MARGULIS FUNCTIONS AND THEIR APPLICATIONS

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1. Definition and basic properties

Motivation. In many cases, one wants to show that trajectories of some dynamical system spend most of the time in compact sets, or more generally, avoid on average a certain subset of the space. The construction of a Margulis function allows one to obtain remarkably sharp estimates of this type. The first construction is due to Margulis in [EMM98] to show quantitative recurrence for the action of $SO(p,q)$ on $SL(p+q,\mathbb{R})/SL(p+q,\mathbb{Z})$; this is used in the proof of the "quantitative Oppenheim conjecture". The difficulty in this problem is related to the complicated geometry of of the noncompact part of the space. However, the method is remarkably versatile, and has seen many other applications.

We now proceed with the formal definition, and give examples later. The reader is encouraged to skip ahead to the examples as necessary.

Let X be the space where our dynamics takes place. First we need an averaging operator A. This is formally just a linear map from the space $C(X)$ of continuous functions on X to itself, where X is the space where the dynamics takes place. We always assume that A is a Markov operator, i.e. that A takes non-negative functions to non-negative functions and takes the constant function 1 to itself.

Let Y be a possibly empty subset of X. If Y is not empty, we assume that it is invariant in the sense that if $h \in C(X)$ is supported on $X \setminus Y$ then so is Ah.

Definition 1.1. A continuous function $f : X \to [1,\infty]$ is called a Margulis function for Y if the following hold:

(a) $f(x) = \infty$ if and only if $x \in Y$. For each $\ell > 0$, the set $\{x \;:\; f(x) \le \ell\}$ is a compact subset of $X \setminus Y$.
(b) There exists $c < 1$ and $b < \infty$ such that for all $x \in X$,

$$(Af)(x) \le cf(x) + b. \tag{1.1}$$

The continuity assumption on f is often modified; this will be mentioned below.

We now state an immediate consequence of the definition:

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Lemma 1.2. *Suppose $x \in X \setminus Y$. Then, there exists $N = N(x)$ such that for all $n > N$,*

$$(A^n f)(x) \leq \frac{2b}{1-c} < \infty. \tag{1.2}$$

The constant $N(x)$ depends only on $f(x)$ and can thus be chosen uniformly over the compact sets $\{x \; : \; f(x) \leq \ell\}$.

Proof. By iterating (1.1) we obtain

$$(A^n f) \leq c^n f(x) + c^{n-1} b + \cdots + cb + b \leq c^n f(x) + \frac{b}{1-c},$$

where for the last estimate we summed the geometric series. Now choose n so that $c^n f(x) < b/(1-c)$. □

2. Random Walks

In this setting, a Margulis function is also called a Foster-Lyapunov (or drift) function and has been used extensively. See the book [MT09] for further references.

Suppose we are considering a random walk on X. This means that for each $x \in X$ we have a probability measure μ_x on X so that the probability of moving in one step of the random walk from x into some subset $E \subset X$ is $\mu_x(E)$. Now, for $h \in C(X)$, let

$$(Ah)(x) = \int h \, d\mu_x,$$

so A is the averaging operator with respect to one step of the random walk. Then A^n is the averaging operator with respect to n steps of the random walk, and we can write $A^n h = \int_X h \, d\mu_x^n$, where $\mu_x^n(E)$ is the probability of moving in n steps of the random walk into some set E.

Lemma 2.1. *Suppose $Y \subset X$ and that a Margulis function f can be constructed for Y. Then, for any $\epsilon > 0$ there exists a compact subset F_ϵ of $X \setminus Y$ such that for any $x \in X \setminus Y$, for all sufficiently large n (depending on x and ϵ), $\mu_x^n(F_\epsilon) > 1 - \epsilon$.*

In particular, Lemma 2.1 shows that any weak-star limit μ_x^∞ of the measures μ_x^n is a probability measure satisfying $\mu_x^\infty(Y) = 0$.

Proof. The equation (1.2) has the interpretation that for any $x \in X$, for large enough n,

$$\int f \, d\mu_x^n \leq 2b/(1-c).$$

Now suppose $\epsilon > 0$ is given, and choose $\ell > \frac{2b}{(1-c)\epsilon}$. By Markov's inequality we have

$$\mu_x^n(\{x \; : \; f(x) > \ell\}) \leq \frac{2b}{(1-c)\ell} < \epsilon.$$

Thus, the μ_x^n measure of the compact set $\{x \ : \ f(x) \leq \ell\}$ is at least $1 - \epsilon$. □

The existence of a Margulis function also implies certain large deviation results, see e.g. §6.

3. Actions of semisimple groups

Suppose the space X admits a continuous action of a semisimple group G. For simplicity of presentation, we will assume in this section that $G = SL(2, \mathbb{R})$. For the case where $G = SO(p, q)$ see the original paper [EMM98].

Recall that G acts on the upper half plane $\mathbb{H}$ by Möbius transformations. It is convenient for to write this action as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{dz - b}{-cz + a}.$$

This is a right action of G, and the stabilizer of $i \in \mathbb{H}$ is $K = SO(2)$. Thus, $\mathbb{H}$ is canonically identified with $K \backslash G$. This action is by hyperbolic isometries, thus $d_{\mathbb{H}}(Kg_1 g, Kg_2 g) = d_{\mathbb{H}}(Kg_1, Kg_2)$ for all $g_1, g_2, g \in G$, where $d_{\mathbb{H}}$ is the hyperbolic metric on $\mathbb{H}$. We will also use this action to identify the unit tangent bundle $T^1(\mathbb{H})$ of $\mathbb{H}$ with G.

