used by Margulis to prove arithmeticity of higher rank lattice subgroups of semisimple Lie groups. Several years later Dani [Dan79] obtained a quantitative strengthening of the initial nondivergence result by showing that such orbits return into a suitably chosen compact set with positive frequency. To be more precise, Dani proved that for any $\Lambda \in \mathcal{L}_n$ there are $0 < \epsilon, c < 1$ such that for any $T > 0$ one has

$$\left| \left\{ t \in [0, T] : u_t \Lambda \notin \mathcal{L}_n(\epsilon) \right\} \right| < cT\,. \tag{2.2}$$

(Here and hereafter for a set $E \subset \mathbb{R}$, $|E|$ denotes the Lebesgue measure of E.) These ideas were developed during later work on the Oppenheim conjecture and related topics, see [Dan81, Dan84, Dan86, DanMar89, DanMar90a, DanMar91, DanMar93]. In this section we only present the result of [KlMar98] which gives an explicit dependence of c on ϵ in (2.2) and includes the earlier results on quantitative non-divergence as special cases. A more detailed account is given in [Kl10, BK22].

2.1.1 An elementary non-divergence result.

Even though it does not capture much of the difficulty of the problem, we start with the $\mathrm{SL}(2, \mathbb{R})$ case as a motivation.

Lemma 2.1. *Suppose $T > 0$, $\Lambda \in \mathcal{L}_2$ and $0 < \rho < 1/\sqrt{2}$ are such that*

$$\forall\, v \in \Lambda \smallsetminus \{0\} \qquad \sup_{t \in [0,T]} \|u_t v\| \geq \rho. \tag{2.3}$$

Then for any $\epsilon < \rho$,

$$|\{t \in [0, T] \ : \ u_t\Lambda \notin \mathcal{L}_2(\epsilon)\}| \leq 2\left(\frac{\epsilon}{\rho}\right) T. \tag{2.4}$$

This can be interpreted as follows. Suppose ρ is the length of the shortest vector in Λ. Then (2.3) holds. Thus for any $\epsilon < \rho$ the lemma gives the quantitative statement (2.4), which says that the trajectory $\{u_t\Lambda\}$, where t ranges from 0 to T, spends little time outside of $\mathcal{L}_2(\epsilon)$.

Proof. Recall that a vector $v \in \Lambda$ is said to be primitive in Λ if $\mathbb{R}v \cap \Lambda$ is generated by v as a $\mathbb{Z}$-module. Now for $r > 0$ and a primitive $v \in \Lambda$ consider

$$B_v(r) \stackrel{\text{def}}{=} \{t \in B \ : \ \|u_t v\| < r\},$$

where $\|\cdot\|$ is the supremum norm. Let $v = \begin{pmatrix} a \\ b \end{pmatrix} \in P(\Lambda)$ be such that $B_v(\epsilon) \neq \varnothing$. Then, since $u_t v = \begin{pmatrix} a + bt \\ b \end{pmatrix}$, it follows that $|b| < \epsilon$, and (2.3) implies that b is nonzero. Therefore, if we denote $f(t) = a + bt$, we have

$$B_v(\epsilon) = \{t \in [0, T] \ : \ |f(t)| < \epsilon\} \quad \text{and} \quad B_v(\rho) = \{x \in [0, T] \ : \ |f(t)| < \rho\}.$$

Clearly the ratio of lengths of intervals $B_v(\epsilon)$ and $B_v(\rho)$ is bounded from above by $2\epsilon/\rho$ (by looking at the worst case when $B_v(\epsilon)$ is close to one of the endpoints of B). Since

$$\begin{array}{c}\text{a unimodular lattice in } \mathbb{R}^2 \text{ cannot contain} \\ \text{two linearly independent vectors each of length } < \rho,\end{array} \tag{2.5}$$

the sets $B_v(\rho)$ are disjoint for different primitive $v \in \Lambda$, and also that $u_t\Lambda \notin \mathcal{L}_2(\epsilon)$ whenever $t \in B_v(\rho) \smallsetminus B_v(\epsilon)$ for some primitive $v \in \Lambda$. Thus we conclude that

$$\begin{aligned} |\{x \in [0, T] \ : \ u_x\Lambda \notin \mathcal{L}_2(\epsilon)\}| &\leq \sum_v |B_v(\epsilon)| \\ &\leq 2\left(\frac{\epsilon}{\rho}\right) \sum_v |B_v(\rho))| \leq 2\left(\frac{\epsilon}{\rho}\right) T. \end{aligned}$$ □

2.1.2 The general case

In this section, we present a generalization of Lemma 2.1, which is in particular valid for any dimension.

First, we note that the group structure of $U = \{u_t : t \in \mathbb{R}\}$ is not used in the proof of Lemma 2.1. In fact it was already observed in [Mar71] that the feature important for the proof is the polynomial nature of the map $t \mapsto u_t$. More generally, Kleinbock and Margulis introduced the following definition:

Definition 2.2. If C and α are positive numbers and B is a subset of $\mathbb{R}^d$, let us say that a function $f : B \mapsto \mathbb{R}$ is (C, α)-*good on* B if for any open ball $J \subset B$ and any $\epsilon > 0$ one has

$$|\{x \in J : |f(x)| < \epsilon\}| \leq C \left(\frac{\epsilon}{\sup_{x \in J} |f(x)|} \right)^{\alpha} |J| \, . \tag{2.6}$$

This definition captures the property of unipotent orbits used in the proof of Lemma 2.1.

Lemma 2.3. (a) *f is (C, α)-good on $B \Leftrightarrow$ so is $|f| \Rightarrow$ so is cf $\forall\, c \in \mathbb{R}$;*

(b) *f_i, $i = 1, \ldots, k$, are (C, α)-good on $B \Rightarrow$ so is $\sup_i |f_i|$;*

(c) *If f is (C, α)-good on B and $c_1 \leq \left|\frac{f(x)}{g(x)}\right| \leq c_2$ for all $x \in B$, then g is $\big(C(c_2/c_1)^{\alpha}, \alpha\big)$-good on B;*

The notion of (C, α)-good functions was introduced in [KlMar98] in 1998, but the importance of (2.6) for measure estimates on the space of lattices was observed earlier. For instance, the next proposition can be traced to [DanMar93, Lemma 4.1].

Proposition 2.4. *For any $k \in \mathbb{N}$, any polynomial of degree not greater than k is $\big(k(k+1)^{1/k}, 1/k\big)$-good on $\mathbb{R}$.*

As a corollary, we have:

Corollary 2.5. *For any $n \in \mathbb{N}$ there exist (explicitly computable) $C = C(n)$, $\alpha = \alpha(n)$ such that for any one-parameter unipotent subgroup $\{u_t\}$ of $\mathrm{SL}_n(\mathbb{R})$, any $\Lambda \in \mathcal{L}_n$ and any subgroup Δ of Λ, the function $t \mapsto \|u_t \Delta\|$ is (C, α)-good.*

The following is the main non-divergence result of Kleinbock and Margulis. In particular, it is a generalization of Lemma 2.1 to the case of arbitrary n.

Theorem 2.6 ([KlMar98]). *Suppose $d, n \in \mathbb{N}$, a lattice $\Lambda \subset \mathbb{R}^n$, a ball $B = B(x_0, r_0) \subset \mathbb{R}^d$, $C, \alpha > 0$, $0 < \rho < 1/n$ and a continuous map $h : \tilde{B} \to \mathrm{SL}(n, \mathbb{R})$ are given, where $\tilde{B} = B(x_0, 3^n r_0)$. Assume that for any primitive subgroup $\Delta \subset \Lambda$,*

(i) *the function $x \mapsto \|h(x)\Delta\|$ is (C, α)-good on $\tilde{B}$, and*

(ii) $\sup_{x \in \tilde{B}} \|h(x)\Delta\| \geq \rho$.

Then for any $\epsilon < \rho$,

$$|\{x \in B \ : \ h(x)\Lambda \notin \mathcal{L}_n(\epsilon)\}| \leq Cc(n,d) \left(\frac{\epsilon}{\rho}\right)^{\alpha} |B| \tag{2.7}$$

where $c(n, d)$ is an explicit constant.

Here is one of the key ideas used in the proof of Theorem 2.6.

Lattices, subspaces and flags. Let Λ be a lattice in $\mathbb{R}^n$. We say that a subspace L of $\mathbb{R}^n$ is Λ-*rational* if $L \cap \Lambda$ is a lattice in L. For any Λ-rational subspace L, we denote by $d_\Lambda(L)$ or simply by $d(L)$ the volume of $L/(L \cap \Lambda)$. Let us note that $d(L)$ is equal to the norm of $e_1 \wedge \cdots \wedge e_\ell$ in the exterior power $\bigwedge^\ell(\mathbb{R}^n)$, where $\ell = \dim L$ and $(e_1, \ldots, e_\ell)$ is a basis over $\mathbb{Z}$ of $L \cap \Lambda$. If $L = \{0\}$ we write $d(L) = 1$.

Recall that a flag is an acsending chain of subspaces $V_1 \subset \cdots \subset V_k$ of $\mathbb{R}^n$. We say that a flag is Λ-rational if all of its subspaces are.

