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# Quasi-isometric rigidity of solvable groups

Alex Eskin and David Fisher *

**Abstract.** In this article we survey recent progress on quasi-isometric rigidity of polycyclic groups. These results are contributions to Gromov's program for classifying finitely generated groups up to quasi-isometry [Gr2]. The results discussed here rely on a new technique for studying quasi-isometries of finitely generated groups, which we refer to as *coarse differentiation*.

We include a discussion of other applications of coarse differentiation to problems in geometric group theory and a comparison of coarse differentiation to other related techniques in nearby areas of mathematics.

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**Keywords.** Quasi-isometry, rigidity, polycyclic groups.

## 1. Introduction, conjectures, and results

For any group $\Gamma$ generated by a subset $S$ one has the associated Cayley graph, $C_\Gamma(S)$. This is the graph with vertex set $\Gamma$ and edges connecting any pair of elements which differ by right multiplication by a generator. There is a natural $\Gamma$ action on $C_\Gamma(S)$ by left translation. By giving every edge length one, the Cayley graph can be made into a (geodesic) metric space. The distance on $\Gamma$ viewed as the vertices of the Cayley graph is the *word metric*, defined via the norm:

$$\|\gamma\| = \inf\{\text{length of a word in the generators } S \text{ representing } \gamma \text{ in } \Gamma.\}$$

Different sets of generators give rise to different metrics and Cayley graphs for a group but one wants these to be equivalent. The natural notion of equivalence in this category is *quasi-isometry*:

**Definition 1.1.** *Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces. Given real numbers $K \geq 1$ and $C \geq 0$, a map $f : X \to Y$ is called a $(K, C)$-*quasi-isometry *if*

1. $\frac{1}{K} d_X(x_1, x_2) - C \leq d_Y(f(x_1), f(x_2)) \leq K d_X(x_1, x_2) + C$ *for all $x_1$ and $x_2$ in $X$, and,*
2. *the $C$ neighborhood of $f(X)$ is all of $Y$.*

If $\Gamma$ is a finitely generated group, $\Gamma$ is canonically quasi-isometric to any finite index subgroup $\Gamma'$ in $\Gamma$ and to any quotient $\Gamma'' = \Gamma/F$ for any finite normal subgroup $F$. The equivalence relation generated by these (trivial) quasi-isometries is called *weak commensurability*. A group is said to *virtually* have a property if some weakly commensurable group does.

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In his ICM address in 1983, Gromov proposed a broad program for studying finitely generated groups as geometric objects, [Gr2]. Though there are many aspects to this program (see [Gr3] for a discussion), the principal question is the classification of finitely generated groups up to quasi-isometry. By construction, any finitely generated group $\Gamma$ is quasi-isometric to any space on which $\Gamma$ acts properly discontinuously and cocompactly by isometries. For example, the fundamental group of a compact manifold is quasi-isometric to the universal cover of the manifold (this is called the Milnor-Svarc lemma). In particular, any two cocompact lattices in the same Lie group $G$ are quasi-isometric. One important aspect of Gromov's program is that it allows one to generalize many invariants, techniques, and questions from the study of lattices to all finitely generated groups.

A major direction in the Gromov program is determining which algebraic properties of groups are quasi-isometry invariants. As consequence of Gromov's theorem on groups of polynomial growth, one has that the property of having a finite index subgroup that is nilpotent is invariant under quasi-isometries [Gr1]. It is then an obvious question whether larger classes of groups might have this property. Erschler showed in [D] that this is not the case for solvable groups. I.e. there are groups quasi-isometric to solvable groups which are not even virtually solvable. However, the following conjecture is plausible and we will spend much of this article discussing progress towards it.

**Conjecture 1.2.** *Let $\Gamma$ be a polycyclic group, then any group $\Gamma'$ quasi-isometric to $\Gamma$ is virtually polycyclic.*

**Remarks:**

1. Conjecture 1.2 can be rephrased as being about lattices in connected, simply connected solvable Lie groups. In particular, by a theorem of Mostow, any polycyclic group is virtually a lattice in a connected, simply connected solvable Lie group, and conversely any lattice in a solvable Lie group is virtually polycyclic [Mo2]. As solvable Lie groups have only cocompact lattices, the conjectures is equivalent to saying that any group quasi-isometric to lattice in a simply connected solvable Lie group is virtually a lattice in a simply connected, solvable Lie group.

2. Some classes of solvable groups which are not polycyclic are known to be quasi-isometrically rigid. See particularly the work of Farb and Mosher on the solvable Baumslag-Solitar groups [FM1, FM2] as well as later work of Farb-Mosher, Mosher-Sageev-Whyte and Wortman [FM3, MSW, W]. The methods used in all of these works depend essentially on topological arguments based on the explicit structure of singularities of the spaces studied and cannot apply to polycyclic groups.

3. Shalom has obtained some evidence for the conjecture by cohomological methods [Sh]. For example, Shalom shows that any group quasi-isometric to a polycyclic group has a finite index subgroup with infinite abelianization. Some of his results have been further refined by Sauer [Sa].

We discuss results that establish Conjecture 1.2 in many cases. We believe our techniques provide a method to attack the conjecture. This is work in progress, joint with Irine Peng.

From an algebraic point of view, solvable groups are generally easier to study than semisimple ones, as the algebraic structure is more easily manipulated. In the present context it is extremely difficult to see that any algebraic structure is preserved and so we are forced to work geometrically. For nilpotent groups the only geometric fact needed is polynomial volume growth. For semisimple groups, the key fact for all approaches is nonpositive curvature. The geometry of solvable groups is quite difficult to manage, since it involves a mixture of positive and negative curvature as well as exponential volume growth.

The simplest non-trivial example for Conjecture 1.2 is the 3-dimensional solvable Lie group Sol. This example has received a great deal of attention. The group Sol $\cong \mathbb{R} \ltimes \mathbb{R}^2$ with $\mathbb{R}$ acting on $\mathbb{R}^2$ via the diagonal matrix with entries $e^{z/2}$ and $e^{-z/2}$. As matrices, Sol can be written as :

$$\mathrm{Sol} = \left\{ \begin{pmatrix} e^{z/2} & x & 0 \\ 0 & 1 & 0 \\ 0 & y & e^{-z/2} \end{pmatrix} \middle| (x, y, z) \in \mathbb{R}^3 \right\}$$

The metric $e^{-z}dx^2 + e^z dy^2 + dz^2$ is a left invariant metric on Sol. Any group of the form $\mathbb{Z} \ltimes_T \mathbb{Z}^2$ for $T \in SL(2, \mathbb{Z})$ with $|tr(T)| > 2$ is a cocompact lattice in Sol.