Let $a_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$, $r_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. Then, under the identification of G with $T^1(\mathbb{H})$, left multiplication by a_t on G corresponds to geodesic flow for time t on $T^1(\mathbb{H})$. In particular,

$$d_{\mathbb{H}}(Ka_t g, Kg) = t.$$

Also, since $Kr_\theta g = Kg$, $r_\theta g$ corresponds to the same point in $\mathbb{H}$ as g but with a different tangent vector. For $g \in G$, let

$$S_\tau(Kg) = \{Ka_\tau r_\theta g \ : \ 0 \leq \theta < 2\pi\} \subset \mathbb{H}.$$

Then, $S_\tau(Kg)$ is the circle of radius τ around the point $Kg \in \mathbb{H}$.

Now suppose X is an arbitrary space with a continuous (left) $SL(2, \mathbb{R})$ action. For a function $h : X \to \mathbb{R}$, we can pull back h to a function h_x on $G \cong T^1(\mathbb{H})$. We then let the averaging operator A_τ be defined as

$$(A_\tau h)(x) = \frac{1}{2\pi} \int_0^{2\pi} h(a_\tau r_\theta x)\, d\theta = \frac{1}{2\pi} \int_0^{2\pi} h_x(a_\tau r_\theta)\, d\theta. \tag{3.1}$$

We will usually take h to be invariant under the action of $K = SO(2) \subset G$. Using the identification of $K \backslash G$ with $\mathbb{H}$, we think of $(A_\tau h)(x)$ as the average of h over a circle of radius τ in the G orbit through x, or more precisely the average of h_x over $S_\tau(K)$, where K is the base point of $K \backslash G \cong \mathbb{H}$.

Suppose $Y \subset X$ is a G-invariant submanifold. (Again, $Y = \emptyset$ is allowed).

Definition 3.1. A K-invariant function $f : X \to [1, \infty]$ is called a Margulis function for Y if it satisfies the following properties:

(a) There exists $\sigma > 1$ such that for all $0 \le t \le 1$ and all $x \in X$,

$$\sigma^{-1} f(x) \le f(a_t x) \le \sigma f(x). \tag{3.2}$$

(This holds if $\log f$ is uniformly continuous along the G-orbits).

(b) For every $c_0 > 0$ there exist $\tau > 0$ and $b_0 > 0$ such that for all $x \in X$,

$$A_\tau f(x) \le c_0 f(x) + b_0.$$

(c) $f(x) = \infty$ if and only if $x \in Y$, and f is bounded on compact subsets of $X \setminus Y$. For any $\ell > 0$, the set $\overline{\{x \;:\; f(x) \le \ell\}}$ is a compact subset of $X \setminus Y$.

Lemma 3.2. *Suppose there exists a Margulis function f for Y. Then,*

(i) *For all $c < 1$ there exists $t_0 > 0$ (depending on σ, and c) and $b > 0$ (depending only on b_0, c_0 and σ) such that for all $t > t_0$ and all $x \in X$,*

$$(A_t f)(x) \le c f(x) + b.$$

(ii) *There exists $B > 0$ (depending only on c_0, b_0 and σ) such that for all $x \in X$, there exists $T_0 = T_0(x, c_0, b_0, \sigma)$ such that for all $t > T_0$,*

$$(A_t f)(x) \le B.$$

(iii) *For every $\epsilon > 0$ there exists a compact subset F_ϵ of $X \setminus Y$ such that for all $x \in X$ there exists $T_0 = T_0(x, c_0, b_0, \sigma)$ such that for all $t > T_0$,*

$$|\{\theta \in [0, 2\pi) \;:\; a_t r_\theta x \in F_\epsilon\}| \ge 2\pi(1 - \epsilon).$$

For completeness, we include the proof of this lemma. It is essentially taken from [EMM98, §5.3], specialized to the case $G = SL(2, \mathbb{R})$.

The basic observation is the following standard fact from hyperbolic geometry:

Lemma 3.3. *There exist absolute constants $0 < \delta' < 1$ and $\delta > 0$ such that for any $p, q \in \mathbb{H}$, for any $t > 0$, for at least δ'-fraction of $z \in S_t(q)$ (with respect to the visual measure from q), we have*

$$d_{\mathbb{H}}(p, q) + t - \delta \le d_{\mathbb{H}}(p, z) \le d_{\mathbb{H}}(p, q) + t. \tag{3.3}$$

Using the identification of G with $T^1(\mathbb{H})$ we can restate Lemma 3.3 as follows:

Corollary 3.4. *There exist absolute constants $0 < \delta' < 1$ and $\delta > 0$ such that for any $t > 0$, any $s > 0$ and any $g \in G$, for at least δ'-fraction of $\phi \in [0, 2\pi]$,*

$$t + s - \delta \le d_{\mathbb{H}}(K a_t r_\phi a_s g, Kg) \le t + s. \tag{3.4}$$