We now present the substitute for (2.5) needed to work with $n > 2$. The following definition is taken from [KLW04] but it is implicit in [KlMar98] and also in [Mar71, Mar75].

Definition 2.7. Suppose $\Lambda \subset \mathbb{R}^n$ is a lattice, $0 < \epsilon < \eta$ are constants, and F is a Λ-rational flag. We say that Λ is *marked* by (F, ϵ, η) if the following hold:

(M1) For any subspace $V \in F$, $d_\Lambda(V) \leq \rho$.

(M2) For any subspace $V \in F$, $d_\Lambda(V) \geq \epsilon$.

(M3) F is maximal among all the Λ-rational flags satisfying (M1).

The higher dimensional analogue of (2.5) is the following:

Proposition 2.8. *Suppose Λ is a lattice, $0 < \epsilon < \eta < 1$, and suppose there exists a Λ-rational flag F such that Λ is marked by (F, ϵ, η). Then $\Lambda \in \mathcal{L}_n(\epsilon)$.*

Proof. Write $F = (V_1, \dots, V_m)$. Then, by (M1) and (M2), for $1 \leq i \leq m$,

$$\epsilon \leq d_\Lambda(V_i) \leq \eta < 1.$$

Suppose $\Lambda \notin \mathcal{L}_n(\epsilon)$, then there exists $v \in \Lambda$ such that $\|v\| < \epsilon$. Let i be such that $v \in V_i$, $v \notin V_{i+1}$, and let $V = V_i + \mathbb{R}v$. Then V is a Λ-rational subspace. Let $v_1, \dots, v_{k-1}$ be a basis for $V_i \cap \Lambda$, and let v_k be such that $v_1, \dots, v_k$ is a basis for $V \cap \Lambda$. Then,

$$d_\Lambda(V) = \|v_1 \wedge \cdots \wedge v_k\| \leq \|v_1 \wedge \cdots \wedge v_{k-1}\| \|v_k\|.$$

Thus, one has

$$d_\Lambda(V) \leq d_\Lambda(V_i)\|v\| < \epsilon.$$

In particular, $d_\Lambda(V) < \rho$, and thus by (M3), $V = V_{i+1}$. Then $d_\Lambda(V_{i+1}) < \epsilon$, contradicting (M1). □

Proof of Theorem 2.6. The proof is a complicated inductive argument based on the idea of the proof of Lemma 2.1 and on Proposition 2.8. We refer the reader to [KlMar98], [KLW04] or [Kl10] for details.

2.1.3 Applications to Diophantine approximation on manifolds

In this section we list some of the applications of Theorem 2.6 to metric diophanitine approximation. A much more detailed and comprehensive survey is given in [BK22].

Mahler's conjecture, proved by Sprindžuk in 1964 [Spr69] is the following statement: for any $n \in \mathbb{N}$, any $\epsilon > 0$ and for almost any real number x, the inequality

$$|p + q_1 x + q_2 x^2 + \cdots + q_n x^n| < \|\mathbf{q}\|^{-n(1+\epsilon)} \tag{2.8}$$

has only finitely many solutions $(p, \mathbf{q}) \in \mathbb{Z} \times \mathbb{Z}^n$, where $\mathbf{q} = (q_1, \dots, q_n)$ and $\|\mathbf{q}\| = \max_{1 \leq i \leq n} |q_i|$.

After Sprindžuk's result, several conjectural improvements were proposed. Baker [Ba75] proposed replacing $\|\mathbf{q}\|^n$ in (2.8) by $\Pi_+(\mathbf{q})^{-(1+\epsilon)}$, where $\Pi_+(\mathbf{q}) = \prod_{i=1}^n |q_i|$. (This is indeed an improvement of (2.8) since $\Pi_+(\mathbf{q}) \leq \|\mathbf{q}\|^n$.) Sprindžuk proposed replacing the powers of x in (2.8) by arbitrary analytic functions, which together with 1 are linearly independent over $\mathbb{R}$. In

[KlMar98], Kleinbock and Margulis prove a combination of both of these conjectures. In fact, the result of Kleinbock and Margulis applies to a more general class of functions which need not be real analytic. We thus make the following:

Definition 2.9. Let $\mathbf{f} = (f_1, \ldots, f_n) : U \to \mathbb{R}^n$ be a map defined on an open subset U of $\mathbb{R}^d$. Given a point $x_0 \in U$, we say that $\mathbf{f}$ is ℓ-non-degenerate at x_0 if $\mathbf{f}$ is ℓ times continuously differentiable on some sufficiently small ball centered at x_0 and the partial derivatives of $\mathbf{f}$ at x_0 of orders up to ℓ span $\mathbb{R}^n$. The map $\mathbf{f}$ is called non-degenerate at x_0 if it is ℓ-non-degenerate at x_0 for some $\ell \in \mathbb{N}$; $\mathbf{f}$ is called nondegenerate almost everywhere (in U) if it is non-degenerate at almost every $x_0 \in U$ with respect to Lebesgue measure. The non-degeneracy of differentiable submanifolds of $\mathbb{R}^n$ is defined via their parameterisation(s).

Note that a real analytic map $\mathbf{f}$ defined on a connected open set is non-degenerate almost everywhere if and only if $1, f_1, \ldots f_n$ are linearly independent over R.

We are now ready to state the main result of Kleinbock and Margulis which solves the Baker and Sprinžuk conjectures in full generality, and also applies to non-degenerate maps.

Theorem 2.10 ([KlMar98, Theorem A])**.** *Let $\mathbf{f} = (f_1, \ldots, f_n)$ be a map defined on an open subset U of $\mathbb{R}^d$ which is non-degenerate almost everywhere. Then for any $\epsilon > 0$, for almost every $x \in U$, the inequality*

$$|p + q_1 f_1(x) + \cdots + q_n f_n(x)| < \Pi_+(\mathbf{q})^{-1-\epsilon} \tag{2.9}$$

has only finitely many solutions $(p, \mathbf{q}) \in \mathbb{Z} \times \mathbb{Z}^n$.

Strategy of proof of Theorem 2.10. Define

$$u_{\mathbf{f}(x)} = \begin{pmatrix} 1 & \mathbf{f}(x) \\ 0 & I_n \end{pmatrix} \in SL_{n+1}(\mathbb{R}),$$

where I_n is the $n \times n$ identity matrix. Also for $\mathbf{t} = (t_1, \ldots, t_n) \in \mathbb{Z}^n_{\geq 0}$, define

$$g_{\mathbf{t}} = \begin{pmatrix} e^t & & & \\ & e^{-t_1} & & \\ & & \ddots & \\ & & & e^{-t_n} \end{pmatrix}, \text{ where } t = t_1 + \cdots + t_n.$$

Given a solution $(p, \mathbf{q})$ to (2.9), define $t_i \in \mathbb{Z}^n_{\geq 0}$ be the smallest integers such that

$$e^{-t_i} \max(1, |q_i|) \leq \Pi_+(\mathbf{q})^{-\epsilon/(n+1)}.$$

Then, an elementary computation using (2.9) shows that

$$e^t|p + q_1 f_1(x) + \cdots + q_n f_n(x)| < e^{n(1+\gamma)} e^{-\gamma t}, \tag{2.10}$$

where $\gamma = \epsilon/(n + 1 + n\epsilon)$.

For $\Lambda \in \mathcal{L}_n$, let $\delta(\Lambda)$ denote the length of the shortest non-zero vector in Λ. (Thus, $\Lambda \in \mathcal{L}_n(\epsilon)$ if and only if $\delta(\Lambda) \geq \epsilon$.)

Therefore, it follows from (2.10) that if $(p, \mathbf{q})$ is a solution to (2.9) and t, $\mathbf{t}$ are as above, then

$$\delta(g_{\mathbf{t}} u_{\mathbf{f}(x)} \mathbb{Z}^{n+1}) < e^{n(1+\gamma)} e^{-\gamma t}. \tag{2.11}$$

Thus it is enough to prove that for any sufficiently small ball B centered at any point x_0 on which $\mathbf{f}$ is non-degenerate,

$$\sum_{\mathbf{t}} |\{x \in B \;\;:\;\; \delta(g_{\mathbf{t}} u_{\mathbf{f}(x)} \mathbb{Z}^{n+1}) < e^{n(1+\gamma)} e^{-\gamma t}\}| < \infty. \tag{2.12}$$

Indeed, if (2.12) holds, then the Borel-Cantelli Lemma ensures that for almost all $x \in B$, (2.11) holds only for finitely many $\mathbf{t}$. The equation (2.12) is proved in [KlMar98] by verifying the conditions of Theorem 2.6.

