MAPPING CLASS GROUPS, 4-MANIFOLDS, AND NIELSEN REALIZATION

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1. Introduction

The theory of mapping classes on surfaces gives insight into several seemingly distinct problems, including Thurston's classification of 3-manifolds, hyperbolic geometry, dynamics on surfaces, and parts of algebraic geometry through its role as the orbifold fundamental group of the moduli space of curves. Many such problems boil down in part to the informal question: What do mapping classes *look* like? That is, does each mapping class contain some optimal representative exhibiting some extra structure relevant to a given problem? The Nielsen-Thurston classification divides mapping classes into periodic, reducible, and pseudo-Anosov categories. Together with Kerchoff's affirmative proof of Nielsen realization, this classification gives a relatively robust understanding of surface mapping classes.

In contrast to surfaces, the mapping class groups of 4-manifolds are poorly understood and much more subtle. Complex and metric structures are no longer equivalent, and even more fundamentally, the smooth and topological mapping class groups are no longer isomorphic. In fact, the map from the smooth mapping class group to its topological counterpart need not be either injective or surjective. But nonetheless we ask: What do mapping classes *look* like?

Below, we review the theory of mapping classes for surfaces and simply-connected 4-manifolds. We then give the basic theory of complex surfaces before exploring the Nielsen realization problem for certain 4-manifolds (including some small amount of original research).

2. Mapping class groups of surfaces

Many of the following proofs in this section draw from [FM12]. First recall the mapping class group of $S^b_{g,n}$, which denotes a surface of genus g with n punctures and b boundary components. Recall that these three parameters classify compact oriented surfaces up to homeomorphism.

Definition 2.1. The mapping class group of $S^b_{g,n}$ is

$$\mathrm{Mod}(S^b_{g,n}) := \mathrm{Diff}^+(S^b_{g,n}, \partial S^b_{g,n})/\mathrm{Diff}^+_0(S^b_{g,n}, \partial S^b_{g,n}) = \pi_0(S^b_{g,n}, \partial S^b_{g,n}).$$

In other words, $\mathrm{Mod}(S^b_{g,n})$ is the set of all isotopy classes of orientation-preserving diffeomorphisms of $S_{g,n}$ which fix the boundary point-wise, considered as a group under composition. By $\mathrm{PMod}(S^b_{g,n})$ denote the subgroup which fixes punctures point-wise.

Recall the following fundamental theorem about the mapping class group.

Theorem 2.1. *$PMod(S_{g,n})$ is generated by a finite set of Dehn twists about non-separating simple closed curves for $g \geq 1$ and $n \geq 0$.*

Before proving Theorem 2.1, note the following lemma.

Lemma 2.2 (Birman exact sequence)**.** *There is an exact sequence*

$$1 \to \pi_1(S_{g,n}) \to \mathrm{Mod}(S_{g,n+1}) \to \mathrm{Mod}(S_{g,n}) \to 1,$$

where the embedding $\pi_1(S_{g,n}) \hookrightarrow \mathrm{Mod}(S_{g,n+1})$ is given by a 'point-pushing' procedure

Proof sketch. There is a fiber bundle

$$\mathrm{Homeo}^+(S,x) \to \mathrm{Homeo}^+(S) \to S,$$

where the fiber over $x \in S$ consists of all orientation-preserving homeomorphisms of S which fix x. Now apply the long exact sequence of homotopy groups. Noting that $\pi_1(\mathrm{Homeo}^+(S)) \simeq \pi_0(S) \simeq 0$ gives the Birman exact sequence. □

Proof sketch of Theorem 2.1. Induct on both g, n. For induction on n, one can leverage the Birman exact sequence of Lemma 2.2. Notice that each point-push is itself the product of two Dehn twists. For the g-induction one uses the 'curve-complex' $\widehat{\mathcal{N}}(S_g)$, which is a graph whose nodes correspond to the non-separating simple closed curves. Two vertices are connected if the corresponding curves intersect exactly once (in minimal position). One can show that $\widehat{\mathcal{N}}(S_g)$ is connected for $3g+n \geq 5$ and then use a result of geometric group theory in order to reduce to showing that the stabilizer of a nonseparating closed curve is finitely generated by Dehn twists. This is easily accomplished by cutting along such a curve and applying a second short exact sequence

$$1 \to \langle T_a \rangle \to \mathrm{Mod}(S_g, \overrightarrow{a}) \to \mathrm{PMod}(S_g - \alpha) \to 1,$$

where T_a is a Dehn twist about a, $\mathrm{Mod}(S_g, \overrightarrow{a})$ is a subgroup of mapping classes which fix a with its orientation, and α is a curve representing a. □

2.1. **The Teichmüller and moduli spaces.** In order to answer our guiding question of what mapping classes look like, we turn to Teichmüller and moduli spaces, whose definitions we now recall.