The following theorem by Eskin, Fisher and Whyte proves a conjecture of Farb and Mosher [EFW0, EFW1, EFW2, FM4]:

**Theorem 1.3.** *Let $\Gamma$ be a finitely generated group quasi-isometric to* Sol*. Then $\Gamma$ is virtually a lattice in* Sol*.*

Peng's thesis contains a far reaching generalization of this result [Pe1, Pe2]. In addition to generalizing the methods introduced in [EFW0, EFW1, EFW2], Peng's thesis makes use of generalizations of some results of Farb and Mosher by Dymarz and Dymarz-Peng [FM1, Dy, DP]. We require some vocabulary to formulate Peng's results. We call a solvable Lie group *abelian by abelian* if it is of the form $\mathbb{R}^k \ltimes \mathbb{R}^n$. Such a group is defined by a linear represenation $\rho : \mathbb{R}^k \to GL(\mathbb{R}^n)$. Note that the image $\rho(\mathbb{R}^k)$ is an abelian subgroup of $GL(\mathbb{R}^n)$ and as such it's elements admit a common Jordan form. The Jordan form gives rise to a collection of functionals, called *weights*, on $\mathbb{R}^k$. Each weight $\omega$ corresponds to a subspace $W$ of $\mathbb{R}^n$ that is common generalized eigenspace for the $\mathbb{R}^k$ action and $\omega(v)$ for $v$ in $\mathbb{R}^k$ is the norm of the generalized eigenvalue for the action of $v$ on $W$. We call an abelian by abelian solvable group non-degenerate if $\rho$ is faithful and no weight $w$ has $w(\mathbb{R}^k)$ contained in $\{\pm 1\}$. Recall that a Lie group is unimodular if it has a bi-invariant Haar measure. For a group of the form $\mathbb{R}^k \ltimes \mathbb{R}^n$, unimodularity is equivalent to $\rho$ taking values in $SL(\mathbb{R}^n)$.

**Theorem 1.4.** *Let $G = \mathbb{R}^k \ltimes \mathbb{R}^n$ be a non-degenerate abelian by abelian unimodular, solvable Lie group. Then any group $\Gamma$ quasi-isometric to $G$ is virtually a lattice in a solvable Lie group $G' = \mathbb{R}^k \ltimes \mathbb{R}^n$ which is also abelian by abelian, non-degenerate and unimodular.*

**Remark:** One can in fact say more about the relation between $G$ and $G'$, but we will not pursue this here.

All our theorems stated above are proved using a new technique, which we call *coarse differentiation.* Even though quasi-isometries have no local structure and conventional derivatives do not make sense, we essentially construct a "coarse derivative" that models the large scale behavior of the quasi-isometry. Coarse differentiation is quite similar to a number of notions that arise in various forms of differentiation theory. However, this construction is quite different from the more conventional method of passing to the asymptotic cone and then applying a differentiation theorem to either the full asymptotic cone or some subspace of it, see §4.5 for more discussion.

## 2. Quasi-isometries are height respecting

A typical step in the study of quasi-isometric rigidity of groups is the identification of all quasi-isometries of some space $X$ quasi-isometric to the group, see §4.6 for a brief explanation. For us, the space $X$ is always a solvable Lie group. To pursue Conjecture 1.2, the goal is to show that all self quasi-isometries of the solvable Lie group $G$ permutes the cosets of a certain subgroup.

For Sol the group whose cosets we show are preserved is exactly the kernel of the homomorphism $h : \mathrm{Sol} \to \mathbb{R}$ which we call the height function. There is a foliation of Sol by level sets of the height function which is also the foliation by cosets of the normal $\mathbb{R}^2$. We will call a quasi-isometry of any of these spaces *height respecting* if it permutes the height level sets to within bounded distance (In [FM4], the term used is horizontal respecting). In our coordinates for Sol, the height function is $h(x, y, z) = z$.

**Theorem 2.1.** *Any $(K, C)$-quasi-isometry $\varphi$ of* Sol *is within bounded distance of a height respecting quasi-isometry $\hat{\varphi}$. Furthermore, this distance can be taken uniform in $(K, C)$ and therefore, in particular, $\hat{\varphi}$ is a $(K', C')$-quasi-isometry where $K', C'$ depend only on $K$ and $C$.*

**Remark:** In fact, Theorem 2.1 can be used to identify the quasi-isometries of Sol completely. Possibly after composing with the map $(x, y, z)\to(y, x, -z)$, any height respecting quasi-isometry (and in particular, any isometry) is at bounded distance from a quasi-isometry of the form $(x, y, z)\to(f(x), g(y), z)$ where $f$ and $g$ are bilipschitz functions. Given a metric space $X$, one defines $\mathrm{QI}(X)$ to be the group of quasi-isometries of $X$ modulo the subgroup of those at finite distance from the identity. The previous statement can then be taken to mean that $\mathrm{QI(Sol)} = \mathrm{Bilip}(\mathbb{R})^2 \ltimes \mathbb{Z}/2\mathbb{Z}$. This explicit description was conjectured by Farb and Mosher.

If we take a group of the form $\mathbb{R}^k \ltimes \mathbb{R}^n$ as in Theorem 1.4, we can write coordinates $(z, \vec{x})$ where $z$ is the coordinate in $\mathbb{R}^k$ and $\vec{x}$ is the coordinate in $\mathbb{R}^n$. Here $h(z, \vec{x}) = z$ and level sets of $h$ are $\mathbb{R}^n$ cosets. We can again call a quasi-isometry height respecting if it permutes level sets of $h$. The following is the main result of [Pe1, Pe2].

**Theorem 2.2.** *Let $X = \mathbb{R}^k \ltimes \mathbb{R}^n$ be as in Theorem 1.4. Then any $(K, C)$-quasi-isometry $\varphi$ of $\mathbb{R}^k \ltimes \mathbb{R}^n$ is within a bounded distance of a height respecting quasi-isometry $\hat{\varphi}$. Furthermore, the bound is uniform in $K$ and $C$.*

**Remark:** There is an explicit description of $QI(\mathbb{R}^k \ltimes \mathbb{R}^n)$ in this context as well, but it is somewhat involved so we omit it.

We now describe a conjecture that is a key step in our approach to Conjecture 1.2. We note here that this conjecture does not suffice to prove that one, but that one requires in addition generalizations of the results in [Dy, DP].

We begin by reviewing some structure theory of simply connected solvable Lie groups. Most of the basics are contained in work of Auslander [A1, A2]. Let $G$ be a solvable Lie group. Then $G$ is part of a short exact sequence:

$$1 \to N \to G \to \bar{H} \to 1$$

where $N$ and $\bar{H}$ are nilpotent. In general this exact sequence does not split but there is a nilpotent group $H$, called a Cartan subgroup, that is minimal among all groups mapping onto $\bar{H}$. The group $N$ is the maximal normal nilpotent subgroup of $G$, also known as it's nilradical. We are particularly interested in a subgroup $\exp(G)$ of $N$, defined independently by Guivarch and Osin, called the exponential radical of $G$ [Gu, Os]. This can be taken to be the subgroup of $G$ generated by all exponentially distorted elements in $G$. ( Guivarch calls it the *unstable* subgroup, the terminology *exponential radical* is due to Osin.) We believe that the following conjecture is a key step for proving Conjecture 1.2.

**Conjecture 2.3.** *Given a unimodular solvable Lie group $G$, then any self quasi-isometry of $G$ is at bounded distance from one which preserves the foliation by cosets of* $\exp(G)$.

We remark here that the assumption that $G$ be unimodular is necessary. To see this consider the group $SL(2,\mathbb{R})$. This group is quasi-isometric to the affine group of the line, which is a two dimensional solvable Lie group of the form $\mathrm{Aff}(\mathbb{R}) = \mathbb{R} \ltimes \mathbb{R}$. It is easy to see that the normal $\mathbb{R}$, i.e. the group of translations, is the exponential radical of $\mathrm{Aff}(\mathbb{R})$. However, $\mathrm{Aff}(\mathbb{R})$ is quasi-isometric to $SL(2,\mathbb{R})/O(2) = \mathbb{H}^2$ and the group of quasi-isometries of $\mathbb{H}^2$ is known to be the group of quasi-symmetric maps of $S^1 = \partial\mathbb{H}^2$. The foliation by cosets of $\mathbb{R}$ identifies naturally with the horocyclic foliation corresponding to the fixed point for $\mathrm{Aff}(\mathbb{R}) < SL(2,\mathbb{R})$ on $S^1$ and it is easy to see that the cosets of this foliation are not permuted by all quasi-symmetric maps.