Khintchine-Groshev type results. The following generalization of Theorem 2.10 is proved in [BKM01]:

Theorem 2.11. *Let $\mathbf{f} = (f_1, \ldots, f_n)$ be a map defined on an open subset U of $\mathbb{R}^d$ which is non-degenerate almost everywhere. Let $\Psi : \mathbb{Z}^n \to \mathbb{R}_+$ be any function such that*

$$\Psi(q_1, \ldots, q_i, \ldots, q_n) \leq \Psi(q_1, \ldots, q_i', \ldots, q_n) \qquad \text{if } |q_i| > |q_i'| \text{ and } q_i q_i' > 0$$

Suppose that

$$\sum_{\mathbf{q} \in \mathbb{Z}^n} \Psi(q) < \infty.$$

Then for almost every $x \in U$, the inequality

$$|p + q_1 f_1(x) + \cdots + q_n f_n(x)| < \Psi(\mathbf{q})$$

has only finitely many solutions $(p, \mathbf{q}) \in \mathbb{Z}^{n+1}$.

Note that $\Psi(\mathbf{q}) = \Pi_+(\mathbf{q})^{-1-\epsilon}$ for $\epsilon > 0$ satisfies the conditions of Theorem 2.11, and thus Theorem 2.11 is indeed a generalization of Theorem 2.10. We note the following corollaries:

Corollary 2.12. *Let* $\mathbf{f}$ *be as in Theorem 2.11 and suppose* $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ *be any monotonic function satisfying*

$$\sum_{k=1}^{\infty} \psi(k) < \infty.$$

Let $\mathbf{r} = (r_1, \dots, r_n) \in \mathbb{R}^n_{>0}$ *be such that* $r_1 + \cdots + r_n = 1$*. For* $\mathbf{q} \in \mathbb{Z}^n$*, write* $\|\mathbf{q}\|_r = \max_{1 \le i \le n} |q_i|^r$*. Then, for almost every* $x \in U$*, the equation*

$$|p + q_1 f_1(x) + \cdots + q_n f_n(x)| < \psi(\|q\|_r)$$

has only finitely many solutions $(p, \mathbf{q}) \in \mathbb{Z}^{n+1}$*.*

Note that if $\mathbf{r} = (1/n, \dots, 1/n)$ then $\|q\|_r = \|q\|^n$. In this case Corollary 2.12 is was proved previously by in [B02] by a different method which does not involve quantitative non-divergence. However, without new ideas, this approach does not seem to be possible to extend to in order to prove the full version of Corollary 2.12 or Corollary 2.13 below.

Corollary 2.13. *Let* $\mathbf{f}$ *be as in Theorem 2.11 and suppose* $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ *be any monotonic function satisfying*

$$\sum_{k=1}^{\infty} (\log k)^{n-1} \psi(k) < \infty.$$

Then, for almost every $x \in U$*, the equation*

$$|p + q_1 f_1(x) + \cdots + q_n f_n(x)| < \psi(\Pi_+(q))$$

has only finitely many solutions $(p, \mathbf{q}) \in \mathbb{Z}^{n+1}$*.*

Strategy of Proof of Theorem 2.11. The idea is to break up into two cases, depending on the size of the gradient $\nabla(\mathbf{q} \cdot \mathbf{f})$. If $\|\nabla(\mathbf{q} \cdot \mathbf{f})\|$ is large, a direct argument is used. If it is small, the authors eventually find a way to reduce to Theorem 2.6.

Maybe a few more references to more recent applications.

2.2 Values of indefinite quadratic forms at integral points

2.2.1 The Oppenheim Conjecture.

In the 1970s unipotent flows made a dramatic appearance with regards to another, seemingly unrelated, problem rooted in number theory. Let

$$Q(x_1, \ldots, x_n) = \sum_{1 \le i \le j \le n} a_{ij} x_i x_j$$

be an indefinite quadratic form in n variables. It is clear that if Q is a multiple of a form with rational coefficients, then the set of values $Q(\mathbb{Z}^n)$ is a discrete subset of $\mathbb{R}$. Much deeper is the following conjecture:

Conjecture 2.14 (Oppenheim, 1931). *Suppose Q is not proportional to a rational form and $n \ge 5$. Then for every $\epsilon > 0$ there exists $x \in \mathbb{Z}^n \smallsetminus \{0\}$ such that $|Q(x)| < \epsilon$.*

This conjecture was later extended by Davenport to $n \ge 3$. Note that it is easy to construct counterexamples when $n = 2$; see e.g. [Esk10, Proposition 1.3].

In the mid 1970s Raghunathan observed a remarkable connection between the Oppenheim Conjecture and flows on the space of lattices $\mathcal{L}_n = G/\Gamma$, where $G = \mathrm{SL}(n, \mathbb{R})$ and $\Gamma = \mathrm{SL}(n, \mathbb{Z})$. (Implicitly this observation was made several decades earlier by Cassels and Swinnerton-Dyer, see [CSD55].) It can be summarized as follows:

Observation 2.15 (Raghunathan). *Let Q be an indefinite quadratic form, and let $H = SO(Q)$ denote its orthogonal group. Consider the orbit of the standard lattice $\mathbb{Z}^n \in \mathcal{L}_n$ under H. Then the following are equivalent:*

(a) *The orbit $H\mathbb{Z}^n$ is not relatively compact in $\mathcal{L}_n$.*

(b) *For all $\epsilon > 0$ there exists $x \in \mathbb{Z}^n \smallsetminus \{0\}$ such that $|Q(x)| < \epsilon$.*

Proof. Suppose (a) holds, so some sequence $h_k\mathbb{Z}^n$ leaves all compact sets. Then in view of the Mahler compactness criterion there exist $v_k \in h_k\mathbb{Z}^n \smallsetminus \{0\}$ such that $\|v_k\| \to 0$. Then also by continuity, $Q(v_k) \to 0$. But then $h_k^{-1}v_k \in \mathbb{Z}^n \smallsetminus \{0\}$, and $Q(h_k^{-1}v_k) = Q(v_k) \to 0$. Thus (b) holds.

On the other hand, assuming (b) we get a sequence of nonzero integer vectors x_k such that $Q(x_k) \to 0$ as $k \to \infty$; then, using the transitivity of the H-action on the level sets of Q one find $h_k \in H$ such that $h_k x_k \to 0$ as $k \to \infty$, proving (a). □

Raghunathan also explained why the case $n = 2$ is different: in that case $H = \mathrm{SO}(Q)$ has no unipotent elements. On the other hand, H is generated by its unipotent one-parameter subgroups when $n > 2$. Margulis's proof of the Oppenheim conjecture, given in [Mar87, Mar89a, Mar89b] uses Raghunathan's observation. In fact Margulis showed that that any relatively compact orbit of $\mathrm{SO}(2,1)$ in $\mathcal{L}_3$ is compact; this implies the Oppenheim Conjecture. For an account which stays reasonably close to Margulis's original proof see [BM00].

2.2.2 Ratner's Theorems

Raghunathan also conjectured the following:

Theorem 2.16 (Raghunathan's topological conjecture)**.** *Let G be a Lie group, $\Gamma \subset G$ a lattice, and $U \subset G$ a one-parameter unipotent subgroup. Suppose $x \in G/\Gamma$. Then there exists a subgroup F of G (generated by unipotents) such that the closure $\overline{Ux}$ of the orbit Ux is Fx.*

In the literature this conjecture was first stated in the paper [Dan81] and in a more general form in [Mar89a] (when the subgroup U is not necessarily unipotent but generated by unipotent elements). Raghunathan's conjecture was eventually proved in full generality by Ratner, see [Ra91b]. Yet, prior to Ratner's proof, Dani and Margulis established Theorem 2.16 in the special case when $G = \mathrm{SL}(3, \mathbb{R})$ and $U = \{u(t)\}$ is a "generic" one-parameter unipotent sugroup of G; that is, such that $u(t) - I$ has rank 2 for all $t \neq 0$. The work done in [DanMar90a], together with the methods developed in [Mar87, Mar89a, Mar89b, DanMar89] suggested an approach for proving the Raghunathan conjecture in general by studying the minimal invariant sets, and the limits of orbits of sequences of points tending to a minimal invariant set.

This strategy can be outlined as follows: Let x be a point in G/Γ, and U a connected unipotent subgroup of G. Denote by X the closure of Ux and consider a minimal closed U-invariant subset Y of X. Suppose that Ux is not closed (equivalently X is not equal to Ux). Then X should contain "many" translations of Y by elements from the normalizer $N(U)$ of U not belonging to U. After that one can try to prove that X contains orbits of bigger and bigger unipotent subgroups until one reaches horospherical subgroups. The basic tool in this strategy is the following fact. Let y be a point in X, and let g_n be a sequence of elements in G such that g_n converges to 1, g_n does not belong to $N(U)$, and $y_n = g_n y$ belongs to X. Then X contains AY where A is a nontrivial connected subset in $N(U)$ containing 1 and "transversal" to U. To prove this one has to observe that the orbits Uy_n and Uy are "almost parallel" in the direction of $N(U)$ most of the time in "the intermediate range".

In fact the set AU as a subset of $N(U)/U$ is the image of a nontrivial rational map from U into $N(U)/U$. Moreover this rational map sends 1 to 1 and also comes from a polynomial map from U into the closure of G/U in the affine space V containing G/U. This affine space V is the space of the rational representation of G such that V contains a vector the stabilizer of which is U (Chevalley theorem). Some elements of this proof are key to the current program of Lindenstrauss, Mohammadi, Margulis and Shah [LMMS19] of giving a fully effective version of Ratner's theorems.