Definition 2.2. The *Teichmüller space* of S_g for $g \geq 2$ is defined to be

$$\mathrm{Teich}\,(S_g) := \{h : h \text{ is a hyperbolic Riemannian metric on } S_g\}/\mathrm{Diff}_0^+(S_g),$$

where $\mathrm{Diff}_0^+(S_g)$ acts by pulling back metrics. Hence $\mathrm{Mod}(S_g)$ acts on $\mathrm{Teich}\,(S_g)$, and denote the resulting orbifold quotient the *moduli space* $\mathcal{M}_g := \mathrm{Teich}\,(S_g)/\mathrm{Mod}(S_g)$.

Endowed with a suitable topology, the homeomorphism type of $\mathrm{Teich}\,(S_g)$ turns out to be surprisingly simple. Recall that to any hyperbolic surface $\mathfrak{X} \in \mathrm{Teich}\,(S_g)$, one can attach a length function $\ell_{\mathfrak{X}}$ which takes in an isotopy class of closed curves an returns the length of the shortest geodesic in that class. Topologize $\mathrm{Teich}\,(S_g)$ with just enough open sets so that the map $\mathfrak{X} \mapsto \ell_{\mathfrak{X}}(c)$ for each fixed c is continuous.

Theorem 2.3. *With the aforementioned topology, $\mathrm{Teich}\,(S_g)$ is homeomorphic to $\mathbb{R}^{6g-6}$ for $g \geq 2$.*

Proof sketch. Decompose S_g into pairs of pants by cutting along $3g-3$ pairwise-disjoint simple closed curves. A hyperbolic metric on S_g is then determined exactly by the lengths of each curve (which fixes a hyperbolic metric on each pair of pants), together with a 'twist parameter' for each curve, which specifies gluing data. Since both length parameters take values in $\mathbb{R}^+$ and twist parameters take values in $\mathbb{R}$, the parameter space as a whole (and hence the Teichmüller space) is homeomorphic to $(\mathbb{R}^+)^{3g-3} \times \mathbb{R}^{3g-3} = \mathbb{R}^{6g-6}$. □

A statement and proof sketch of the Nielsen-Thurston classification requires two more preliminary results.

Theorem 2.4. Mod(S_g) *acts properly discontinuously on Teich* (S_g) *with finite point-stabilizers.*

Proof sketch. Using analytic machinery, one can construct the 'Teichmüller metric' on Teich (S_g). Let $K \subset$ Teich (S_g) be compact of diamter D, and let c_1, c_2 be two simple closed non-separating curved which fill S_g (meaning their complement is a union of disks), and let $L = \max\{\ell_{\mathfrak{X}}(c_1), \ell_{\mathfrak{X}}(c_2)\}$. Suppose that $f \cdot K \cap K \neq \varnothing$. So in the Teichmüller metric, $d(\mathfrak{X}, f \cdot \mathfrak{X}) \leq 2D$. Wolpert's lemma then gives a constant C independent of f such that

$$\ell_{\mathfrak{X}}(f^{-1}(c_i)) = \ell_{f \cdot \mathfrak{X}}(c_i) \leq CL.$$

Since the length spectrum of a hyperbolic surface is discrete there are finitely many possibilities for $f^{-1}(c_i)$. The Alexander method then implies there are finitely many choices of f. Finally, the stabilizer of a point is just the isometry group of that metric, which is finite. □

Corollary 2.5. $\mathcal{M}_g$ *is connected.*

Theorem 2.6 (Mumford Compactness Criterion)**.** *For any* $\varepsilon > 0$*, let* $\mathcal{M}_g(\epsilon)$ *denote the subset of metrics in* $\mathcal{M}_g$ *for which all geodesics have length at least* ϵ*. Then* $\mathcal{M}_g(\epsilon)$ *is compact.*