# 3. Geometry of Sol

In this subsection we describe the geometry of Sol and related spaces in more detail, with emphasis on the geometric facts used in our proofs.

The upper half plane model of the hyperbolic plane $\mathbb{H}^2$ is the set $\{(x,\xi) \mid \xi > 0\}$ with the length element $ds^2 = \frac{1}{\xi^2}(dx^2 + d\xi^2)$. If we make the change of variable $z = \log\xi$, we get $\mathbb{R}^2$ with the length element $ds^2 = dz^2 + e^{-z}dx^2$. This is the *log model* of the hyperbolic plane $\mathbb{H}^2$.

The length element of Sol is:

$$ds^2 = dz^2 + e^{-z}dx^2 + e^{z}dy^2.$$

Thus planes parallel to the $xz$ plane are hyperbolic planes in the log model. Planes parallel to the $yz$ plane are *upside-down* hyperbolic planes in the log model. All of these copies of $\mathbb{H}^2$ are isometrically embedded and totally geodesic .

We will refer to lines parallel to the $x$-axis as $x$-horocycles, and to lines parallel to the $y$-axis as $y$-horocycles. This terminology is justified by the fact that each ($x$ or $y$)-horocycle is indeed a horocycle in the hyperbolic plane which contains it.

We now turn to a discussion of geodesics and quasi-geodesics in Sol. Any geodesic in an $\mathbb{H}^2$ leaf in Sol is a geodesic. There is a special class of geodesics, which we call *vertical geodesics*. These are the geodesics which are of the form $\gamma(t) = (x_0, y_0, t)$ or $\gamma(t) = (x_0, y_0, -t)$. We call the vertical geodesic *upward oriented* in the first case, and *downward oriented* in the second case. In both cases, this is a unit speed parametrization. Each vertical geodesic is a geodesic in two hyperbolic planes, the plane $y = y_0$ and the plane $x = x_0$.

Certain quasi-geodesics in Sol are easy to describe. Given two points $(x_0, y_0, t_0)$ and $(x_1, y_1, t_1)$, there is a geodesic $\gamma_1$ in the hyperbolic plane $y = y_0$ that joins $(x_0, y_0, t_0)$ to $(x_1, y_0, t_1)$ and a geodesic $\gamma_2$ in the plane $x = x_1$ that joins $(x_1, y_0, t_1)$ to a $(x_1, y_1, t_1)$. It is easy to check that the concatenation of $\gamma_1$ and $\gamma_2$ is a quasi-geodesic. In first matching the $x$ coordinates and then matching the $y$ coordinates, we made a choice. It is possible to construct a quasi-geodesic by first matching the $y$ coordinates and then the $x$ coordinates. This immediately shows that any pair of points not contained in a hyperbolic plane in Sol can be joined by two distinct quasi-geodesics which are not close together. This is an aspect of positive curvature. One way to prove that the objects just constructed are quasi-geodesics is to note the following: The pair of projections $\pi_1, \pi_2 : \mathrm{Sol} \to \mathbb{H}^2$ onto the $xt$ and $yt$ coordinate planes can be combined into a quasi-isometric embedding $\pi_1 \times \pi_2 : \mathrm{Sol} \to \mathbb{H}^2 \times \mathbb{H}^2$.

We state here the simplest version of a key geometric fact used at various steps in the proof.

**Lemma 3.1** (Quadrilaterals). *Suppose $p_1, p_2, q_1, q_2 \in \mathrm{Sol}$ and $\gamma_{ij} : [0, \ell_{ij}] \to \mathrm{Sol}$ are vertical geodesic segments parametrized by arclength. Suppose $C > 0$. Assume that for $i = 1, 2$, $j = 1, 2$,*

$$d(p_i, \gamma_{ij}(0)) \leq C \qquad \text{and} \qquad d(q_i, \gamma_{ij}(\ell_{ij})) \leq C,$$

*so that $\gamma_{ij}$ connects the $C$-neighborhood of $p_i$ to the $C$-neighborhood of $q_j$. Further assume that for $i = 1, 2$ and all $t$, $d(\gamma_{i1}(t), \gamma_{i2}(t)) \geq (1/10)t - C$ (so that for each $i$, the two segments leaving the neighborhood of $p_i$ diverge right away) and for $j = 1, 2$ and all $t$, $d(\gamma_{1j}(l_{1j} - t), \gamma_{2j}(l_{2j} - t)) \geq (1/10)t - C$ . Then there exists $C_1$ depending only on $C$ such that exactly one of the following holds:*

(a) *All four $\gamma_{ij}$ are upward oriented, $p_2$ is within $C_1$ of the $y$-horocycle passing through $p_1$ and $q_2$ is within $C_1$ of the $x$-horocycle passing through $\phi(q_1)$.*

(b) *All four $\gamma_{ij}$ are downward oriented, $p_2$ is within $C_1$ of the $x$-horocycle passing through $p_1$ and $q_2$ is within $C_1$ of the $y$-horocycle passing through $q_1$.*

We think of $p_1, p_2, q_1$ and $q_2$ as defining a quadrilateral. The content of the lemma is that any quadrilateral has its four "corners" in pairs that lie essentially along horocycles. In particular, if we take a quadrilateral with geodesic segments $\gamma_{ij}$ and with $h(p_1) = h(p_2)$ and $h(q_1) = h(q_2)$ and map it forward under a $(K, C)$-quasi-isometry $\phi$, and if we would somehow know that $\phi$ sends each of the four $\gamma_{ij}$ close to a vertical geodesic, then Lemma 3.1 would imply that $\phi$ sends the $p_i$ (resp. $q_i$) to a pair of points at roughly the same height.

We now define certain useful subsets of Sol. Let $B(L, \vec{0}) = [-e^L, e^L] \times [-e^L, e^L] \times [-L, L]$. Then $|B(L, \vec{0})| \approx Le^{2L}$ and $Area(\partial B(L, \vec{0})) \approx e^{2L}$, so $B(L)$ is a Fölner set. We call $B(L, \vec{0})$ a box of size $L$ centered at the identity. We define the box of size $L$ centered at a point $p$ by $B(L, p) = T_p B(L, \vec{0})$ where $T_p$ is left translation by $p$. Since left translation is an isometry, $B(L, p)$ is also a Fölner set. We frequently omit the center of a box in our notation and write $B(L)$.

**Approximating a box by a graph.** Notice that the top of $B(L)$, meaning the set $[-e^L, e^L] \times [-e^L, e^L] \times \{L\}$, is not at all square - the sides of this rectangle are horocyclic segments of lengths $2e^{2L}$ and 2 - in other words it is just a small metric neighborhood of a horocycle. Similarly, the bottom is also essentially a horocycle but in the transverse direction. Further, we can connect the 1-neighborhood of any point of the top horocycle to the 1-neighborhood of any point of the bottom horocycle by a vertical geodesic segment, and these segments essentially sweep out the box $B(L)$. Thus a box contains an extremely large number of quadrilaterals. If we discretize the top and bottom horocycle, we can think of this process as giving a description of a graph which we call $G_L$. This graph is essentially a complete bipartite graph with $4e^{2L}$ vertices. Throughout the proof of our results on Sol, this highly connected graph plays a key role.