It is worth pointing out that Ratner derived Theorem 2.16 from her measure classification theorem, conjectured earlier by Dani. Loosely speaking, it says that all U-invariant ergodic measures are very nice.

Theorem 2.17 (Ratner's measure classification theorem)**.** [Ra91a] *Let G be a Lie group, $\Gamma \subset G$ a lattice. Let U be a one-parameter unipotent subgroup of G. Then, any ergodic U-invariant measure is algebraic; namely, there exists $x \in G/\Gamma$ and a subgroup F of G such that Fx is closed, and μ is the F-invariant probability measure supported on Fx. (Also the group F is generated by unipotent elements and contains U).*

We note that following the publication of Ratner's papers, Margulis and Tomanov [MarTom94] gave a different proof of the measure classification theorem which in particular made use of entropy considerations. This proof turned out to be extremely influential for future developments in the area.

2.2.3 A quantitative version of the Oppenheim Conjecture.

Fix an indefinite quadratic form Q. Let ν be a continuous positive function on the sphere $\{v \in \mathbb{R}^n \mid \|v\| = 1\}$, and let $\Omega := \{v \in \mathbb{R}^n \mid \|v\| < \nu(v/\|v\|)\}$. We denote by $T\Omega$ the dilate of Ω by T. Define the following set:

$$V_{(a,b)}(\mathbb{R}) := \{x \in \mathbb{R}^n \mid a < Q(x) < b\}.$$

Also let $V_{(a,b)}(\mathbb{Z}) := \{x \in \mathbb{Z}^n \mid a < Q(x) < b\}$. The set $T\Omega \cap \mathbb{Z}^n$ consists of $O(T^n)$ points, and the set of values $Q(T\Omega \cap \mathbb{Z}^n)$ is contained in an interval of the form $[-\mu T^2, \mu T^2]$, where $\mu > 0$ is a constant depending on Q and Ω. Thus one might expect that for any interval (a, b), as $T \to \infty$,

$$|V_{(a,b)}(\mathbb{Z}) \cap T\Omega| \sim c_{Q,\Omega}(b-a)T^{n-2} \tag{2.13}$$

where $c_{Q,\Omega}$ is a constant depending on Q and Ω. This may be interpreted as "uniform distribution" of the sets $Q(\mathbb{Z}^n \cap T\Omega)$ in the real line. The main result of this section is that (2.13) holds if Q is not proportional to a rational form, and has signature (p, q) with $p \geq 3$, $q \geq 1$. We also determine the constant $c_{Q,\Omega}$.

If Q is an indefinite quadratic form in n variables, Ω is as above and (a, b) is an interval, it can be shown that there exists a constant $\lambda = \lambda_{Q,\Omega}$ so that as $T \to \infty$,

$$\operatorname{Vol}(V_{(a,b)}(\mathbb{R}) \cap T\Omega) \sim \lambda_{Q,\Omega}(b-a)T^{n-2} \tag{2.14}$$

The main result of [EMM98] is the following:

Theorem 2.18. *Let Q be an indefinite quadratic form of signature (p, q), with $p \geq 3$ and $q \geq 1$. Suppose Q is not proportional to a rational form. Then for any interval (a, b), as $T \to \infty$,*

$$|V_{(a,b)}(\mathbb{Z}) \cap T\Omega| \sim \lambda_{Q,\Omega}(b-a)T^{n-2} \tag{2.15}$$

where $n = p + q$, and $\lambda_{Q,\Omega}$ is as in (2.14).

The asymptotically exact lower bound was proved in [DanMar93]. For that Dani and Margulis introduced a *linearization method*, which we wil describe in more detail in §2.3.

If the signature of Q is $(2, 1)$ or $(2, 2)$ then no universal formula like (2.13) holds. In fact, the following theorem holds:

Theorem 2.19. *Let Ω_0 be the unit ball, and let $q = 1$ or 2. Then for every $\epsilon > 0$ and every interval (a, b) there exists a quadratic form Q of signature $(2, q)$ not proportional to a rational form, and a constant $c > 0$ such that for an infinite sequence $T_j \to \infty$,*

$$|V_{(a,b)}(\mathbb{Z}) \cap T\Omega_0| > cT_j^q(\log T_j)^{1-\epsilon}.$$

The case $q = 1$, $b \leq 0$ of Theorem 2.19 was noticed by P. Sarnak and worked out in detail in [Bre]. The quadratic forms constructed are of the form $x_1^2 + x_2^2 - \alpha x_3^2$, or $x_1^2 + x_2^2 - \alpha(x_3^2 + x_4^2)$, where α is extremely well approximated by squares of rational numbers.

2.2.4 More on signature (2,2).

Recall that a subspace is called isotropic if the restriction of the quadratic form to the subspace is identically zero. Observe also that whenever a form of signature $(2,2)$ has a rational isotropic subspace L then $L \cap T\Omega$ contains on the order of T^2 integral points x for which $Q(x) = 0$, hence $N_{Q,\Omega}(-\epsilon, \epsilon, T) \geq cT^2$, independently of the choice of ϵ. Thus to obtain an asymptotic formula similar to (2.15) in the signature $(2,2)$ case, we must exclude the contribution of rational isotropic subspaces. We remark that an irrational quadratic form of signature $(2,2)$ may have at most 4 rational isotropic subspaces (see [EMM05, Lemma 10.3]).

The space of quadratic forms in 4 variables is a linear space of dimension 10. Fix a norm $\|\cdot\|$ on this space.

Definition 2.20. (EWAS) A quadratic form Q is called *extremely well approximable by split forms (EWAS)* if for any $N > 0$ there exists a split integral form Q' and $2 \leq k \in \mathbb{R}$ such that

$$\left\| Q - \frac{1}{k}Q' \right\| \leq \frac{1}{k^N}.$$

The main result of [EMM05] is:

Theorem 2.21. *Suppose Ω is as above. Let Q be an indefinite quadratic form of signature $(2,2)$ which is not EWAS. Then for any interval (a,b), as $T \to \infty$,*

$$\tilde{N}_{Q,\Omega}(a,b,T) \sim \lambda_{Q,\Omega}(b-a)T^2, \tag{2.16}$$

where the constant $\lambda_{Q,\Omega}$ is as in (2.14), and $\tilde{N}_{Q,\Omega}$ counts the points not contained in isotropic subspaces.

Theorem 2.21 has implications for eigenvalue spacings on a flat 2-dimensional torus.

2.3 Linearization

In this section, we give a partial description of the "linearization" technique introduced in [DanMar93] and used for the proof of the lower bounds in the quantitative version of the Oppenheim conjecture. This technique, and in particular Theorem 2.24 below, is used in a multitude of applications of the theory of unipotent flows.

2.3.1 Non-ergodic measures invariant under a unipotent.

The collection $\mathcal{H}$. (Up to conjugation, this should be the collection of groups which appear in the definition of algebraic measure in the statement of Theorem 2.17).

Let G be a Lie group, Γ a discrete subgroup of G, and $\pi : G \to G/\Gamma$ the natural quotient map. Let $\mathcal{H}$ be the collection of all closed subgroups F of G such that $F \cap \Gamma$ is a lattice in F and the subgroup generated by unipotent one-parameter subgroups of G contained in F acts ergodically on $\pi(F) \cong F/(F \cap \Gamma)$ with respect to the F-invariant probability measure. This collection is countable (see [Ra91a, Theorem 1.1] or [DanMar93, Proposition 2.1] for different proofs of this result).

Let U be a unipotent one-parameter subgroup of G and $F \in \mathcal{H}$. Define

$$\begin{aligned} N(F,U) &= \{g \in G : U \subset gFg^{-1}\} \\ S(F,U) &= \bigcup \{N(F',U) : F' \in \mathcal{H},\ F' \subset F,\ \dim F' < \dim F\}. \end{aligned}$$

It is clear that if $g \in N(F,U)$ and $F \in \mathcal{H}$, then the orbit $U\pi(g)$ is contained in the closed subset $\pi(gF)$. More precisely, it is possible to prove the following (cf. [MozSha95, Lemma 2.4]):

Lemma 2.22. *Let $g \in G$ and $F \in \mathcal{H}$. Then $g \in N(F,U) \smallsetminus S(F,U)$ if and only if the group gFg^{-1} is the smallest closed subgroup of G which contains U and whose orbit through $\pi(g)$ is closed in G/Γ. Moreover in this case the action of U on $g\pi(F)$ is ergodic with respect to a finite gFg^{-1}-invariant measure.*

As a consequence of this lemma, one has

$$\pi\big(N(F,U) \smallsetminus S(F,U)\big) = \pi\big(N(F,U)\big) \smallsetminus \pi\big(S(F,U)\big) \qquad \forall F \in \mathcal{H}. \tag{2.17}$$

Theorem 2.17 states that given any U-ergodic invariant probability measure on G/Γ, there exists $F \in \mathcal{H}$ and $g \in G$ such that μ is $g^{-1}Fg$-invariant and $\mu\big(\pi(F)g\big) = 1$. Now decomposing any finite invariant measure into its ergodic component and using Lemma 2.22, one obtains the following description for any U-invariant probability measure on G/Γ (see [MozSha95, Theorem 2.2]).