Proof sketch. Bers's theorem gives at constant L such that for every hyperbolic structure on S_g, there is a pants decomposition whose length parameters are all smaller than L. Given a sequence of $X_i \in \mathcal{M}_g$, pick a set of lifts $\mathfrak{X}_i \in$ Teich (S_g). Since the number of topological types of pants decompositions is a finite constant depending on g, pick a subsequence so that this type is constant. Then if needed, change each lift by the action of a Dehn twist around the curves of our pants decomposition until all twist parameters of the $\mathfrak{X}_i$ are in $[0, 2\pi]$. Thus, the sequence $\mathfrak{X}_i$ is contained in the box $[\epsilon, L] \times [0, 2\pi] \in$ Teich (S_g) and hence has a convergent subsequence. Thus the sequence X_i has a convergent subsequence as well. □

2.2. **Nielsen realization and the Nielsen-Thurston classification.**

Theorem 2.7 (Nielsen realization)**.** *Let* $g \geq 1$*, and consider the projection* π : Diff$^+(S_g) \to$ Mod(S_g)*. For any finite group* $G \leq$ Mod(S_g)*, there is a finite group* $\tilde{G} <$ Diff$^+(S_g)$ *such that* π *induces an isomorphism* $G \simeq \tilde{G}$*.*

Proof sketch for cyclic groups. Let $[f] \in$ Mod(S_g) have finite order n. It suffices to show that f fixes a point in Teichmüller space, since then f is an isometry of some hyperbolic surface, and the isometry groups of compact hyperbolic surfaces are finite. Notice that f cannot act freely on Teich (S_g), or else the quotient Teich $(S_g)/\langle f \rangle$ would be a finite-dimensional $K(\mathbb{Z}/nZ, 1)$. So f^k fixes from $\mathfrak{X} \in$ Teich (S_g). Induct on the number of (not necessarily distinct) prime factors of n. If n is prime, then f is a power of f^k, and so f fixes $\mathfrak{X}$.

□

Finally, we outline the Nielsen-Thurston classification and its proof.

Theorem 2.8 (Nielsen-Thurston classification)**.** *Any mapping class* $[f] \in$ Mod(S_g) *has a representative diffeomorphism* f *belonging to one of the following categories:*

(1) $[f]$ *is periodic:* $f \in \mathrm{Diff}^+(S_g)$ *has finite order,*
(2) $[f]$ *is reducible:* f *preserves (setwise) some finite collection of simple closed curves,*
(3) $[f]$ *is pseudo-Anosov:* f *preserves a transverse pair of singular measured foliations* $(\mathcal{F}_s, \mu_s), (\mathcal{F}_u, \mu_u)$*, and there is some dilitation constant* $\lambda > 1$ *so that* $g \cdot \mu_s = \frac{1}{\lambda}\mu_s$*,* $g \cdot \mu_u = \lambda\mu_u$*.*

Proof sketch. Let $\tau(f) \coloneqq \inf_{\mathfrak{X} \in \text{Teich}\,(S_g)}\{d(\mathfrak{X}, f \cdot \mathfrak{X})\}$ denote the translation distance of f. There are three possibilities, which turn out to be equivalent (respectively) to the three categories in the theorem statement:

(1) $\tau(f) = 0$ and is realized,
(2) $\tau(f)$ is not realized,
(3) $\tau(f) > 0$ and is realized.

Case (1): f fixes a complex structure and hence acts as an isometry. Then f is finite order (and by Nielsen realization also has a finite-order representative).
Case (2): Take a sequence of $\mathfrak{X}_i$ such that $d(\mathfrak{X}_i, f \cdot \mathfrak{X}_i) \to \tau(f)$, and one can show that it leaves every compact set in $\mathcal{M}_g$. Applying Theorem 2.6 Mumford's Compactness Criterion, Wolpert's lemma, and the Collar Lemma then gives the result.
Case (3): Use the Teichmüller metric to show that f preserves the Teichmüller line γ that passes through $\mathfrak{X}$ and $f \cdot \mathfrak{X}$, for any $\mathfrak{X}$ with $d(\mathfrak{X}, f \cdot \mathfrak{X}) = \tau(f)$. One then shows that f acts on γ by translation. The Teichmüller existence theorem then gives a pseudo-Anosov representative. □

3. Smooth 4-manifolds and intersection forms

We now give an introduction to simply-connected smooth 4-manifolds, with an eye towards their mapping class groups. Many proofs in this section draw from [Sco05]. From now on, assume all our 4-manifolds are closed, oriented, and simply connected (even though this is not quite needed in every case). Note that this implies an isomorphism $H_2(M;\mathbb{Z}) \simeq H^2(M,\mathbb{Z})$.