# 4. On proofs

In this section, we give some of the key ideas in the proofs. In the first two subsections we indicate the key new ideas behind our proof of Theorem 2.1. The first contains quantative estimates on the behavior of quasi-geodesics. The second subsection averages this behavior over families of quasi-geodesics. In §4.3 we sketch the proof of Theorem 2.1. Subsection 4.4 briefly discusses the ideas needed to adapt the proof of Theorem 2.1 to prove the other results in Section 2 and indicates obstructions and progress in the general case of Conjecture 2.3. Before continuing with discussion of proofs, we include a discussion of how to axiomatize the methods of §4.1 and §4.2 into a general method of *coarse differentiation* in §4.5. In subsection §4.6, we discuss deducing results in §1 from results in §2.

**4.1. Behavior of quasi-geodesics.** We begin by discussing some quantative estimates on the behavior of quasi-geodesic segments in Sol. Throughout the discussion we assume $\alpha : [0, r] \to \mathrm{Sol}$ is a $(K, C)$-quasi-geodesic segment for a fixed choice of $(K, C)$, i.e. $\alpha$ is a quasi-isometric embedding of $[0, r]$ into Sol. A quasi-isometric embedding is a map that satisfies point (1) in Definition 1.1 but not point (2).

**Definition 4.1** (**$\epsilon$-monotone).** *A quasigeodesic segment $\alpha : [0, r] \to \mathrm{Sol}$ is $\epsilon$-monotone if for all $t_1, t_2 \in [0, r]$ with $h(\alpha(t_1)) = h(\alpha(t_2))$ we have $|t_1 - t_2| < \epsilon r$.*

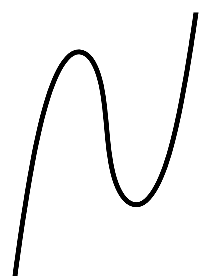

Figure 1. A quasigeodesic segment which is not $\epsilon$-monotone.

The following fact about $\varepsilon$-monotone geodesics is an easy exercise in hyperbolic geometry:

**Lemma 4.2** ($\epsilon$-monotone is close to vertical). *If $\alpha : [0, r] \to \mathrm{Sol}$ is $\epsilon$-monotone, then there exists a vertical geodesic segment $\lambda$ such that $d(\alpha, \lambda) = O(\epsilon r)$.*

**Remark:** The distance $d(\alpha, \lambda)$ is the Hausdorff distance between the sets and does not depend on parametrizations.

**Lemma 4.3** (Subdivision). *Suppose $\alpha : [0, r] \to \mathrm{Sol}$ is a quasi-geodesic segment which is not $\epsilon$-monotone. Suppose $n \gg 1$ (depending on $\epsilon$, $K$, $C$). Then*

$$\sum_{j=0}^{n-1} \left| h(\alpha(\tfrac{(j+1)r}{n})) - h(\alpha(\tfrac{jr}{n})) \right| \geq |h(\alpha(0)) - h(\alpha(r))| + \frac{\epsilon r}{8K^2}.$$

*Outline of Proof.* If $n$ is sufficiently large, the total variation of the height increases after the subdivision by a term proportional to $\epsilon$. See Figure 2. □

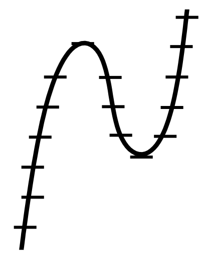

Figure 2. Proof of Lemma 4.3

**Choosing Scales:** Choose $1 \ll r_0 \ll r_1 \ll \cdots \ll r_M$. In particular, $C \ll r_0$ and $r_{m+1}/r_m > n$.

**Lemma 4.4.** *Suppose $L \gg r_M$, and suppose $\alpha : [0, L] \to \mathrm{Sol}$ is a quasi-geodesic segment. For each $m \in [1, M]$, subdivide $[0, L]$ into $L/r_m$ segments of length $r_m$. Let $\delta_m(\alpha)$ denote the fraction of these segments whose images are not $\epsilon$-monotone. Then,*

$$\sum_{m=1}^{M} \delta_m(\alpha) \leq \frac{16K^3}{\epsilon}.$$

*Proof.* By applying Lemma 4.3 to each non-$\epsilon$-monotone segment on the scale $r_M$, we get

$$\sum_{j=1}^{L/r_{M-1}} |h(\alpha(jr_{M-1})) - h(\alpha((j-1)r_{M-1}))| \geq$$
$$\geq \sum_{j=1}^{L/r_M} |h(\alpha(jr_M)) - h(\alpha((j-1)r_M))| + \delta_M(\alpha)\frac{\epsilon L}{8K^2}.$$

Doing this again, we get after $M$ iterations,

$$\sum_{j=1}^{L/r_0} |h(\alpha(jr_0)) - h(\alpha((j-1)r_0))| \geq$$
$$\geq \sum_{j=1}^{L/r_M} |h(\alpha(jr_M)) - h(\alpha((j-1)r_M))| + \frac{\epsilon L}{8K^2}\sum_{m=1}^{M} \delta_m(\alpha).$$

But the left-hand-side is bounded from above by the length and so bounded above by $2KL$. □

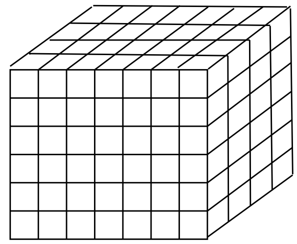

Figure 3. The box $B(L)$.

**4.2. Averaging.** In this subsection we apply the estimates from above to images of geodesics under a quasi-isometry of Sol. The idea is to average the previous estimates over families of quasi-geodesics. This results in a coarse analogue of Rademacher's theorem, which says that a bilipschitz map of $\mathbb{R}^n$ is differentiable almost everywhere, see below for discussion.

**Setup and Notation.**

- Suppose $\phi : \text{Sol} \to \text{Sol}$ is a $(K, C)$ quasi-isometry. Without loss of generality, we may assume that $\phi$ is continuous.
- Let $\gamma : [-L, L] \to \text{Sol}$ be a vertical geodesic segment parametrized by arclength where $L \gg C$.
- Let $\overline{\gamma} = \phi \circ \gamma$. Then $\overline{\gamma} : [-L, L] \to \text{Sol}$ is a quasi-geodesic segment.