Theorem 2.23 (Ratner)**.** *Let U be a unipotent one-parameter subgroup of G and let μ be a finite U-invariant measure on G/Γ. For every $F \in \mathcal{H}$, let μ_F denote the restriction of μ to $\pi(N(F,U) \setminus S(F,U))$. Then μ_F is U-invariant, and any U-ergodic component of μ_F is a gFg^{-1}-invariant measure on the closed orbit $g\pi(F)$ for some $g \in N(F,U) \setminus S(F,U)$.*

In particular, for all Borel measurable subsets A of G/Γ,

$$\mu(A) = \sum_{F \in \mathcal{H}^*} \mu_F(A),$$

where $\mathcal{H}^* \subset \mathcal{H}$ is a countable set consisting of one representative from each Γ-conjugacy class of elements in $\mathcal{H}$.

Remark. One often uses Theorem 2.23 in the following form: suppose μ is any U-invariant measure on G/Γ which is not G-invariant. Then there exists $F \in \mathcal{H}$ such that μ gives positive measure to some compact subset of $N(F,U) \setminus S(F,U)$.

2.3.2 The theorem of Dani–Margulis on uniform convergence

The "linearization" technique of Dani and Margulis was devised to understand which measures give positive weight to compact subsets subsets of $N(F,U) \setminus S(F,U)$. Using this technique Dani and Margulis proved the following theorem, which is important for many applications.

Theorem 2.24 ([DanMar93], Theorem 3)**.** *Let G be a connected Lie group and let Γ be a lattice in G. Let μ be the G-invariant probability measure on G/Γ. Let $U = \{u_t\}$ be an Ad-unipotent one-parameter subgroup of G and let f be a bounded continuous function on G/Γ. Let $\mathcal{D}$ be a compact subset of G/Γ and let $\epsilon > 0$ be given. Then there exist finitely many proper closed subgroups $F_1 = F_1(f, \mathcal{D}, \epsilon), \ldots, F_k = F_k(f, \mathcal{D}, \epsilon)$ such that $F_i \cap \Gamma$ is a lattice in F_i for all i, and compact subsets $C_1 = C_1(f, \mathcal{D}, \epsilon), \ldots, C_k = C_k(f, \mathcal{D}, \epsilon)$ of $N(F_1, U), \ldots, N(F_k, U)$ respectively, for which the following holds: For any compact subset $\mathcal{K}$ of $\mathcal{D} \setminus \bigcup_{1 \le i \le k} \pi(C_i)$ there exists a $T_0 \ge 0$ such that for all $x \in \mathcal{K}$ and $T > T_0$*

$$\Big| \frac{1}{T} \int_0^T f(u_t x)\, dt - \int_{G/\Gamma} f\, d\mu \Big| < \epsilon. \tag{2.18}$$

This theorem can be informally stated as follows: Fix f and $\epsilon > 0$. Then (2.18) holds uniformly in the base point x, as long as x is restricted to compact sets away from a finite union of "tubes" $N(F, U)$; the latter are associated with orbits which do not become equidistributed in G/Γ, because their closure is strictly smaller.)

Note that only finitely many F_k are needed in Theorem 2.24. This has the remarkable implication that if $F \in \mathcal{H} \smallsetminus \{F_1, \dots, F_k\}$, then (2.18) holds for $x \in N(F, U)$ even though Ux is not dense in G/Γ (the closure of Ux is Fx). Informally, this means that the non-dense orbits of U are themselves becoming equidistributed as they get longer.

In the rest of this subsection, we present some of the ideas developed for the proof of Theorem 2.24.

Linearization of neighborhoods of singular subsets. Let $F \in \mathcal{H}$. Let $\mathfrak{g}$ denote the Lie algebra of G and let $\mathfrak{f}$ denote its Lie subalgebra associated to F. For $d = \dim \mathfrak{f}$, put $V_F = \wedge^d \mathfrak{f}$, and consider the linear G-action on V_F via the representation $\wedge^d \operatorname{Ad}$, the d-th exterior power of the Adjoint representation of G on $\mathfrak{g}$. Fix $p_F \in \wedge^d \mathfrak{f} \smallsetminus \{0\}$, and let $\eta_F : G \to V_F$ be the map defined by $\eta_F(g) = g \cdot p_F = (\wedge^d \operatorname{Ad} g) \cdot p_F$ for all $g \in G$. Note that

$$\eta_F{}^{-1}(p_F) = \{g \in N_G(F) : \det(\operatorname{Ad} g|_{\mathfrak{f}}) = 1\}.$$

The idea of Dani and Margulis is to work in the representation space V_F (or more precisely $\bar{V}_F$, which is the quotient of V_F by the involution $v \to -v$) instead of G/Γ. In fact, for most of the argument one works only with the oribit $G \cdot p_F \subset V_F$. The advantage is that F is collapsed to a point (since it stabilizes p_F). The difficulty is that the map $\eta_F : G \to \bar{V}_F$ is not Γ-equivariant, and so becomes multivalued if considered as a map from G/Γ to V_F. Dani and Margulis showed that the orbit $\Gamma \cdot p_F$ is discrete in V_F [DanMar93, Theorem 3.4], and that

$$\eta_F{}^{-1}(A_F) = N(F, U) \tag{2.19}$$

[DanMar93, Prop. 3.2], where A_F be the linear span of $\eta_F(N(F, U))$ in V_F.

Let $N_G(F)$ denote the normalizer in G of F. Put $\Gamma_F = N_G(F) \cap \Gamma$. Then for any $\gamma \in \Gamma_F$, we have $\gamma\pi(F) = \pi(F)$, and hence γ preserves the volume of $\pi(F)$. Therefore $|\det(\operatorname{Ad} \gamma|_{\mathfrak{f}})| = 1$, and thus $\gamma \cdot p_F = \pm p_F$. Now define

$$\bar{V}_F = \begin{cases} V_F/\{\operatorname{Id}, -\operatorname{Id}\} & \text{if } \Gamma_F \cdot p_F = \{p_F, -p_F\} \\ V_F & \text{if } \Gamma_F \cdot p_F = p_F \end{cases}$$

The action of G factors through the quotient map of V_F onto $\bar{V}_F$. Let $\bar{p}_F$ denote the image of p_F in $\bar{V}_F$, and define $\bar{\eta}_F : G \to \bar{V}_F$ as $\bar{\eta}_F(g) = g \cdot \bar{p}_F$ for all $g \in G$. Then $\Gamma_F = \bar{\eta}_F{}^{-1}(\bar{p}_F) \cap \Gamma$. Let $\bar{A}_F$ denote the image of A_F in $\bar{V}_F$. Note that the inverse image of $\bar{A}_F$ in V_F is A_F.

For every $x \in G/\Gamma$, define the set of representatives of x in $\bar{V}_F$ to be

$$\mathrm{Rep}(x) = \bar{\eta}_F\big(\pi^{-1}(x)\big) = \bar{\eta}_F(x\Gamma) \subset \bar{V}_F.$$

The following lemma allows us to understand the map Rep in a special case:

Lemma 2.25. *If $x = \pi(g)$ and $g \in N(F,U) \smallsetminus S(F,U)$*

$$\mathrm{Rep}(x) \cap \bar{A}_F = \{g \cdot p_F\}.$$

Thus x has a single representative in $\bar{A}_F \subset V_F$.

Proof. Indeed, using (2.19),

$$\mathrm{Rep}\,\big(\pi(g)\big) \cap \bar{A}_F = \big(g\Gamma \cap N(F,U)\big) \cdot \bar{p}_F$$

Now suppose $\gamma \in \Gamma$ is such that $g\gamma \in N(F,U)$. Then g belongs to $N(\gamma F \gamma^{-1}, U)$ as well as to $N(F,U)$. Since $g \notin S(F,U)$, we must have $\gamma F \gamma^{-1} = F$, so $\gamma \in \Gamma_F$. Then $\gamma \bar{p}_F = \bar{p}_F$, so $\big(g\Gamma \cap N(F,U)\big) \cdot \bar{p}_F = \{g \cdot \bar{p}_F\}$ as required. □

We extend this observation in the following result (cf. [Sha91, Prop. 6.5]).