Lemma 3.1. *Let X be a closed, oriented 4-manifold. Then every class in* $H_2(X;\mathbb{Z})$ *can be represented by an embedded submanifold.*

Proof sketch. Since $\mathbb{CP}^\infty$ is a $K(\mathbb{Z}, 2)$-space, then elements of $H^2(M;\mathbb{Z})$ correspond to homotopy classes of maps $M \to \mathbb{CP}^\infty$. Any such map can be deformed off higher dimensional cells, so that the image lies in $\mathbb{CP}^2 \subset \mathbb{CP}^\infty$. Letting f_α be the map corresponding to $\alpha \in H^2(M;\mathbb{Z})$, assume f_α is differentiable and transverse to $\mathbb{CP}^1 \in \mathbb{CP}^2$. Then the surface $f_\alpha^{-1}(\mathbb{CP}^1)$ represents the Poincaré dual of α. □

Recall that any smooth, oriented 4-manifold M produces an intersection form on $H^2(M;\mathbb{Z})$ given by evaluating the cup product on the fundamental class $[M]$. This pairing has the following two basic properties.

Lemma 3.2. *The cup product pairing on* $H^2(M;\mathbb{Z})$ *given by* $(\alpha^*, \beta^*) \mapsto (\alpha^* \cup \beta^*)[M]$ *corresponds via Poincaré duality to the intersection pairing* Q *on* $H_2(M;\mathbb{Z})$*.*

Proof sketch. Let $\alpha, \beta \in H_2(M)$, and let $\alpha^*, \beta^* \in H^2(M)$ be their Poincaré duals. That is, $\alpha^* \cap [M] = \alpha$ (resp. β). Let S_α, S_β be embedded surfaces representing α, β. Using the equation $(\alpha^* \cup \beta^*)[M] = \alpha^*[\beta^* \cap [M]]$ gives $\alpha^* \cup \beta^* = \alpha^*[S_\beta]$. So it remains to show $\alpha^*[S_\beta] = S_\alpha \cdot S_\beta$. Since M is simply-connected, $H^2(M)$ is free, and hence it can be interpreted as deRham cohomology.

Choosing local coordinates $\{x, y, z, w\}$ on M so that S_α coincides locally with the (x, y)-plane, α^* can be written locally as $f(x, y)dz \wedge dw$, where f is a bump function which integrates to 1. Given some S_β transverse to S_α, arrange S_β to coincide locally with the (z, w)-plane. Then $\int_{S_b} \alpha^* = S_\alpha \cdot S_\beta$, since each intersection points adds ± 1 according to its orientation. □

Lemma 3.3. *The intersection form Q is an integral, symmetric, unimodular (i.e., invertible over $\mathbb{Z}$) bilinear form on $H_2(M; \mathbb{Z})$.*

Proof sketch. Unimodularity follows by Poincaré duality and symbol-pushing, together with elements of the previous proof. The other properties are clear. □

Remarkably, the intersection form of a simply connected, closed, oriented 4-manifold is a complete homeomorphic invariant. First, it is a complete homotopy invariant:

Theorem 3.4 (Whitehead's Theorem)**.** *Two simply-connected 4-manifolds are homotopy-equivalent if and only if they have isomorphic intersection forms.*

Proof sketch. From $\pi_1(M) = 0$, it follows that $H_1(M; \mathbb{Z}) = H_3(M; \mathbb{Z}) = 0$. Hurewicz's theorem then gives the isomorphism $\pi_2(M) \simeq H_2(M; \mathbb{Z})$. Again since $\pi_1(M) = 0$, then $H_2(M; \mathbb{Z})$ is torsion-free by the universal coefficient theorem, and hence is free of rank m. The Hurewicz isomorphism is then realized by some map

$$f : \vee_m S^2 \to M.$$

In fact, $\vee_m S^2$ is homotopy equivalent to the complement of a 4-ball in M. This is because the map $f : \vee_m S^2 \to M \setminus D^4$, defined as above but with restricted codomain, induces an isomorphism on all homotopy groups. In summary, the homotopy type of M is determined by the homotopy class of the attaching map $S^3 \to \vee_m S^2$, where S^3 is the boundary of the 4-ball which was removed from M. In other words, the homotopy type of M depends on a choice of element in $\pi_3(M)$.