Proof. This is indeed Lemma 3.3 with $p = Kg$, $q = K a_s g$ (so $d_{\mathbb{H}}(p, q) = s$). As ϕ varies, the points $K a_t r_\phi a_s g$ trace out $S_t(q) = S_t(K a_s g) \subset \mathbb{H}$. □

Corollary 3.5. *Suppose $f : X \to [1, \infty]$ is a K-invariant function satisfying (3.2). Then, there exists $\sigma' > 1$ depending only on σ such that for any $t > 0$, $s > 0$ and any $x \in X$,*

$$(A_{t+s} f)(x) \le \sigma' (A_t A_s f)(x). \tag{3.5}$$

Outline of proof. Fix $x \in X$. For $g \in SL(2,\mathbb{R})$, let $f_x(g) = f(gx)$, and let

$$\tilde{f}_x(g) = \frac{1}{2\pi}\int_0^{2\pi} f(gr_\theta x)\, d\theta.$$

Then, $\tilde{f}_x : \mathbb{H} \to [1,\infty]$ is a spherically symmetric function, i.e. $\tilde{f}_x(g)$ depends only on $d_{\mathbb{H}}(Kg, Ke)$, where e is the identity of G.

We have

$$(A_t A_s f)(x) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{2\pi}\int_0^{2\pi} f(a_t r_\phi a_s r_\theta x)\, d\phi\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} \tilde{f}_x(a_t r_\phi a_s). \tag{3.6}$$

By Corollary 3.4, for at least δ'-fraction of $\phi \in [0, 2\pi]$, (3.4) holds (with $g = e$). Then, by (3.2), for at least δ'-fraction of $\phi \in [0, 2\pi]$,

$$\tilde{f}_x(a_t r_\phi a_s) \geq \sigma_1^{-1} \tilde{f}_x(a_{t+s})$$

where $\sigma_1 = \sigma_1(\sigma, \delta) > 1$. Plugging in to (3.6), we get

$$(A_t A_s f)(x) \geq (\delta'\sigma_1^{-1})\tilde{f}_x(a_{t+s}) = (\delta'\sigma_1^{-1})(A_{t+s} f)(x),$$

as required. □

Proof of Lemma 3.2. By condition (b) of Definition 3.1 we can choose τ large enough in (b) so that c_0 is sufficiently small so that $\kappa \equiv c_0\sigma' < 1$, where σ' is as in Corollary 3.5. Then, for any $s \in \mathbb{R}$ and for all x,

$$\begin{aligned}
(A_{s+\tau} f)(x) &\leq \sigma' A_s(A_\tau f)(x) && \text{by (3.5)}\\
&\leq \sigma' A_s(c_0 f(x) + b_0) && \text{by condition (b)}\\
&= \kappa(A_s f)(x) + \sigma' b_0 && \text{since } \sigma' c_0 = \kappa.
\end{aligned}$$

Iterating this we get, for $n \in \mathbb{N}$

$$(A_{n\tau} f)(x) \leq \kappa^n f(x) + \sigma' b_0 + \kappa\sigma' b_0 + \cdots + \kappa^{n-1}\sigma' b_0 \leq \kappa^n f(x) + B,$$

where $B = \frac{\sigma' b_0}{1-\kappa}$. Since $\kappa < 1$, $\kappa^n f(x) \to 0$ as $n \to \infty$. Therefore both (i) and (ii) follow for $t \in \tau\mathbb{N}$. The general case of both (i) and (ii) then follows by applying again condition (a). The derivation of (iii) from (ii) is the same as in the random walk case. □

As in the random walk setting, the existence of a Margulis function implies certain large deviation results, see §6.

4. Construction of Margulis functions I: easy cases

In this section, we construct Margulis functions in the simplest possible settings. A much more elaborate (and useful) construction is done in the next section.

We begin with the following elementary calculation:

Lemma 4.1. *Fix $0 \le \delta < 1$. Then there exists a constant $c(\delta)$ such that for any $\tau > 0$ and any $v \in \mathbb{R}^2 - \{(0,0)\}$,*

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta}{\|a_\tau r_\theta v\|^{1+\delta}} \le \frac{c(\delta) e^{-\tau(1-\delta)}}{\|v\|^{1+\delta}} \tag{4.1}$$

Proof. By rescaling and rotating may assume that $v = (0,1)$. Then, the left-hand-side of (4.1) becomes:

$$\frac{1}{2\pi}\int_0^{2\pi} (e^{2\tau}\sin^2\theta + e^{-2\tau}\cos^2\theta)^{-(1+\delta)/2}\, d\theta$$

We decompose $[0,2\pi] = R_1 \cup R_2$, where $R_1 = \{\theta \ : \ e^{2\tau}\sin^2\theta \le e^{-2\tau}\cos^2\theta\}$ and R_2 is the set where the opposite inequality holds. Note that there exist absolute constants $0 < c_1 < c_2$ such that

$$c_1 e^{-2\tau} \le |R_1| \le c_2 e^{-2\tau} \tag{4.2}$$

On R_1, the integrand is bounded by a constant multiple of $e^{\tau(1+\delta)}$. Hence, in view of (4.2), the integral over R_1 is $O(e^{-\tau(1-\delta)})$ as required. Now the integral over R_2 is bounded by

$$e^{-\tau(1+\delta)} \int_{R_2} |\sin\theta|^{-(1+\delta)}\, d\theta = O(e^{-\tau(1-\delta)}),$$

where in the last estimate we used (4.2). □

Interpretation in the hyperbolic upper half plane. Given $g \in SL(2,\mathbb{R})$, we may write

$$g^{-1} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

In view of our conventions at the beginning of §3, $g \cdot i = x + iy$, and let $\phi(g) = x + iy$. Then, ϕ gives an identification between $SO(2)\backslash SL(2,\mathbb{R})$ and the hyperbolic upper half plane $\mathbb{H}$. Under this identification, the right multiplication action of $SL(2,\mathbb{R})$ on $SO(2)\backslash SL(2,\mathbb{R})$ becomes action by Möbius transformations on $\mathbb{H}$.