Proposition 2.26 ([DanMar93, Corollary 3.5])**.** *Let D be a compact subset of $\bar{A}_F$. Then for any compact set $\mathcal{K} \subset G/\Gamma \smallsetminus \pi\big(S(F,U)\big)$ there exists a neighborhood Φ of D in $\bar{V}_F$ such that any $x \in \mathcal{K}$ has at most one representative in Φ.*

Using this proposition, one can uniquely represent in Φ the parts of the unipotent trajectories in G/Γ lying in $\mathcal{K}$. Then one also has a "polynomial divergence" estimate similar to the ones used in §2.1:

Proposition 2.27 ([DanMar93, Proposition 4.2])**.** *Let a compact set $C \subset \bar{A}_F$ and an $\epsilon > 0$ be given. Then there exists a (larger) compact set $D \subset \bar{A}_F$ with the following property: For any neighborhood Φ of D in $\bar{V}_F$ there exists a neighborhood Ψ of C in $\bar{V}_F$ with $\Psi \subset \Phi$ such that the following holds: For any unipotent one parameter subgroup $\{u(t)\}$ of G, an element $w \in \bar{V}_H$ and and interval $I \subset \mathbb{R}$, if $u(t_0)w \notin \Phi$ for some $t_0 \in I$ then,*

$$|\{t \in I : u(t)w \in \Psi\}| \leq \epsilon \cdot |\{t \in I : u(t)w \in \Phi\}|. \tag{2.20}$$

As a consequence, Dani and Margulis derive the following result of independent interest:

Theorem 2.28 ([DanMar93, Theorem 1])**.** *Let G be a connected Lie group and let Γ be a discrete subgroup of G. Let U be any closed connected subgroup of G which is generated by the Ad-unipotent elements contained in it. Let $\mathcal{K}$ be a compact subset of $G/\Gamma \smallsetminus \bigcup_{F \in \mathcal{H}} N(F,U)$. Then for any $\epsilon > 0$ there exists a neighbourhood Ω of $\bigcup_{F \in \mathcal{H}} N(F,U)$ such that for any Ad-unipotent one-parameter subgroup $\{u_t\}$ of G, any $x \in \mathcal{K}$ and any $T \geq 0$,*

$$|\{t \in [0,T] : u_t x \in \Omega\}| < \epsilon T.$$

Proof of Theorem 2.24. The proof relies on Ratner's measure classification theorem (Theorem 2.17) as well as on a refined version of Theorem 2.28, which carefully handles trajectories of points in some $N(F,U)$.

2.4 Upper bounds in the Oppenheim conjecture

We now return to the set-up of the quantitative version of the Oppenheim conjecture stated in §2.2.3, and describe some ideas involved in the proof of the upper bounds.

2.4.1 Passage to the space of lattices.

Here we relate the counting problem of Theorem 2.18 to a certain integral expression involving the orthogonal group of the quadratic form and the space $\mathcal{L}_n$. Roughly this is done as follows. Let f be a bounded function on $\mathbb{R}^n \smallsetminus \{0\}$ vanishing outside a compact subset. For a lattice $\Lambda \in \mathcal{L}_n$ let

$$\tilde{f}(\Lambda) = \sum_{v \in \Lambda \smallsetminus \{0\}} f(v) \tag{2.21}$$

(the function $\tilde{f}$ is called the *Siegel Transform* of f). The proof is based on the identity of the form

$$\int_K \tilde{f}(a_t k \Lambda)\, dk = \sum_{v \in \Lambda \smallsetminus \{0\}} \int_K f(a_t k v)\, dk \tag{2.22}$$

obtained by integrating (2.21). In (2.22) $\{a_t\}$ is a certain diagonal subgroup of the orthogonal group of Q, and K is a maximal compact subgroup of the orthogonal group of Q. Then for an appropriate function f, the right hand side is related to the number of lattice points $v \in [e^t/2, e^t]\partial\Omega$ with $a < Q(v) < b$. The asymptotics of the left-hand side is then established using the ergodic theory of unipotent flows and some other techniques. Namely it is shown in [EMM98] that Theorem 2.18 can be reduced to the following theorem:

Theorem 2.29. *Suppose $p \geq 3$, $q \geq 1$. Let $\Lambda \in \mathcal{L}_n$ be a unimodular lattice such that $H\Lambda$ is not closed. Let ν be any continuous function on K. Then*

$$\lim_{t \to +\infty} \int_K \tilde{f}(a_t k \Lambda)\nu(k)\, dm(k) = \int_K \nu\, dm \int_{\mathcal{L}_n} \tilde{f}(\Delta)\, d\mu(\Delta). \tag{2.23}$$

Note that if we replace $\tilde{f}$ by a bounded continuous function ϕ, then (2.23) follows easily from Theorem 2.24. (This was the original motivation for Theorem 2.24.) The fact that Theorem 2.24 deals with unipotents and Theorem 2.29 deals with large spheres is not a serious obstacle, since large spheres can be approximated by unipotents. In fact, the integral in (2.23) can be rewritten as

$$\int_B \left(\frac{1}{T(x)} \int_0^{T(x)} \phi(u_t x)\, dm(k) \right) dx,$$

where B is a suitable subset of G and U is a suitable unipotent. Now by Theorem 2.24, the inner integral tends to $\int_{G/\Gamma} \phi$ uniformly as long as x is in a compact set away from an explicitly described set E, where E is a finite union of neighborhoods of sets of the form $\pi(C)$ where C is a compact subset of some $N(F, U)$. By direct calculation one can show that only a small part of B is near E, hence (2.23) holds.

However, for a non-negative bounded continuous function f on $\mathbb{R}^n$, the function $\tilde{f}$ defined in (2.21) is unbounded (it is in $L^s(\mathcal{L}_n)$ for $1 \leq s < n$). As it was done in [DanMar93], it is possible to obtain asymptotically exact lower bounds by considering bounded continuous functions $\phi \leq \tilde{f}$. But to prove the upper bounds in the theorems stated above one needs to examine carefully the situation at the "cusp" of G/Γ, i.e. outside of compact sets.

The functions α_i and α. Let Λ be a lattice in $\mathbb{R}^n$. Recall that the notion of a Λ-rational subspace and the function d_Λ was defined in §2.1 (following the statement of Theorem 2.6). Let us introduce the following notation:

$$\alpha_i(\Lambda) = \sup\left\{ \frac{1}{d_\Lambda(L)} \,\middle|\, L \text{ is a } \Lambda\text{-rational subspace of dimension } i \right\}, \quad 0 \le i \le n,$$
$$\alpha(\Lambda) = \max_{0\le i\le n} \alpha_i(\Lambda). \tag{2.24}$$

By [Sch68, Lemma 2], for any bounded compactly supported function f on $\mathbb{R}^n$ there exists a positive constant $c = c(f)$ such that $\tilde{f}(\Lambda) < c\alpha(\Lambda)$ for any $\Lambda \in \mathcal{L}_n$. The upper bound in Theorem 2.18 is proved by combining the above observation with the following integrability estimate:

Theorem 2.30 ([EMM98])**.** *If $p \ge 3$, $q \ge 1$ and $0 < s < 2$, or if $p = 2$, $q \ge 1$ and $0 < s < 1$, then for any $\Lambda \in \mathcal{L}_n$*

$$\sup_{t>0} \int_K \alpha(a_t k\Lambda)^s \, dm(k) < \infty.$$

The upper bound is uniform as Λ varies over compact subsets of $\mathcal{L}_n$.

This result can be interpreted as follows. For $\Lambda \in \mathcal{L}_n$ and $h \in H$, let $f(h) = \alpha(h\Lambda)$. Since α is left-$\hat{K}$ invariant, f is a function on the symmetric space $X = K\backslash H$. Theorem 2.30 is the statement that if if $p \ge 3$, then the averages of f^s, $0 < s < 2$ over the sets Ka_tK in X remain bounded as $t \to \infty$, and the bound is uniform as one varies the base point Λ over compact sets.

If $(p,q) = (2,1)$ or $(2,2)$, Theorem 2.30 does not hold even for $s = 1$. The following result is, in general, best possible:

Theorem 2.31 ([EMM98])**.** *If $p = 2$ and $q = 2$, or if $p = 2$ and $q = 1$, then for any $\Lambda \in \mathcal{L}_n$*

$$\sup_{t>1} \frac{1}{t} \int_K \alpha(a_t k\Lambda) \, dm(k) < \infty, \tag{2.25}$$

The upper bound is uniform as Λ varies over compact subsets of $\mathcal{L}_n$.

2.4.2 Margulis functions.

We now present some ideas from the proof of Theorem 2.30 and Theorem 2.31. We recall the notation from §2.2.1 and §2.4.1: $G = \mathrm{SL}(n,\mathbb{R})$, $\Gamma = \mathrm{SL}(n,\mathbb{Z})$, $\hat{K} \cong \mathrm{SO}(n)$ is a maximal compact subgroup of G, $H \cong \mathrm{SO}(p,q) \subset G$, $K = H \cap \hat{K} = \mathrm{SO}(p) \times \mathrm{SO}(q)$ is a maximal compact subgroup of H, and X is the symmetric space $K\backslash H$. Let $m(\cdot)$ denote the normalized Haar measure on K. Let $\{a_t : t \in \mathbb{R}\}$ be a self-adjoint one-parameter subgroup of $\mathrm{SO}(2,1)$, where $\mathrm{SO}(2,1)$ is embedded into $\mathrm{SO}(p,q)$, so that a_t is conjugate to the diagonal matrix with entries $(e^t, 1, \dots, 1, e^{-t})$.

The strategy of the proof is to construct what we now call a *Margulis function.* This idea has been extremely influential, see for example the survey [EskMoz22].