It follows from Lemma 4.4, that for every $\theta > 0$ and every geodesic segment $\gamma$, assuming that $M$ is sufficiently large, there exists $m \in [1, M]$ such that $\delta_m(\overline{\gamma}) < \theta$. The difficulty is that $m$ may depend on $\gamma$. For Sol, this is overcome as follows:

Recall that $B(L) = [-e^L, e^L] \times [-e^L, e^L] \times [-L, L]$. Then $|B(L)| \approx Le^{2L}$ and $Area(\partial B(L)) \approx e^{2L}$, so $B(L)$ is a Fölner set. Average the result of Lemma 4.4 over $Y_L$, the set of vertical geodesics in $B(L)$ and let $|Y_L|$ denote the measure/cardinality of $Y_L$. Changing order, we get:

$$\sum_{m=1}^{M} \left( \frac{1}{|Y_L|} \sum_{\gamma \in Y_L} \delta_m(\overline{\gamma}) \right) \leq \frac{32K^3}{\epsilon}.$$

Thus, given any $\theta > 0$, (by choosing $M$ sufficiently large) we can make sure that there exists $1 \leq m \leq M$ such that

$$\frac{1}{|Y_L|} \sum_{\gamma \in Y_L} \delta_m(\overline{\gamma}) < \theta. \tag{1}$$

**Conclusion.** On the scale $R \equiv r_m$, at least $1 - \theta$ fraction of all vertical geodesic segments in $B(L)$ have nearly vertical images under $\phi$. See Figure 3.

The difficulty is that, at this point, it may be possible that some of the (upward oriented) vertical segments in $B(L)$ may have images which are going up, and some may have images which are going down.

We think of the process we have just described as a form of "coarse differentiation". For further discussion of this process and a more general variant on the discussion in the last two subsections, see subsection 4.5.

**4.3. The scheme of the proof of Theorem 2.1..** Roughly speaking, the proof proceeds in the following steps:

**Step 1.** *For all $\theta > 0$ there exists $L_0$ such that for any box $B(L)$ where $L \geq L_0$, there exists $0 \ll r \ll R \ll L_0$ such that for the tiling:*

$$B(L) = \bigsqcup_{i=1}^{N} B_i(R)$$

*there exists $I \subset \{1, \dots, N\}$ with $|I| \geq (1-\theta)N$ and for each $i \in I$ there exists a height-respecting map $\hat{\phi}_i : B_i(R) \to \mathrm{Sol}$ and a subset $U_i \subset B_i(R)$ with $|U_i| \geq (1-\theta)|B_i(R)|$ such that*

$$d(\phi|_{U_i}, \hat{\phi}_i) = O(r).$$

Roughly, Step 1 asserts that every sufficiently large box can be tiled into small boxes, in such a way that for most of the small boxes $B_i(R)$, the restriction of $\phi$ to $B_i(R)$ agrees, on most of the measure of $B_i(R)$, with a height-respecting map $\hat{\phi}_i : B_i(R) \to \mathrm{Sol}$. There is no assertion in Step 1 that the height-respecting maps $\hat{\phi}_i$ on different small boxes match up to define a height-respecting map on most of the measure on $B(L)$; the main difficulty is that some of the $\hat{\phi}_i$ may send the "up" direction to the "down" direction, while other $\hat{\phi}_i$ may preserve the up direction.

Step 1 follows from a version of (1) and some geometric arguments using Lemma 3.1. The point is that any $\epsilon$-monotone quasi-geodesic is close to a vertical geodesic by Lemma 4.2. By the averaging argument in subsection 4.2, we find a scale $R$ at which most segments have $\epsilon$-monotone image under $\phi$. More averaging implies that on most boxes $B_i(R)$ most geodesic segments joining the top of the box to the bottom of the box have $\epsilon$-monotone images. We then apply Lemma 3.1 to the images of these geodesics and use this to show that the map is roughly height preserving on each $B_i(R)$. This step also uses the geometric description of $B_i(R)$ given in the last paragraph of §3, i.e. the fact that a box is coarsely a complete bipartite graph $G_R$ on nets in the "top" and "bottom" of the box.

**Step 2.** *For all $\theta > 0$ there exists $L_0$ such that for any box $B(L)$ where $L \geq L_0$, $\exists$ subset $U \subset B(L)$ with $|U| \geq (1-\theta)|B(L)|$ and a height-respecting map $\hat{\phi} : B(L) \to \mathrm{Sol}$ such that*

$$d(\phi|_U, \hat{\phi}) = O(l),$$

*where $l \ll L_0$.*

This is the essentially the assertion that the different maps $\hat{\phi}_i$ from Step 1 are all oriented in the same way, and can thus be replaced by one standard map $\hat{\phi} : B(L) \to \mathrm{Sol}$.

Step 2 is the most technical part of the proof. The problem here derives from exponential volume growth. In Euclidean space, given a set of almost full measure $U$ in a box, every point in the box is close to a point in $U$. This is not true in Sol because of exponential volume growth. Another manifestation of this difficulty is that Sol does not have a Vitali covering lemma. The proof involves using refinements of Lemma 3.1 and further averaging on the image of $\phi$.

**Step 3.** *The map $\phi$ is $O(L_0)$ from a standard map $\hat{\phi}$.*

This follows from Step 2 and some geometric arguments using variants of Lemma 3.1. The large constant, $O(L_0)$, arises because we pass to very large scales to ignore the sets of small measure that arise in Steps 1 and 2.

**4.4. Remarks on the proof of Theorem 1.4 and the general case:.** Peng's proof of Theorem 1.4 proceeds roughly using the same strategy as the proof of Theorem 1.3. The main difference is that instead of vertical geodesics one has "vertical flats" (which are the orbits of $\mathbb{R}^k$ acting on $\mathbb{R}^k \ltimes \mathbb{R}^n$). These flats are equipped with a foliation by hyperplanes which are parallel to the kernels of the weights on $\mathbb{R}^k$ defined by the map $\rho : \mathbb{R}^k \to \mathbb{R}^n$. Peng shows that the quasi-isometry roughly preserves the vertical flats, and also the restriction of the map to a flat preserves the foliation by hyperplanes. In particular a geodesic in a vertical flat which is transverse to the root hyperplanes maps roughly to another such object. This allows Peng to show that the map roughly preserves subsets of the space whose geometry is quite similar to Sol geometry. This fact then allows her to use the geometry of the graphs $G_L$ described at the end of §3 in her arguments.

The case $G = \mathbb{R} \ltimes N$ where $N$ is a nilpotent group and $N$ is equal to the exponential radical of $G$ can already present considerable extra difficulties. In this case we can split the Lie algebra $\mathfrak{n}$ of $N$ into an expanding subspace $\mathfrak{n}^+$ and a contracting subspace $\mathfrak{n}^-$. In the case where $[\mathfrak{n}^+, \mathfrak{n}^-] = 0$, the geometry of $G$ is quite similar to the geometry of $Sol$ and the proof that the $N$ coset foliation is preserved can be carried out in a very similar fashion. However if $[\mathfrak{n}^+, \mathfrak{n}^-] \neq 0$, the geometry is quite different and the the quadrilateral lemma (Lemma 3.1) fails to be true. This is closely related to the fact the the graphs $G_L$, which are the $\mathbb{R} \ltimes N$ analogues of the graph given the same name at the end of §3, are no longer complete bipartite. One can make progress in this direction by replacing Lemma 3.1 by a sort of averaged version, but this requires the detailed study of the graphs $G_L$, including proving a uniform spectral gap as the size of the box $L$ tends to infinity. This is done in [FP].

Once the $\mathbb{R} \ltimes N$ case is complete, the proof in split polycyclic case $G = \mathbb{R}^k \ltimes N$ would presumably involve incorporating the ideas of [Pe1], [Pe2]. Even the split case where $G = N_1 \ltimes N_2$ with $N_2 = \exp(G)$ will be quite similar with vertical flats replaced by vertical copies of $N_1$. For the general polycyclic group $G$, the exact sequence $1 \to \exp(G) \to G \to G/\exp(G) \to 1$ may not split, and "vertical flats" are no longer defined. However, there is a geometric splitting of the exact sequence which defines a foliation of $G$ by sets diffeomorphic to $G/\exp(G)$ where the maps sending leaves into the space $G/\exp(G) \to G$ preserves distances up to a logarithmic error and thus the methods described here are still relevant. This geometric splitting is used by de Cornulier in his work on the asymptotic geometry of solvable Lie groups [dC1, dC2].