Let $\beta : \mathbb{H} \to \mathbb{R}^+$ be defined by $\beta(x+iy) = y^{1/2}$. Note that in view of the definitions of β and ϕ,

$$\beta(\phi(g)) = \left\| g \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\|^{-1}.$$

Thus, Lemma 4.1 is equivalent to the following well known:

Lemma 4.2. *Fix $0 \le \delta < 1$. Then there exists a constant $c(\delta)$ such that for any $\tau > 0$ and any $z \in \mathbb{H}$,*

$$\int_{S_\tau(z)} \beta^{1+\delta}(w)\, dm_z(w) \le c(\delta) e^{-\tau(1-\delta)} \beta(z)^{1+\delta}, \tag{4.3}$$

where $S_\tau(z)$ is the sphere in $\mathbb{H}$ centered at z and with radius τ, and the measure $m_z$is the normalized visual measure (from z) on the sphere $S_\tau(z)$.

(In other words, most of the measure of $S_\tau(z)$ is concentrated closer to the x axis than z). If, as in §3, we let

$$(A_\tau h)(z) = \int_{S_\tau(z)} h\, dm_z, \tag{4.4}$$

then (4.3) may be rewritten as

$$(A_\tau \beta^{1+\delta})(z) \le c(\delta) e^{-\tau(1-\delta)} \beta^{1+\delta}(z). \tag{4.5}$$

Taking the quotient by $SL(2,\mathbb{Z})$. Let $\alpha(z) = \sup_{\gamma \in SL(2,\mathbb{Z})} \beta(\gamma z)$.

Lemma 4.3. *For any $0 \le \delta < 1$ and any τ large enough depending on δ, the function $\alpha^{1+\delta}$ is a Margulis function for the averaging operator (4.4) on $X = \mathbb{H}/SL(2,\mathbb{Z})$ with $Y = \emptyset$.*

Proof. The property (a) of Definition 1.1 is immediate from the description of the fundamental domain of the action of $SL(2,\mathbb{Z})$ on $\mathbb{H}$. To show (1.2), fix $\tau > 0$ large enough so that $c_0 \equiv c(\delta)e^{-(1-\delta)\tau} < 1$.

Note that if $\operatorname{Im} z \ge 1$, then $\alpha(z) = \beta(z)$. Thus, if $\operatorname{Im} z$ is large enough so that $S(z,\tau) \subset \{x + iy \ : \ y \ge 1\}$, then, in view of (4.5), we have

$$(A_\tau \alpha^{1+\delta})(z) \le c_0 \alpha^{1+\delta}(z).$$

If $\operatorname{Im} z$ is not large enough, then $\alpha(z) \le C(\tau)$, and then for all $w \in S(z,\tau)$, $\alpha(z)^{1+\delta} \le b(\delta,\tau)$, where $b(\delta,\tau)$ is some constant. Thus, in this case,

$$(A_\tau \alpha^{1+\delta})(z) \le c_0 \alpha^{1+\delta}(z) + b(\delta,\tau).$$

Thus, for all $z \in \mathbb{H}$,

$$(A_\tau \alpha^{1+\delta})(z) \le c_0 \alpha^{1+\delta}(z) + b(\delta,\tau),$$

and $c_0 < 1$. This verifies condition (b) of Definition 1.1. □

The space $SL(2,\mathbb{R})/SL(2,\mathbb{Z})$. The space $\mathcal{L}_2$ of unimodular lattices in $\mathbb{R}^2$ admits a transitive action by $SL(2,\mathbb{R})$ and the stabilizer of the square lattice is $SL(2,\mathbb{Z})$; thus $\mathcal{L}_2$ is isomorphic to the quotient space $SL(2,\mathbb{R})/SL(2,\mathbb{Z})$.

Note that the map ϕ is $SL(2,\mathbb{Z})$-equivariant. Let $d(L)$ denote the length of the shortest vector in the lattice L. From the definitions we see the following:

Lemma 4.4. *For any $g \in SL(2,\mathbb{R})$,*

$$\alpha(\phi(g)) = d(g\mathbb{Z}^2)^{-1}.$$

Then, as a corollary of Lemma 4.3, we get the following:

Lemma 4.5. *For any $0 \le \delta < 1$, the function $d^{-(1+\delta)}$ is a Margulis function (in the sense of Definition 3.1 with the averaging operator A_τ given by (3.1)) for the action of $SL(2,\mathbb{R})$ on $X = \mathcal{L}_2$, with $Y = \emptyset$.*