Let Y be a space on which H acts. (In our case, $Y = \mathcal{L}_n$). For $t > 0$, let A_t be the averaging operator taking a function $\phi : Y \to \mathbb{R}$ to the function $A_t\phi : Y \to \mathbb{R}$ defined by

$$(A_t\phi)(x) = \int_K \phi(a_t k x)\, dm(k). \tag{2.26}$$

Definition 2.32. A K-invariant function $f : Y \to [1, \infty]$ is called a *Margulis function* (for the averages A_t) if it satisfies the following properties:

(a) There exists $\sigma > 1$ such that for all $0 \le t \le 1$ and all $x \in Y$,

$$\sigma^{-1} f(x) \le f(a_t x) \le \sigma f(x). \tag{2.27}$$

(This holds if $\log f$ is uniformly continuous along the H-orbits).

(b) For every $c_0 > 0$ there exist $\tau > 0$ and $b_0 > 0$ such that for all $x \in Y$,

$$A_\tau f(x) \le c_0 f(x) + b_0. \tag{2.28}$$

(c) f is bounded on compact subsets of Y. For any $\ell > 0$, the set $\{x : f(x) \le \ell\}$ is a compact subset of Y.

We have the following abstract lemma:

Lemma 2.33. *Suppose f is a Margulis function on Y. Then, for all $c < 1$ there exists $t_0 > 0$ (depending on σ and c) and $b > 0$ (depending only on b_0, c_0 and σ) such that for all $t > t_0$ and all $x \in Y$,*

$$(A_t f)(x) \le c f(x) + b. \tag{2.29}$$

A more general version of this lemma is proved in [EMM98, §5.3]. The reader may also refer to a simplified proof in [EskMoz22, §3], specialized to the case $H = \mathrm{SL}(2, \mathbb{R})$.

From the proof of Lemma 2.33, one can deduce the following variant:

Lemma 2.34. *For every $\sigma > 1$ there exists $c_0 > 0$ such that the following holds. Suppose $f : Y \to [1, \infty)$ is a K-invariant function satisfying* (a) *and* (c) *of Definition 2.32, and let A_t be as in* (2.26)*. Suppose also that there exists $\tau > 0$ and $b_0 > 0$ such that* (2.28) *holds. Then f is a Margulis function for the averages A_t.*

For a wider perspective on Margulis functions and many related results see the survey [EskMoz22].

Strategy of the proof of Theorem 2.30. Suppose $0 < s < 2$. If the function α^s were a Margulis function on G/Γ, then Theorem 2.30 would follow immediately from (2.29). Even though this is not true, the idea is to construct a Margulis function f on G/Γ which is within a bounded multiple of α^s.

If $p \geq 3$ and $0 < s < 2$, or if $(p, q) = (2, 1)$ or $(2, 2)$ and $0 < s < 1$, it is shown in [EMM98, §5.3] that for any $c > 0$ there exist $t > 0$ such that the functions α_i^s satisfy the following system of integral inequalities:

$$A_t \alpha_i^s \leq c_i \alpha_i^s + e^{2t} \max_{0 < j \leq \min(n-i, i)} \sqrt{\alpha_{i+j}^s \alpha_{i-j}^s}, \tag{2.30}$$

where A_t is the averaging operator $(A_t f)(\Delta) = \int_K f(a_t k \Lambda)$ and $c_i \leq c$. If $(p, q) = (2, 1)$ or $(2, 2)$ and $s = 1$, then (2.30) also holds (for suitably modified functions α_i), but some of the constants c_i cannot be made smaller than 1.

In §2.4.4 we will show that if (2.30) holds, then for any $\epsilon > 0$, the function $f = f_{\epsilon,s} = \sum_{0 \leq i \leq n} \epsilon^{i(n-i)} \alpha_i^s$ is the desired Margulis function, and it follows from (2.24) that the ratio of α^s and f is uniformly bounded between two positive constants.

We now outline the proof of (2.30).

2.4.3 A system of inequalities.

By a direct calculation one can prove the following:

Proposition 2.35. *Let $\{a_t \mid t \in \mathbb{R}\}$ be a self-adjoint one-parameter subgroup of* $\mathrm{SO}(2,1)$*. Let $p, q \in \mathbb{N}$ and let $0 < i < p + q = n$. Let*

$$F(i) = \{x_1 \wedge \cdots \wedge x_i \mid x_1, \ldots, x_i \in \mathbb{R}^n\} \subset \bigwedge^i(\mathbb{R}^n).$$

Then, if $p \geq 3$, or if $p = 2$, $q = 2$ and $i \neq 2$, for any $0 < s < 2$ one has

$$\lim_{t \to \infty} \sup_{v \in F(i),\ \|v\|=1} \int_K \frac{dm(k)}{\|a_t k v\|^s} = 0. \tag{2.31}$$

where $K = \mathrm{SO}(p) \times \mathrm{SO}(q)$ and $\mathrm{SO}(2,1)$ is embedded into $\mathrm{SO}(p,q)$. If $p = 2$ and $q = 1$, or if $p = 2$, $q = 2$ and $i = 2$, then (2.31) holds for any $0 < s < 1$.

Lemma 2.36. *Let $\{a_t\}$, p, q and n be as in Proposition 2.35. Denote $\mathrm{SO}(p) \times \mathrm{SO}(q)$ by K. Suppose $p \geq 3$, $q \geq 1$ and $0 < i < n$, or $p = 2$, $q = 2$ and $i = 1$ or 3. Then for any $0 < s < 2$, and any $c > 0$ there exist $t > 0$ such that for any $\Lambda \in \mathcal{L}_n$,*

$$\int_K \alpha_i(a_t k\Lambda)^s \, dm(k) < \frac{c}{2}\alpha_i(\Lambda)^s + e^{2t} \max_{0<j\leq \min\{n-i,i\}} \left(\sqrt{\alpha_{i+j}(\Lambda)\alpha_{i-j}(\Lambda)}\,\right)^s. \tag{2.32}$$

If $p = 2$, $q = 1$ and $i = 1, 2$, or if $p = 2$, $q = 2$ and $i = 2$, then for any $0 < s < 1$ and any $c > 0$ there exist $t > 0$ such that (2.32) holds.

Proof. Fix $c > 0$. In view of Proposition 2.35 one can find $t > 0$ such that

$$\int_K \frac{dm(k)}{\|a_t k v\|^s} < \frac{c}{2} \cdot \frac{1}{\|v\|^s} \tag{2.33}$$

for any $v \in F(i) \smallsetminus \{0\}$. Let $\Lambda \in \mathcal{L}_n$. There exists a Λ-rational subspace L_i of dimension i such that

$$\frac{1}{d_\Lambda(L_i)} = \alpha_i(\Lambda). \tag{2.34}$$

The inequality (2.33) therefore implies

$$\int_K \frac{dm(k)}{d_{a_t k\Lambda}(a_t k L_i)^s} < \frac{c}{2} \cdot \frac{1}{d_\Lambda(L_i)^s}. \tag{2.35}$$

Observe that

$$e^{-t} \leq \frac{\|a_t v\|}{\|v\|} \leq e^t \qquad \forall\, 0 < j < n \text{ and } \forall\, v \in F(j) \smallsetminus \{0\}. \tag{2.36}$$

Let us denote by Ψ_i the set of Λ-rational subspaces L of dimension i with $d_\Lambda(L) < e^{2t} d_\Lambda(L_i)$. We get from (2.36) that for a Λ-rational i-dimensional subspace $L \notin \Psi_i$

$$d_{a_t k\Lambda}(a_t k L) > d_{a_t k\Lambda}(a_t k L_i), \qquad k \in K. \tag{2.37}$$

It follows from (2.35), (2.37) and the definition of α_i that

$$\int_K \alpha_i(a_t k\Lambda)^s \, dm(k) < \frac{c}{2}\alpha_i(\Lambda)^s \text{ if } \Psi_i = \{L_i\}. \tag{2.38}$$

Assume now that $\Psi_i \neq \{L_i\}$, and let $M \in \Psi_i \smallsetminus \{L_i\}$. Then $\dim(M + L_i)$ is equal to $i + j$ where $j > 0$. Now using (2.34), (2.36) and the fact that

$$d_\Lambda(L) d_\Lambda(M) \geq d_\Lambda(L \cap M) d_\Lambda(L + M)$$

(see [EMM98, Lemma 5.6]), we get that for any $k \in K$

$$\begin{aligned} \alpha_i(a_t k \Lambda) < e^t \alpha_i(\Lambda) = \frac{e^t}{d_\Lambda(L_i)} < \frac{e^{2t}}{\sqrt{d_\Lambda(L_i) d_\Lambda(M)}} \\ \leq \frac{e^{2t}}{\sqrt{d_\Lambda(L_i \cap M) d_\Lambda(L_i + M)}} \leq e^{2t} \sqrt{\alpha_{i+j}(\Lambda) \alpha_{i-j}(\Lambda)}\,. \end{aligned} \tag{2.39}$$

Hence if $\Psi_i \neq \{L_i\}$

$$\int_K \alpha_i(a_t k \Lambda)^s \, dm(k) \leq e^{2t} \max_{0<j\leq \min\{n-i,i\}} \left(\sqrt{\alpha_{i+j}(\Lambda) \alpha_{i-j}(\Lambda)} \right)^s. \tag{2.40}$$

Combining (2.38) and (2.40), we obtain (2.32). □

2.4.4 Averages over large spheres.

In this subsection we complete the proof of Theorem 2.30.