Embed $\vee_m S^2 \hookrightarrow \times_m \mathbb{CP}^\infty$, thinking of each S^2 as the $\mathbb{CP}^1$ in each corresponding $\mathbb{CP}^\infty$. Since $\mathbb{CP}^\infty$ is a $K(\mathbb{Z}, 2)$-space, then the only non-trivial homotopy group of $\times_m \mathbb{CP}^2$ is π_2. Then the long exact sequence of homotopy groups for the pair $(\times_m \mathbb{CP}^\infty, \vee_m S^2)$ gives

$$\pi_4(\vee_m S^2, \times_m \mathbb{CP}^\infty) \simeq \pi_3(\vee_m S^2).$$

Since the inclusion $\vee_m S^2 \hookrightarrow \times_m \mathbb{CP}^\infty$ induces an isomorphism on π_2, the same exact sequence implies that π_2 and π_3 of the pair both vanish. Then from Hurewicz's theorem applied to the pair,

$$\pi_4(\times_m \mathbb{CP}^\infty, \vee_m S^2) \simeq H_4(\times_m \mathbb{CP}^\infty, \vee_m S^2; \mathbb{Z}).$$

One last exact sequence on homology finally gives

$$H_4(\times_m \mathbb{CP}^\infty, \vee_m S^2; \mathbb{Z}) \simeq H_4(\times_m \mathbb{CP}^\infty; \mathbb{Z}).$$

Overall, the isomorphism $\pi_3(\vee_m S^2) \simeq H_4(\times_m \mathbb{CP}^\infty; \mathbb{Z})$ pairs the attaching map of a 4-cell with the homotopy class of that cell in $\times_m \mathbb{CP}^\infty$.

Notice that

$$H^4(\times_m \mathbb{CP}^\infty; \mathbb{Z}) = \mathrm{Hom}(H_4(\times_m \mathbb{CP}^\infty; \mathbb{Z}); \mathbb{Z}).$$

In other words, a class of $H_4(\times_m \mathbb{CP}^\infty; \mathbb{Z})$ is determined by how $H^4(\times_m \mathbb{CP}^\infty; \mathbb{Z})$ acts on it. But notice that by the Kunneth formula, a natural $\mathbb{Z}$-basis for $H^4(\vee_m \mathbb{CP}^\infty)$ is given by $\{\omega_i \cup \omega_j\}_{i,j}$, where the ω_i run over the Poincaré duals to a copy of $\mathbb{CP}^1$ in each $\mathbb{CP}^\infty$. Since $H^2(\times_m \mathbb{CP}^\infty; \mathbb{Z}) \simeq H^2(M; \mathbb{Z})$, it is convenient to regard these basis elements as cohomology classes of M. Finally, notice that evaluating $\omega_i \cup \omega_j$ on an element of $H_4(\times_m \mathbb{CP}^\infty; \mathbb{Z})$ is exactly equivalent to evaluating $\omega_i \cup \omega_j$ (as classes on M) on the fundamental class $[M]$. Thus the intersection form, which records precisely this data, determines the homotopy type of M. □

Using Freedman's h-cobordism theorem (which we do not state or prove here), achieves the following corollary.

Corollary 3.5. *Two simply-connected 4-manifolds are homeomorphic if and only if they have isomorphic intersection forms.*

The intersection form also gives direct insight into mapping class groups. Denote the topological mapping class group of M by $\text{Mod}_{top}(M) := \pi_0(\text{Homeo}^+(M))$. Specifically, there is an isomorphism $\text{Mod}_{top}(M) \simeq \text{Aut}(Q)$ given by taking the action on $H_2(M; \mathbb{Z})$.

Even though we are mostly interested in smooth mapping class groups, the relation between intersection forms and topological mapping class groups provides an important foothold.

4. Complex Surfaces

We now take a moment to consider complex algebraic surfaces, which constitute an important class of smooth 4-manifolds with an enormous amount of extra structure. For the remainder of this section, let M be such a compact complex surface.

4.1. **Towards a classification.** One important order of business is to somehow classify complex surfaces. One approach, the Enriques-Kodaira classification, requires a few preliminary results. Recall that the canonical bundle of M is the line bundle K_M defined by

$$K_m = \det T^*M,$$

the determinant of the complex cotangent bundle. Note that $c_1(K_M) = -c_1(TM)$, where c_1 denotes the first Chern class.