We now come full circle by indicating a direct proof of Lemma 4.5 (i.e. without thinking of the hyperbolic plane). Note that a unimodular lattice in $\mathbb{R}^2$ can have at most one (linearly independent) vector of length < 1 (otherwise the covolume is too small). If the shortest vector v of a lattice L is sufficiently short (depending on τ) then for all $\theta \in [0, 2\pi)$, $d(a_\tau r_\theta L) = \|a_\tau r_\theta v\|$. Then, by Lemma 4.1,

$$(A_\tau d^{-(1+\delta)})(L) \le c(\delta) e^{-(1-\delta)\tau} d^{-(1+\delta)}(L).$$

If not, then $d(L)^{-(1+\delta)} \le C(\delta, \tau)$ and then

$$(A_\tau d^{-(1+\delta)})(L) \le b(\delta, \tau) d^{-(1+\delta)}(L).$$

Then, in all cases, provided τ is large enough so that $c_0 \equiv c(\delta) e^{-(1-\delta)\tau} < 1$, we have

$$(A_\tau d^{-(1+\delta)})(L) \le c_0 d^{-(1+\delta)}(L) + b_0,$$

where $c_0 < 1$. Thus, (b) of Definition 3.1 holds. The condition (c) holds by Mahler compactness, and (a) follows immediately from the definitions. □

Ball averages. For $h : \mathbb{H} \to \mathbb{R}$, let $(B_\tau h)(z)$ denote the average of h over the ball $B(z, \tau)$ of radius τ centered at z, with respect to the hyperbolic volume. Thus, B_τ is similar to A_τ, but is doing ball averages instead of sphere averages. In view of hyperbolic geometry (and in particular the fact that most of the hyperbolic volume of a ball is concentrated near its outer radius) and the results for the sphere averages A_τ, we see that for all $0 \le \delta < 1$, assuming τ is sufficiently large depending on δ, we have for all $z \in \mathbb{H}$,

$$(B_\tau \alpha^{1+\delta})(z) \le c_0 \alpha^{1+\delta}(z) + b_0,$$

where $c_0 < 1$ and $b_0 = b_0(\delta, \tau)$.

Products of upper half planes. Suppose $X = \mathbb{H} \times \mathbb{H}$ is a product of two copies of the hyperbolic plane. We consider X with the supremum metric (i.e. distance on X is the supremum of the distances in the two factors). Then, the ball of radius τ in X is the product of the balls of radius τ in the two factors. Hence if B_τ^X is the averaging operator over the ball in X of radius τ, then $B_\tau^X = B_\tau^1 B_\tau^2$, where B_τ^1 is the averaging operator over the ball of radius τ in the first factor, and B_τ^2 is the analogous thing in the second factor.

For $z = (z_1, z_2) \in X$, let $\alpha_1(z) = \alpha(z_1)$, $\alpha_2(z) = \alpha(z_2)$.

Lemma 4.6. *Suppose $0 \le \delta < 1$. Let*

$$u(z) = \epsilon(\alpha_1(z)\alpha_2(z))^{1+\delta} + \alpha_1(z)^{1+\delta} + \alpha_2(z)^{1+\delta},$$

Then, (provided τ is large enough depending on δ) and ϵ is chosen sufficiently small depending on δ and τ, u is a Margulis function for the averages B_τ^X on X, with $Y = \emptyset$.

Proof. We have

$$B_\tau^X(\epsilon\alpha_1^{1+\delta}\alpha_2^{1+\delta}) = \epsilon(B_\tau^1\alpha_1^{1+\delta})(B_\tau^2\alpha_2^{1+\delta}) \le \epsilon(c_0\alpha_1^{1+\delta}+b_0)(c_0\alpha_2^{1+\delta}+b_0) \le \le \epsilon c_0^2\alpha_1^{1+\delta}\alpha_2^{1+\delta} + \epsilon b_0\alpha_1^{1+\delta} + \epsilon b_0\alpha_2^{1+\delta} + \epsilon b_0^2. \tag{4.6}$$

Also, for $i = 1, 2$,

$$B_\tau^X(\alpha_i^{1+\delta}) \le c_0\alpha_i^{1+\delta} + b_0.$$

Thus,

$$B_\tau^X u \le \epsilon c_0^2\alpha_1^{1+\delta}\alpha_2^{1+\delta} + (\epsilon b_0 + c_0)\alpha_1^{1+\delta} + (\epsilon b_0 + c_0)\alpha_2^{1+\delta} + \epsilon b_0^2 + 2b_0.$$

We now choose ϵ is sufficiently small so that $c_1 \equiv \epsilon b_0 + c_0 < 1$. We get

$$B_\tau^X u \le c_1 u + b_1,$$

where $c_1 < 1$ and $b_1 = \epsilon b_0^2 + 2b_0$. This completes the proof. □

Similar constructions work for the product of any number of copies of $\mathbb{H}$, but the coefficients ϵ become more complicated. The Minsky product region theorem [Mi96] states that the geometry at infinity of Teichmüller space is similar to that of products of hyperbolic planes (with the supremum metric). In view of this an analogue of the function u of Lemma 4.6 was used in [EMi11] to show that most closed geodesics return to a given compact set. A more refined version (which can deal with random geodesics on strata of quadratic or abelian differentials) was proved in [EMR12].