**4.5. Remarks on coarse differentiation:.** If a map is differentiable, then it is locally at sub-linear error from a map which takes lines to lines. This is roughly the conclusion of the argument above for the vertical geodesics in Sol, at least on an appropriately chosen large scale and off of a set of small measure. The ideas employed here can be extended to general metric spaces, by replacing the notion of $\epsilon$-monotone with a more general notion of $\epsilon$-efficient which we will describe below. The ideas in our proof are not so different from the proof(s) of Rademacher's theorem that a bilipschitz map of $\mathbb{R}^n$ is differentiable almost everywhere. In fact, our method applied to quasi-isometries of $\mathbb{R}^n$ gives roughly the same information as the application of Rademacher's theorem to the induced bilipschitz map on the asymptotic cone of $\mathbb{R}^n$ (which is again $\mathbb{R}^n$). In this context the presence of sets of small measure can be eliminated by a covering lemma argument. In the context of solvable groups, passage to the asymptotic cone is complicated by the exponential volume growth. The asymptotic cone for these groups is not locally compact, which makes it difficult to find useful notions of sets of zero or small measure there.

We now formulate somewhat loosely a more general form of the "differentiation theorem" given in subsections 4.1 and 4.2. Throughout this subsection $Y$ will be a general metric space, though it may be most useful to think of $Y$ as a complete, geodesic metric space. First we generalize the notion of $\epsilon$-monotone.

**Definition 4.5.** *A quasigeodesic segment $\alpha : [0, L] \to Y$ is $\epsilon$-efficient on the scale $r$ if*

$$\sum_{j=1}^{L/r} d(\alpha(jr), \alpha((j-1)r)) \leq (1+\epsilon) d(\alpha(L), \alpha(0)).$$

The fact is that a quasi-geodesic, unless it is a $(1+\epsilon)$ quasi-geodesic, fails to be $\epsilon$-efficient at some scale some fraction of the time. The observation embedded in subsection 4.1 is that this cannot happen everywhere on all scales and in fact cannot happen too often on too many scales.

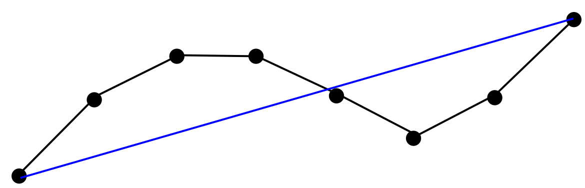

Figure 4. The definition of $\epsilon$-efficient.

With this definition, the following variant on Lemma 4.3 becomes a tautology.

**Lemma 4.6** (Subdivision II). *Given $\epsilon > 0$, there exist $r \gg C$ and $n \gg 1$ (depending on $K, C$ and $\epsilon$) such that any $(K, C)$-quasi-geodesic segment $\alpha : [0, r] \to X$ which is not $\epsilon$-efficient on scale $\frac{r}{n}$ we have:*

$$\sum_{j=0}^{n-1} d(\alpha(\tfrac{(j+1)r}{n}), \alpha(\tfrac{jr}{n})) \geq d(\alpha(0), \alpha(r)) + \frac{\epsilon r}{2K}.$$

We now state a variant of Lemma 4.4 whose proof is verbatim the proof of that lemma.
**Choosing Scales:** Choose $1 \ll r_0 \ll r_1 \ll \cdots \ll r_M$. In particular, $C \ll r_0$ and $r_{m+1}/r_m > n$.

**Lemma 4.7.** *Suppose $L \gg r_M$, and suppose $\alpha : [0, L] \to X$ is a quasi-geodesic segment. For each $m \in [1, M]$, subdivide $[0, L]$ into $L/r_m$ segments of length $r_m$. Let $\delta_m(\alpha)$ denote the fraction of these segments whose images are not $\epsilon$-efficient on scale $r_{m-1}$. Then,*

$$\sum_{m=1}^{M} \delta_m(\alpha) \leq \frac{4K^2}{\epsilon}.$$

Let $X$ be a geodesic metric space. Coarse differentiation amounts to the following easy lemma.

**Lemma 4.8** (Coarse Differentiation). *Let $\phi : X \to Y$ be a $(K, C)$-quasi-isometry. For all $\theta > 0$ there exists $L_0 \gg 1$ such that for any $L > L_0$ and any family $\mathcal{F}$ of geodesics of length $L$ in $X$, there exist scales $r, R$ with $C \ll r \ll R \ll L_0$ such that if we divide each geodesic in $\mathcal{F}$ into subsegments of length $R$, then at least $(1-\theta)$ fraction of these subsegments have images which are $\epsilon$-efficient at scale $r$.*

This lemma and its variants seem likely to be useful in other settings. In fact, the lemma holds only assuming that $\phi$ is coarsely lipschitz. A map $\phi : X \to Y$ is a $(K, C)$ *coarsely lipschitz* if $d_Y(\phi(x_1), \phi(x_2)) \leq K d_X(x_1, x_2) + C$. We now describe the relation to taking derivatives and also to the process of taking a "derivative at infinity" of a quasi-isometry by passing to asymptotic cones.

We first discuss the case of maps $\mathbb{R}^n \to \mathbb{R}^n$. Suppose $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is a quasi-isometry. Suppose one chooses a net $N$ on the unit circle and takes $\mathcal{F}$ to be the set of all lines of length $L$ in a large box, whose direction vector is in $N$. Lemma 4.8 applied to $\mathcal{F}$ then states that most of these lines, on the appropriate scale, map under $\phi$ close to straight lines, which implies that the map $\phi$ (in a suitable box) can be approximated by an affine map. Thus, in this context, Lemma 4.8 is indeed analogous to differentiation (or producing points of differentiability).

An alternative approach for analyzing quasi-isometries $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is to pass to the asymptotic cone to obtain a bilipschitz map $\tilde{\phi} : \mathbb{R}^n \to \mathbb{R}^n$ and then apply Rademacher's theorem to $\tilde{\phi}$. If one attempts to pull the information this yields back to $\phi$ one gets statements that are similar to those one would obtain directly using Lemma 4.8. This is not surprising, since averaging arguments like those used in the proof of Lemma 4.8 are implicit in the proofs of Rademacher's theorem.

Passing to the asymptotic cone has obvious advantages because it allows one to replace a $(K, C)$ quasi-isometry from $X$ to $Y$ with a $(K, 0)$-quasi-isometry (i.e. a bilipschitz map) from the asymptotic cone of $X$ to the asymptotic cone of $Y$. One can then try to use analytic techniques to study the bilipshitz maps. However, a major difficulty which occurs is that the asymptotic cones are typically not locally compact and notions of measure and averaging on such spaces are not clear. This difficulty arises as soon as one has exponential volume growth. In particular it is not clear if there is a useful version of Rademacher's theorem for the asymptotic cones of the spaces which we consider in this paper.