Example 1. Observe that $K_{\mathbb{CP}^2} = -3[\mathbb{CP}^1]$. To see, this consider some meromorphic section, like the one given locally on an affine patch $\mathbb{C}^2 \subset \mathbb{CP}^2$ by $dz_1 \wedge dz_2$. Changing coordinates to a different affine patch with coordinates (w_1, w_2) gives the form $(\frac{1}{w_1} dw_2 - \frac{w_2}{w_1^2} dw_1) \wedge -\frac{1}{dw_1^2} dw_1 = -w_1^{-3} dw_1 \wedge dw_2$. That is, this meromorphic form (section of the cotangent bundle) on $\mathbb{CP}^2$ globally has a triple pole along a hyperplane and no zeros. the same is true in the third affine patch Thus, the first Chern class of T^*M is $-3[\mathbb{CP}^1]$, from the structure of $H^2(\mathbb{CP}^2)$.

Proposition 4.1 (Adjunction formula)**.** *For C a nonsingular curve in M,*

$$\chi(C) + C \cdot C = -K_M \cdot C.$$

Proof.

$$\begin{aligned} -\langle K_M, C\rangle &= \langle c_1(TM), C\rangle = c_1(TM|_C) \\ &= c_1(T_C \oplus N_{C/M}) = c_1(T_C) + c_1(N_{C/M}) = \chi(C) + C \cdot C. \end{aligned}$$

□

The adjunction formula is a powerful tool. For example, it gives an easy proof of the following theorem.

Theorem 4.2. *A smooth degree-d polynomial in $\mathbb{CP}^2$ has as its vanishing locus a surface of genus $g = (d-1)(d-2)/2$.*

Proof sketch. Simply rearranging the sought formula gives $(2-2g)+d^2 = -3d$. But now the left side matches the left side of the adjunction formula by Bezout's theorem. And since a degree-d curve represents the homology class $d[\mathbb{CP}^1]$ (for example by Bezout and the intersection form of $\mathbb{CP}^2$), our example above shows that the right sides match, too. $\square$

In order to classify surfaces, we need the notion of *nef*. A line bundle L is said to be *nef* if $c_1(L) \cdot C \geq 0$ for every curve C. One other important notion is that of a blow-up/blow-down. Since the definition of a blow-up is standard, we don't recall it again here. A blow-down is simply the inverse of a blow-up.

Lemma 4.3. *Topologically, blowing up at a point is diffeomorphic to connect-summing with a copy of $\overline{\mathbb{CP}^2}$. The (-1)-curve of the blow-up appears as $\overline{\mathbb{CP}^1} \subset \overline{\mathbb{CP}^2}$.*

Proof sketch. Take a tubular neighborhood of the (-1)-curve, which is diffeomorphic to a tubular neighborhood of $\overline{\mathbb{CP}^1} \subset \overline{\mathbb{CP}^2}$. Check that the gluing data match between these two models. $\square$

Notice that by the adjunction formula, $c_1(K_M) \cdot E = -1$. So blowing down E eliminates a curve that has negative intersection with K_M, thus bringing K_M closer to being *nef*. But can we always keep blowing down until we get rid of all curves intersecting negatively with K_M? Miraculously, the following theorem (whose proof falls outside the scope of our exposition) says yes.

Theorem 4.4. *Let M be simply connected. Then blowing down a finite number of times gives a surface M', where either $K_{M'}$ is* nef*, M' is a $\mathbb{CP}^1$-bundle over $\mathbb{CP}^1$ (a 'ruled surface'), or $M' = \mathbb{CP}^2$ (M is a 'rational surface').*

The second and third cases are well understood, so now consider the *nef* case. Let K_M be *nef*. There are three sub-cases, which labelled by a so-called numerical dimension:

- $\mathrm{num}(M) = 0$: for every curve C, we have $K_M \cdot C = 0$,
- $\mathrm{num}(M) = 1$: there is a curve C with $K_M \cdot C > 0$, but $K_M \cdot K_M = 0$,
- $\mathrm{num}(M) = 2$: there is a curve C with $K_M \cdot C > 0$, and $K_M \cdot K_M > 0$.