5. Construction of Margulis functions: $SL(n,\mathbb{R})/SL(n,\mathbb{Z})$.

Let Δ be a lattice in $\mathbb{R}^n$. We say that a subspace L of $\mathbb{R}^n$ is Δ-*rational* if $L \cap \Delta$ is a lattice in L. For any Δ-rational subspace L, we denote by $d_\Delta(L)$ or simply by $d(L)$ the volume of $L/(L\cap\Delta)$. Let us note that $d(L)$ is equal to the norm of $u_1 \wedge \cdots \wedge u_\ell$ in the exterior power $\bigwedge^\ell(\mathbb{R}^n)$ where $\ell = \dim L$, $(u_1, \cdots, u_\ell)$ is a basis over $\mathbb{Z}$ of $L\cap\Delta$, and the norm on $\bigwedge(\mathbb{R}^n)$ is induced from the Euclidean norm on $\mathbb{R}^n$. If $L = \{0\}$ we write $d(L) = 1$. A lattice is Δ *unimodular* if $d_\Delta(\mathbb{R}^n) = 1$. The space of unimodular lattices is canonically identified with $SL(n,\mathbb{R})/SL(n,\mathbb{Z})$.

Let us introduce the following notation:

$$\alpha_i(\Delta) = \sup\left\{\frac{1}{d(L)}\,\middle|\, L \text{ is a } \Delta\text{-rational subspace of dimension } i\right\}, \quad 0 \le i \le n,$$

(5.1)

$$\alpha(\Delta) = \max_{0\le i\le n}\alpha_i(\Delta).$$

The classical Mahler compactness theorem states that for any $M > 0$ the set $\{\Delta \in SL(n,\mathbb{R})/SL(n,\mathbb{Z}) \ : \ \alpha(\Delta) \le M\}$ is compact.

Let $G = SL(n,\mathbb{R})$, $\Gamma = SL(n,\mathbb{Z})$, $\hat{K} \cong SO(n)$ is a maximal compact subgroup of G, $H \cong SO(p,q) \subset G$ and $K = H \cap \hat{K}$ is a maximal compact subgroup of H.

Let G is $SL(n,\mathbb{R})$, $\Gamma = SL(n,\mathbb{Z})$, $\hat{K} \cong SO(n)$ is a maximal compact subgroup of G, $H \cong SO(p,q) \subset G$, $K = H \cap \hat{K}$ be a maximal compact subgroup of H.

For any K-invariant function f on G/Γ, let $(A_t f)(x) = \int_K f(a_t k x)\, dm(k)$, where m is the normalized Haar measure on K. Suppose $x \in G/\Gamma$ and the stabilizer of x in H is trivial. Then $K\backslash Hx$ is isomorphic to the symmetric space $K\backslash H$, with x corresponding to the origin. If $\operatorname{rank} K\backslash H = 1$, then $(A_t f)(x)$ can be interpreted as the average of f over the sphere of radius $2t$ centered at the origin in the symmetric space $K\backslash Hx$.

If $p \geq 3$ and $0 < s < 2$, or if $(p,q) = (2,1)$ or $(2,2)$ and $0 < s < 1$, it is shown in [EMM98, Lemma 5.6] that for any $c > 0$ there exist $t > 0$, and $\omega > 1$ so that the the functions α_i^s satisfy the following system of integral inequalities in the space of lattices:

$$A_t \alpha_i^s \leq c_i \alpha_i^s + \omega^2 \max_{0<j\leq \min(n-i,i)} \sqrt{\alpha_{i+j}^s \alpha_{i-j}^s}, \tag{5.2}$$

where A_t is the averaging operator $(A_t f)(\Delta) = \int_K f(a_t k \Delta)$, and $c_i \leq c$. If $(p,q) = (2,1)$ or $(2,2)$ and $s = 1$, then (5.2) also holds (for suitably modified functions α_i), but some of the constants c_i cannot be made smaller than 1. (The proof of [EMM98, Lemma 5.6] is a much more complicated version of the direct proof of Lemma 4.5 in §4.)

In [EMM98, §5.4] it is shown that if the α_i satisfy (5.2) then for any $\epsilon > 0$, the function $f = f_{\epsilon,s} = \sum_{0\leq i\leq n} \epsilon^{i(n-i)} \alpha_i^s$ satisfies the scalar inequality:

$$A_t f \leq c f + b, \tag{5.3}$$

where t, c and b are constants. (This proof is a more complicated version of the proof of Lemma 4.6 in §4.) If $c < 1$, which occurs in the case $p \geq 3$, it follows that f is a Margulis function (for the case $Y = \emptyset$).

If $c = 1$, which will occur in the $SO(2,1)$ and $SO(2,2)$ cases, then (5.3) implies that $(A_r f)(1)$ is growing at most linearly with the radius.

Throughout [EMM98] one considers the functions $\alpha(g)^s$ for $0 < s < 2$ even though for the application to quadratic forms one only needs $s = 1 + \delta$ for some $\delta > 0$. This yields a better integrability result, and is also necessary for the proof of the convergence results [EMM98, Theorem 3.4] and [EMM98, Theorem 3.5].

Even though the function f is not a strictly speaking a Margulis function for the case $s = 1$, $p = 2$, $q = 2$, it plays a key role in the analysis of the $(2,2)$ case of the quantitative Oppenheim conjecture in [EMM05].