The main advantage of Lemma 4.8 compared to the asymptotic cone approach is that the averaging is done on the (typically locally compact) space $X$, i.e. the domain of the quasi-isometry $\phi$. In other words, we construct a "coarse derivative" without first passing to a limit to get rid of the additive constant. In particular, the information we obtain about Sol and other solvable groups by coarse differentiation is not easily extracted by passage to the asymptotic cone.

We remark again that Lemma 4.8 applies to any quasi-isometric embedding (or any uniform embedding) between any two metric spaces $X$ and $Y$. However its usefulness clearly depends on the situation.

The coarse differentiation approach is closely related to results proved the method of the "iterated midpoint" which is well-known in the theory of Banach spaces, see e.g. [B],[BL], [JLS], [M], [Pr], [BJLPS]. Some results of some of those papers also have a similar flavor, resulting in points where a map between Banach spaces is $\epsilon$-Frechet differentiable, i.e. that the map is sublinear distance from an affine map at some scale. The main difference in proofs is that in our setting it is possible to average the inequality as described in §4.2 to obtain some control on a set of large (but not full) measure.

**4.6. Deduction of rigidity results.** In our setting, the deduction of rigidity results from the classification of quasi-isometries follows a fairly standard outline that is similar to one used for semisimple groups as well as for certain solvable groups in [FM2, FM3, MSW]. As this is standard, we will say relatively little about it. Some of these ideas go back to Mostow's original proof of Mostow rigidity [Mo1, Mo3] and have been developed further by many authors.

Given a group $\Gamma$ any element of $\gamma$ in $\Gamma$ acts on $\Gamma$ by isometries by left multiplication $L_\gamma$. If $X$ is a metric space and $\phi : \Gamma \to X$ is a quasi-isometry, we can conjugate each $L_\gamma$ to a self quasi-isometry $\phi \circ L_\gamma \circ \phi^{-1}$ of $X$. This induces a homomorphism of $\Phi : \Gamma \to \mathrm{QI}(X)$. Here $\mathrm{QI}(X)$ is the group of quasi-isometries of $X$ modulo the subgroup of quasi-isometries a bounded distance from the identity. The approach we follow is to use $\Phi$ to define an action of $\Gamma$ on a "boundary at infinity" of the space $X$. All theorems are then proven by studying the dynamics of this "action at infinity." We are ignoring many important technical points here, such as why $\Phi$ has finite kernel and why $\mathrm{QI}(X)$ acts on either $X$ or the boundary at infinity of $X$.

The deduction of Theorem 1.3 from Theorem 2.1 was known to Farb and Mosher [FM2, FM4]. The action at infinity is studied using a variant of a theorem of Hinkannen due to Farb and Mosher [H, FM2, FM4]. In the context of Theorem 1.4, we deduce the result from Theorem 2.2 using results from the dissertation of Tullia Dymarz and a further paper by Dymarz and Peng [Dy, DP]. These are variants and extensions of the results of Tukia in [Tu]. As remarked above, a proof of Conjecture 1.2 from Conjecture 2.3 will require a further generalization of these results. This generalization is already a significant and difficult problem.

# 5. Further consequences

In this section we discuss some other results that are consequence either of our methods or of our results.

**5.1. Geometry of Diestel-Leader graphs.** In addition our methods yield quasi-isometric rigidity results for a variety of solvable groups which are not polycyclic, in particular the so-called lamplighter groups. These are the wreath products $\mathbb{Z} \wr F$ where $F$ is a finite group. The name lamplighter comes from the description $\mathbb{Z} \wr F = F^{\mathbb{Z}} \rtimes \mathbb{Z}$ where the $\mathbb{Z}$ action is by a shift. The subgroup $F^{\mathbb{Z}}$ is thought of as the states of a line of lamps, each of which has $|F|$ states. The "lamplighter" moves along this line of lamps (the $\mathbb{Z}$ action) and can change the state of the lamp at her current position. The Cayley graphs for the generating sets $F \cup \{\pm 1\}$ depend only on $|F|$, not the structure of $F$. Furthermore, $\mathbb{Z} \wr F_1$ and $\mathbb{Z} \wr F_2$ are quasi-isometric whenever there is a $d$ so that $|F_1| = d^s$ and $|F_2| = d^t$ for some $s, t$ in $\mathbb{Z}$. The problem of classifying these groups up to quasi-isometry, and in particular, the question of whether the 2 and 3 state lamplighter groups are quasi-isometric, were well known open problems in the field, see [dlH].

**Theorem 5.1.** *The lamplighter groups $\mathbb{Z} \wr F$ and $\mathbb{Z} \wr F'$ are quasi-isometric if and only if there exist positive integers $d, s, r$ such that $|F| = d^s$ and $|F'| = d^r$.*

For a rigidity theorem for lamplighter groups, see Theorem 5.2 below.

To state Theorem 5.2 as well as some other results, we need to describe a class of graphs. These are the Diestel-Leader graphs, $DL(m, n)$, which can be defined as follows: let $T_1$ and $T_2$ be regular trees of valence $m + 1$ and $n + 1$. Choose orientations on the edges of $T_1$ and $T_2$ so each vertex has $n$ (resp. $m$) edges pointing away from it. This is equivalent to choosing ends on these trees. We can view these orientations at defining height functions $f_1$ and $f_2$ on the trees (the Busemann functions for the chosen ends). If one places the point at infinity determining $f_1$ at the top of the page and the point at infinity determining $f_2$ at the bottom of the page, then the trees can be drawn as:
The graph $DL(m, n)$ is the subset of the product $T_1 \times T_2$ defined by $f_1 + f_2 = 0$. The analogy with the geometry of Sol is clear from section 3. For $n = m$ the Diestel-Leader graphs arise as

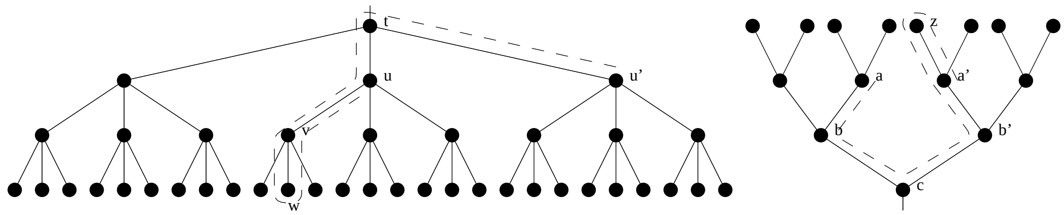


Figure 5. The trees for $DL(3,2)$. Figure borrowed from [PPS].

Cayley graphs of lamplighter groups $\mathbb{Z}\wr F$ for $|F| = n$. This observation was apparently first made by R.Moeller and P.Neumann [MN] and is described explicitly, from two slightly different points of view, in [Wo2] and [W]. We prove the following:

**Theorem 5.2.** *Let $\Gamma$ be a finitely generated group quasi-isometric to the lamplighter group $\mathbb{Z}\wr F$. Then there exists positive integers $d, s, r$ such that $d^s = |F|^r$ and an isometric, proper, cocompact action of a finite index subgroup of $\Gamma$ on the Diestel-Leader graph $DL(d, d)$.*

**Remark:** The theorem can be reinterpreted as saying that any group quasi-isometric to $DL(|F|, |F|)$ is virtually a cocompact lattice in the isometry group $\text{Isom}(DL(d, d)$ of $DL(d, d)$ where $d$ is as above.