Proof of Theorem 2.30. It is easy to see that each of the functions α_i^s is K-invariant and has properties (a) and (c) of Definition 2.32. In particular, there exists $\sigma > 1$ such that for all $1 \leq i \leq n$, the equation (2.27) holds for α_i^s. Let c_0 be such that Lemma 2.34 holds for this σ.

Applying Lemma 2.36, we see that there exists $\tau > 0$ such that for any $0 < i < n$

$$A_\tau \alpha_i^s < \frac{c_0}{2} \alpha_i^s + e^{2\tau} \max_{0<j\leq \min\{n-i,i\}} \sqrt{\alpha_{i+j}^s \alpha_{i-j}^s}. \tag{2.41}$$

Let us denote $q(i) = i(n-i)$. Then by a direct computation

$$2q(i) - q(i+j) - q(i-j) = 2j^2.$$

Therefore we get from (2.41) that for any $0 < i < n$, and any $0 < \epsilon < 1$

$$\begin{aligned} A_\tau(\epsilon^{q(i)} \alpha_i^s) < \frac{c_0}{2} \epsilon^{q(i)} \alpha_i^s + e^{2\tau} \max_{0<j\leq \min\{n-i,i\}} \epsilon^{q(i) - \frac{q(i+j)+q(i-j)}{2}} \sqrt{\epsilon^{q(i+j)} \alpha_{i+j}^s \epsilon^{q(i-j)} \alpha_{i-j}^s} \\ \leq \frac{c_0}{2} \epsilon^{q(i)} \alpha_i^s + \epsilon e^{2\tau} \max_{0<j\leq \min\{n-i,i\}} \sqrt{\epsilon^{q(i+j)} \alpha_{i+j}^s \epsilon^{q(i-j)} \alpha_{i-j}^s}. \end{aligned} \tag{2.42}$$

Consider the linear combination

$$f_{\epsilon,s} = \sum_{0 \le i \le n} \epsilon^{q(i)} \, \alpha_i^s.$$

The function $f_{\epsilon,s}$ then also has properties (a) and (c) of Definition 2.32. Since $\epsilon^{q(i)}\alpha_i^s < f_{\epsilon,s}, \alpha_0 = 1$ and $\alpha_n = 1$, the inequalities (2.42) imply the following inequality:

$$A_\tau f_{\epsilon,s} < 2 + \frac{c_0}{2} f_{\epsilon,s} + n\epsilon e^{2\tau} f_{\epsilon,s}. \tag{2.43}$$

Taking $\epsilon = \frac{c_0}{2n} e^{-2\tau}$, we see that there exists $\tau > 0$ such that

$$A_\tau f_{\epsilon,s} < c_0 f_{\epsilon,s} + 2.$$

Then, by Lemma 2.34, $f_{\epsilon,s}$ is a Margulis function on G/Γ. Since

$$\alpha_i^s \le \epsilon^{-q(i)} f_{\epsilon,s},$$

Lemma 2.33 implies that there exists a constant $B > 0$ so that for each i and all $t > 0$,

$$\int_K \alpha_i(a_t k \Lambda)^s \, dm(k) < B,$$

and that the bound is uniform as Λ varies over compact subsets of G/Γ. From this the theorem follows. □

2.5 Effective Estimates

In this section, we present some work of Margulis and co-authors regarding effective equidistribution and effective estimates on diophantine inequalities. This is far from a comprehensive account; we choose to focus on two papers relating to the Oppenheim conjecture. A much more detailed and comprehensive survey is given in [EinMoh22].

2.5.1 Effective solution of the Oppehneim conjecture

The following was proven by Dani and Margulis in [DanMar89]:

Theorem 2.37. *Let Q be an indefinite ternary quadratic form which is not proportional to an integral form. Then the set*

$$\{Q(v) \;\; : \;\; v \in \mathbb{Z}^3, \; v \text{ primitive}\}$$

is dense in $\mathbb{R}$.

In the following, we implicitly assume all integral quadratic forms we consider are primitive in the sense that they are not a nontrivial integer multiple of another integral quadratic form.

The main result of the paper [LinMar14] is the following quantification of Theorem 2.37.

Theorem 2.38. *Let Q_1 be an indefinite, ternary quadratic form with* $\det Q_1 = 1$ *and suppose* $\epsilon > 0$*. Then for any* $T \geq T_0(\epsilon)\|Q1\|^{K_1}$ *at least one of the following holds:*

(i) *There is an integral quadratic form* Q_2 *with* $|\det(Q_2)| < T$ *and* $\|Q_1 - \lambda Q_2\| \ll \|Q_1\|T^{-1}$ *where* $\lambda = |\det(Q_2)|^{-1/3}$.

(ii) *For any* $\xi \in [-(\log T)\kappa_2, (\log T)\kappa_2]$ *there is a primitive integer vector* $v \in \mathbb{Z}^3$ *with* $0 < \|v\| < T^{K_3}$ *satisfying*

$$|Q_1(v) - \xi| \ll (\log T)^{-\kappa_2}$$

(with K_1*,* κ_2*,* K_3*, and the implicit constants absolute).*

Though there are significant differences, the strategy which is used in the paper has many similarities with the strategy which was used by Margulis in [Mar87, Mar89a] and subsequent papers by Dani and Margulis [DanMar89, DanMar90a, DanMar90b, DanMar91]. The main ingredient in these strategies is to prove that an orbit closure contains orbits of additional subgroups. In the original approach, this is achieved using minimal sets for appropriately chosen subactions, while in [LinMar14] the beginning point of the orbit of the new subgroup is moving. To make this approach work, the authors need to control how this base point changes so it remains sufficiently generic in an appropriate quantitative sense.

2.5.2 Power law estimates in dimension at least 5.

Note that in the above result the dependence on the parameter T is logarithmic. If the number of variables d is greater or equal to 5, power estimates are possible. We now present the main result of [BGHM22], which is based in part on earlier work of Götze and Margulis.

To state the result we use the following notation. Denote by Q the symmetric matrix in $GL(d, \mathbb{R})$ associated with the form $Q(x) := \langle x, Qx \rangle$, where $\langle \cdot, \cdot \rangle$ is the standard Euclidean scalar product on $\mathbb{R}^d$. Let Q_+ denote the unique positive symmetric matrix such that $Q_+^2 = Q^2$ and let $Q_+(x) = \langle x, Q_+ x \rangle$ denote the associated positive form with eigenvalues being the eigenvalues of Q in absolute value. Let q, resp. q_0, denote the largest, resp. smallest, of the absolute value of the eigenvalues of Q and assume $q_0 \geq 1$. In the Oppenheim conjecture, we are concerned with the inequality $|Q(m)| < \epsilon$; we can replace the form Q by Q/ϵ and consider the inequality $|Q(m)| < 1$. The following effective estimate is proved in [BGHM22]:

Theorem 2.39. *For all indefinite and non-degenerate quadratic forms Q of dimension $d \geq 5$ and signature (r, s) there exists for any $\delta > 0$ a non-trivial integral solution $m \in \mathbb{Z}^d \setminus \{0\}$ to the Diophantine inequality $|Q(m)| < 1$ satisfying*

$$\|Q_+^{1/2} m\| \ll_{\delta,d} (q/q_0)^{\frac{d+1}{d-2}} q^{1/2+\max\{\rho d+2,d+1\}/(d-4)+\delta},$$

where the dependency on the signature (r,s) is given by

$$\rho := \rho(r,s) = \begin{cases} \frac{1}{2}\frac{r}{s} & \text{for } r \geq s+3 \\ \frac{1}{2}\frac{s+2}{s-1} & \text{for } r = s+2 \text{ or } r = s+1 \\ \frac{1}{2}\frac{s+1}{s-2} & \text{for } r = s \end{cases}$$

In particular, for indefinite non-degenerate forms in $d \geq 5$ variables of signature (r, s) and eigenvalues in absolute value contained in a compact set $[1, C]$, i.e $1 \leq q_0 \leq q \leq C$, Theorem 2.39 yields non-trivial solutions $m \in \mathbb{Z}^d$ of $|Q(m)| < \epsilon$ of size bounded by

$$\|m\| \ll_{C,\delta} \epsilon^{-\max\{\rho d+2,d+1\}/(d-4)-\delta}.$$

The proof of Theorem 2.39 relies on Götze's analytic approach [Go04] via Theta series, translating the lattice point counting problem into averages of certain functions on the space of lattices, for which the authors extend the mean-value estimates obtained by Eskin-Margulis-Mozes [EMM98].

Other results. We should also remark that in some cases, especially if the acting group has a spectral gap, Margulis and co-authors were able to use a purely dynamical approach to get error estimates which are powers (and not logs). For a sophisticated example, see [EMMV20]. I think we should at least mention [LMMS19] as well as the earlier paper of Einsiedler, Margulis and Venkatesh.

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