6. Large deviation estimates

For simplicity we state the results for the $SL(2,\mathbb{R})$-action setting. For the random walk setting, see [Ath06, Theorem 1.2].

Let $C_\ell = \overline{\{x \in X \;:\; f(x) < \ell\}}$. Let m denote the uniform measure on $SO(2) \subset SL(2,\mathbb{R})$. We refer to the trajectories of the group $\{a_t \;:\; t \in \mathbb{R}\}$ as "geodesics".

Theorem 6.1. *([Ath06, Theorem 1.1])*

(1) *For all l sufficiently large and all $x \notin C_l$, there are positive constants $c_1 = c_1(l, x), c_2(l)$, with*

$$m\{\theta : a_t r_\theta x \notin C_l, 0 \le t \le T\} \le c_1 e^{-c_2 T}$$

for all T sufficiently large. That is, the probability that a random geodesic trajectory has not visited C_l by time T decays exponentially in T.

(2) *For all l, S, T sufficiently large and all $x \in X$, there are positive constants $c_3 = c_3(S, l, x), c_4 = c_4(l)$, with*

$$m\{\theta : a_t r_\theta x \notin C_l, S \le t \le S + T\} \le c_3 e^{-c_4 T}.$$

That is, the probability that a random geodesic trajectory does not enter C_l in the interval $[S, S + T]$ decays exponentially in T.

(3) *Let $x \in X$. For any $0 < \lambda < 1$, there is a $l \ge 0$, and $0 < \gamma < 1$, such that for all T sufficiently large (depending on all the above constants)*

$$m\{\theta : \frac{1}{T}|\{0 \le t \le T : a_t r_\theta x \notin C_l\}| > \lambda\} \le \gamma^T.$$

Result (3) above may be thought of as a large deviations result for the "geodesics". Suppose μ_Q is an $SL(2,\mathbb{R})$-invariant measure on X (which we think of as the volume). While ergodicity guarantees that $\frac{1}{T}|\{0 \le t \le T : a_t x \in C_l\}| \to \mu_Q(C_l)$ for μ_Q-almost every $x \in X$, Theorem 6.1 gives explicit information for any $x \in X$ about the likelihood of bad trajectories starting in the set $SO(2)x$. Notice, however, this is *not* a traditional large deviations result, which estimates the probability of a deviation of any $\epsilon > 0$ from the ergodic average.

7. Other constructions and applications

Homogeneous dynamics. Let G be a semisimple Lie group, and let Γ be a lattice in G. Suppose μ is a probability measure on G; then μ defines a random walk on G/Γ.

In [EMa05], provided that the group generated by the support of μ is Zariski dense in G, a Margulis function for this random walk (and $Y = \emptyset$) was constructed; in the case $G = SL(n, \mathbb{R})$ and $\Gamma = SL(n, \mathbb{Z})$ the function is in fact the same as the function in §5.

In [BQ12], a Margulis function for this random walk (again with $Y = \emptyset$) was constructed under the weaker assumption that the Zariski closure of the group generated by the support of μ is semisimple. For a treatment of the case where Y is a closed orbit of some semisimple subgroup wee [BQ13].

In [GM10] a different Margulis function was used in conjunction with Fourier analysis to give polynomial error terms for the quantitative Oppenheim conjecture in at least five variables. This also gives an alternative proof of the definite case of the Oppenheim conjecture in five or more variables first proved in [G04].

A Margulis function (for $Y = \emptyset$) was constructed for the space of inhomogeneous lattices in [MM11] in order to prove the analogue of the quantitative Oppenheim conjecture for inhomogeneous quadratic forms.

In [HLM17], the construction of the Margulis function on the space of lattices was extended to the S-arithmetic case and used to prove the S-arithmetic version of the quantitative Oppenheim conjecture.

In [EK12] and [KKLM17] a modification of the Margulis function from [EMM98] was used to control the entropy contribution from the thin part of the space of lattices. In [Kh18], a Margulis function was used to study the Hausdorff dimension of the set of diverging trajectories of a diagonalizable element on the space of lattices.

Teichmüller dynamics. The idea of Margulis functions has played a key role in Teichmüller dynamics. In [EMas01], a Margulis function for the action of $SL(2,\mathbb{R})$ for $Y = \emptyset$ on strata of Abelian or quadratic differentials has been constructed. The construction has some parallels to that of §5. This function was used in [Ath06] to prove some exponential large deviation estimates for Teichmüller geodesic rays starting at a given point in the space. Athreya's results were later used in [AthF08] to control deviation of ergodic averages in almost all directions for a billiard flow in a rational polygon. The Margulis function of [EMas01] was later used in [AG13] in their proof of exponential decay of correlations for the Teichmüller geodesic flow. Building on the work of [EMas01] and [Ath06] a Margulis function for the same action but arbitrary $SL(2,\mathbb{R})$ invariant submanifolds Y was constructed in [EMM15]. Together with the measure classification theorem of [EMi18], this function played a key role in the proof that $SL(2,\mathbb{R})$-orbit closures are invariant submanifolds. This function is also used in many related results such as [CE15].

A modified (and independently developed) version of the Margulis function technique was used in [AF07] to prove that that the generic interval exchange transformation is weak mixing.