Recently, the second author, de Cornulier and Kashyp have proven some detailed results concerning the algebraic structure of cocompact lattices in $\text{Isom}(DL(d, d))$. We state here one corollary of that work.

**Theorem 5.3.** *Let $\Gamma < \text{Isom}(DL(d, d))$ be a cocompact lattice. Then $\Gamma$ admits a transitive, proper action on $DL(d^n, d^n)$ for some positive $n$.*

The paper [dCFK] also contains many examples of lattices in $\text{Isom}(DL(d, d))$ which are not weakly commensurable to lamplighters.

In [SW, Wo1], Soardi and Woess ask whether every homogeneous graph is quasi-isometric to a finitely generated group. The graph $DL(m, n)$ is easily seen to be homogeneous (i.e. it has a transitive isometry group). For $m \neq n$ its isometry group is not unimodular, and hence has no lattices. Thus there are no obvious groups quasi-isometric to $DL(m, n)$ in this case. In fact, we have:

**Theorem 5.4.** *There is no finitely generated group quasi-isometric to the graph $DL(m, n)$ for $m \neq n$.*

This theorem was conjectured by Diestel and Leader in [DL], where the Diestel-Leader graphs were introduced for this purpose. Note that Theorem 5.4 can be reinterpreted as the statement that for $m \neq n$, there is no finitely generated group quasi-isometric to the isometry group of $DL(m, n)$.

Recall that $DL(m, n)$ is defined as the subset of $T_{m+1} \times T_{n+1}$ where $f_m(x) + f_n(y) = 0$ where $f_m$ and $f_n$ are Busemann functions on $T_m$ and $T_n$ respectively. Here we simply set $h((x, y)) = f_m(x) = -f_n(y)$ which makes sense exactly on $DL(m, n) \subset T_{m+1} \times T_{n+1}$. The reader can verify that the level sets of the height function are orbits for a subgroup of $\text{Isom}(DL(m, n))$.

**Theorem 5.5.** *Any $(K, C)$-quasi-isometry $\varphi$ of $DL(m,n)$ is within bounded distance from a height respecting quasi-isometry $\hat{\varphi}$. Furthermore, the bound is uniform in $K$ and $C$.*

**Remark:** We can reformulate Theorem 5.5 in terms similar to those of Theorem 2.1. Here the group Bilip($\mathbb{R}$)×Bilip($\mathbb{R}$) will be replaced by Bilip($X_m$)× Bilip($X_n$) for $X_m$ (resp. $X_n$) the complement of a point in the (visual) boundary of $T_{m+1}$ (resp. $T_{n+1}$). These can easily be seen to be the $m$-adic and $n$-adic rationals, respectively.

Note that when $m = n$, this theorem is used to prove Theorem 5.2 and when $m \neq n$ it is used to prove Theorem 5.4. The proofs in these two cases are somewhat different, the proof in the case $m = n$ being almost identical to the proof of Theorem 2.1. In the other case, the argument is complicated by the absence of metric Fölner sets, but simplifications also occur since there is no element in the isometry group that "flips" height. There is an analogue of the above results for the case of the solvable Lie groups which appears in Theorem 5.9.

Another recent dramatic development that uses Theorem 5.5 is the following result of Dymarz [Dy2].

**Theorem 5.6.** *Consider the two lamplighter groups $F^k \wr \mathbb{Z}$ and $F \wr \mathbb{Z}$ where $|F| = m$ and $F^k$ is the direct product of $k$ copies of $F$. Then there does not exist a bijective quasi-isometry between $F^k \wr \mathbb{Z}$ and $F \wr \mathbb{Z}$ if $k$ is not a product of prime factors appearing in $m$.*

The main point of the theorem is that these two groups are quasi-isometric but not bijectively quasi-isometric. A result of Whyte says that any pair of non-amenable groups are quasi-isometric if and only if they are bijectively quasi-isometric [Wh]. Dymarz's result proves that this is no longer true for amenable groups and answers a question that had been open for over ten years.

## 5.2. Low dimensional topology and geometry.

We now state a theorem that is a well-known consequence of Theorem 1.3, Thurston's Geometrization Conjecture and results in [CC, Gr1, KaL1, KaL2, PW, S1, Ri]. We state it assuming that the Geometrization Conjecture is known.

**Theorem 5.7.** *Let $M$ be a compact three manifold without boundary and $\Gamma$ a finitely generated group. If $\Gamma$ is quasi-isometric to the universal cover of $M$, then $\Gamma$ is virtually the fundamental group of $M'$, also a compact three manifold without boundary.*

For more discussion of this theorem and significant progress towards classifying three manifold groups up to quasi-isometry, see work of Behrstock and Neumann [BN1, BN2].

The existence of transitive graphs not quasi-isometric to Cayley graphs as given by Theorem 5.4 gives rise to interesting surfaces with exotic properties. The surfaces are obtained simply by replacing edges in the graphs by tubes and vertices by spheres to which one attaches the tubes. This construction is used by Bonafert, Canary, Souto and Taylor to construct uniformly quasiconformally homogeneous Riemann surfaces which are not quasiconformal deformations of regular covers of closed orbifolds [BCST].

## 5.3. Lie groups not quasi-isometric to discrete groups.

The following is a basic question:

**Question 5.8.** *Given a Lie group $G$, is there a finitely generated group quasi-isometric to $G$?*

It is clear that the answer is yes whenever $G$ has a cocompact lattice. However, many solvable locally compact groups, and in particular, many solvable Lie groups do not have any lattices. The simplest examples are groups which are not unimodular. However, it is possible for Question 5.8 to have an affirmative answer even if $G$ is not unimodular. For instance, the non-unimodular group solvable group $\left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \middle| \, a > 0, b \in \mathbb{R} \right\}$ acts simply transitively by isometries on the hyperbolic plane, and thus is quasi-isometric to the fundamental group of any closed surface of genus at least 2. Thus the answer to Question 5.8 can be subtle. Our methods give:

**Theorem 5.9.** *Let $G = \mathbb{R} \ltimes \mathbb{R}^2$ be a solvable Lie group where the $\mathbb{R}$ action on $\mathbb{R}^2$ is defined by $z \cdot (x, y) = (e^{az}x, e^{-bz}y)$ for $a, b > 0$, $a \neq b$. Then there is no finitely generated group $\Gamma$ quasi-isometric to $G$.*

If $a > 0$ and $b < 0$, then $G$ admits a left invariant metric of negative curvature. The fact that there is no finitely generated group quasi-isometric to $G$ in this case is a result of Kleiner [K], see also [Pa2]. Kleiner's result has recently been generalized by Shanmugalingam and Xie [SX]. It is possible to generalize the theorem above further using our techniques and the results in [Dy, DP]. Nilpotent Lie groups not quasi-isometric to any finitely generated group where constructed in [ET].

### 5.4. Distortion of embeddings and multi-commodity flow problems.

The technique of coarse differentiation has also been applied by Lee and Raghavendra to a problem arising from theoretical computer science [LR]. Their work is motivated by the multi-commodity version of min cut- max flow. This problem is known to be related to problems concerning distortion of embeddings into $L^1$ spaces. They use coarse differentiation to obtain bounds on the distortion of $L^1$ embeddings for a certain family of graphs.

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Department of Mathematics, University of Chicago, 5734 S. University Avenue, Chicago, Illinois 60637.

Department of Mathematics, Indiana University, Rawles Hall, Bloomington, IN, 47405.