Other applications. Suppose μ is a probability measure on $SL(n,\mathbb{R})$. We may then consider random products of independent matrices, each with the distribution μ. We can then ask if the Lyapunov exponents $\lambda_i(\mu)$ of these random products depend continuously on μ. In [V14, Chapter 10] a new version of the Margulis function technique, due to Avila and Viana, which involves a modification of the natural averaging operator so that a Margulis function can be constructed, was used to show that in dimension 2, a natural continuity statement holds; namely if $\mu_j \to \mu$ in the weak star topology and also the support of μ_j tends to the support of μ in the Hausdorff topology then for $i = 1, 2$, $\lambda_i(\mu_j) \to \lambda_i(\mu)$. (The assumption about the support is necessary, see [V14, Chapter 10] for a counterexample). A more complicated proof was given previously in [BV10] without use of Margulis functions.

The result of [V14, Chapter 10] was extended in [MV14] to the case of Markov processes. (For the case $n \geq 2$ see the next paragraph).

Additive Margulis functions. Suppose we have a decomposition of $X = C \cup D$ where $C \cap D = \emptyset$. Let A be an averaging operator. An additive Margulis function (relative to this decomposition) is a function $\phi : X \to \mathbb{R}^+$ with the following properties:

(a) There exists a constant $\kappa_C > 0$ such that for $x \in C$,

$$(A\phi)(x) < \phi(x) - \kappa_C \tag{7.1}$$

(b) There exists a constant $\kappa_D > 0$ such that for $x \in D$,

$$(A\phi)(x) < \phi(x) + \kappa_D \tag{7.2}$$

Suppose f is a Margulis function, and choose $\Lambda > b/c$. Let

$$C = \{x \in X \;:\; f(x) > \Lambda\} \qquad D = \{x \in X \;:\; f(x) \leq \Lambda\}.$$

Then, it follows from Jensen's inequality that $\log f$ is an additive Margulis function relative to the decomposition $X = C \cup D$.

However, it is not true that if ϕ is an additive Margulis function, then e^ϕ is a multiplicative one. In fact, the inequality (1.1) is very sensitive to the "worst case behavior" of e^ϕ on the support of the measure μ_x defining A; on the other hand, the inequalities (7.1) and (7.2) depend more on the "average case" behavior of ϕ. Because of this effect, it is often much easier to construct an additive Margulis function than a multiplicative one. (In fact we do not know how to construct a useful multiplicative Margulis function in the setting of [V14, Chapter 10] beyond the case $n = 2$).

Additive Margulis functions are useful because of the following:

Lemma 7.1. *Suppose ϕ is an additive Margulis function for A relative to the decomposition $X = C \cup D$, and suppose η is a measure on X with $\int_X \phi\, d\eta < \infty$. Suppose also $\int_X (A\phi)(x)\, d\eta(x) \geq \int_X \phi(x)\, d\eta(x)$ (for example this holds if η is A-invariant). Then,*

$$\eta(D) \geq \frac{\kappa_C}{\kappa_C + \kappa_D} \eta(X). \tag{7.3}$$

Proof. We have

$$\int_X \phi(x)\, d\eta(x) \leq \int_X (A\phi)(x)\, d\eta(x) < \int_X \phi(x)\, d\eta(x) - \kappa_C \eta(C) + \kappa_D \eta(D).$$

Thus, $-\kappa_C \eta(C) + \kappa_D \eta(D) > 0$, which implies (7.3). ☐

This circle of ideas was used in [AEV] in order to extend the results on continuity of Lyapunov exponents in [V14, Chapter 10] to arbitrary dimensions, and also in [BBB15] in a non-linear setting.

8. Comparasion to other techniques

In the homogeneous dynamics setting, there is another technique for proving results similar in flavor to what can be obtained using Margulis functions. For the case $Y = \emptyset$, this originates with the paper [Mar71], and was further developed in [Dan84], [Dan86]. These ideas were used in many of the foundational papers in homogeneous dynamics such as [DM89, DM90] and [Ra91]. For other Y, the key result is the "linearization" technique of [DM93] (in which in particular the asymptotically exact lower bounds for the quantitative Oppenheim conjecture were proved). An abstract framework for these methods is terms of "(C, α)-good" functions defined in [EMS97] is developed in [KMar98]. These techniques (and in particular the framework in [KMar98]) have numerous applications to diophantine approximations and other areas, which are beyond the scope of this survey.

The "(C, α)-good" techniques rely essentially on the variants of polynomial nature of the unipotent flow, and have limited applicability outside of homogeneous dynamics. (An exception are [MW02], [MW14], where the authors manage to obtain results on non-divergence in the Teichmüller dynamics setting using essentially polynomial techniques). In the homogeneous setting, one usually obtains sharper estimates if one manages to construct a Margulis function; for example the quantitative Oppenheim conjecture can not be proved by (C, α)-good techniques since the estimates one obtains that way are too weak. (This is in fact the original motivation for Margulis functions). However, a construction of a Margulis function is not always possible, e.g. for the action of a single unipotent. This is related to the fact that (1.1) has to hold for *all* $x \in X$. This can be easier to do if one considers additive Margulis functions instead, but then the results are even weaker than what is obtained by (C, α)-good methods. In general, (non-additive) Margulis functions are an extremely powerful tool, but in many cases, their construction is a difficult engineering challenge.

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