Theorem 4.5. *The above three cases have the following additional structure.*

- *If $\mathrm{num}(M) = 0$, then M is a K3 surface (discussed below).*
- *If $\mathrm{num}(M) = 1$, then M admits a possibly singular fibration $M \to \mathbb{CP}^1$, with generic torus fiber.*
- *If $\mathrm{num}(M) = 2$, then M is called of general type. (This is sort of a wild west, an uncontrolled territory of 'everything else'.)*

4.2. **The case of K3 surfaces.**

Definition 4.1. A K3 surface is defined to be a closed simply-connected complex surface which has a non-vanishing holomorphic 2-form. Below, we show that this data is enough to calculate the intersection form. It is a much deeper theorem that all K3 surfaces are diffeomorphic (see, eg, [Bar+04] Cor. 8.6).

Since the canonical bundle for any complex surface is $K_M = \Lambda^{2,0} = \det_{\mathbb{C}} T_M^*$, requiring a non-vanishing holomorphic 2-form is the same as requiring that K_M has a non-vanishing section as a complex line bundle. That is, $c_1(K_M) = -c_1(TM) = 0$.

Example 2. Let M be a smooth quartic in $\mathbb{CP}^3$ (for example, the Fermat quartic $X^4 + Y^4 + Z^4 + W^4 = 0$). The Veronese embedding and the Lefschetz hyperplane theorem imply that any smooth degree d hypersurface $M^n \subset \mathbb{CP}^{n+1}$ is simply connected. So it suffices to show there is a non-vanishing holomorphic 2-form. Let U_i be the chart with $x_i = 1$, so on this chart our polynomial is $F = 1 + x_1^4 + x_2^4 + x_3^4$. Then consider the form

$$\theta = \frac{dx_j \wedge dx_k}{\partial F/\partial x_\ell}$$

for distinct $j, k, \ell \neq 0$. Notice that this is well-defined when $x_\ell \neq 0$. Since $(0, 0, 0)$ is not a solution to F, then there is always a choice of i, j, k for which θ is well-defined. A calculation shows that θ is invariant under permutations of the indices, and furthermore that it is compatible with a change of affine patches. The following proposition extends from the Fermat quartic to show that all smooth cubic surfaces are K3 surfaces.

Proposition 4.6. *Any two smooth degree d hyper-surfaces in $\mathbb{CP}^3$ are diffeomorphic.*

Proof sketch. The space of all degree d hypersurfaces in $\mathbb{CP}^n$ is just all homogeneous degree d polynomials in $n + 1$ variables, which is $\mathbb{P}(V_{d,n+1})$. Since $\dim_{\mathbb{C}} V_{d,n+1} = \binom{d+n}{n}$, and so $\dim_{\mathbb{C}} \mathbb{P}V_{d,n+1} = \binom{d+n}{n} - 1$. Notice that to be singular is for there to exist a point at which all partial derivatives vanish, which is an algebraically closed condition. Thus the space $X_{d,n}$ of *non-singular* degree d hyper-surfaces is a quasi-projective variety. Furthermore, it is smooth since as a manifold, one can vary coefficients in a little ball around any point without becoming singular. (Hence $S_{d,n}$ is locally diffeomorphic to $\mathbb{P}(V_{d,n+1})$.) Since $\mathbb{P}(V_{d,n+1})$ is connected, and the real codimension of locus of singular curves is at least 2, then $X_{d,n}$ is connected. Let $U_{d,n} = \{(S, p) : S \in X_{d,n}, p \in S\}$. Since the projection map is a proper submersion, then by Ehressman, it is a locally trivial fibration! Then by connectedness, all fibers must be diffeomorphic. □

Example 3. Consider now another kind of K3 surface: the Kummer surface K. Start with the torus T^4, and an action i, which can be seen as $(z, w) \mapsto (-z, -w)$ on $\mathbb{C}^2/\Lambda$. There are sixteen fixed points, and all the others descend to smooth points of the quotient. To obtain a complex surface, simply blow up at each of these 16 points before quotienting. A K3 surface requires there to be a non-vanishing holomorphic 2-form. There is a nonvanishing 2-form on the torus, given by $dz_1 \wedge dz_2$ on the universal cover $\mathbb{C}^2$. This form descends to a holomorphic non-vanishing form on $(\mathbb{C}^2/\Lambda)/\sim$, where this quotient is by the action of i. To extend this form over the blow-ups, it suffices to locally consider one blow-up, and the others follow similarly. Take the underlying singularity to be at the origin in $\mathbb{C}^2$. The ring of invariant polynomials is $\mathbb{C}[z_1, z_2]^{\{\pm 1\}} = \mathbb{C}[z_1^2, z_1 z_2, z_2^2] = \mathbb{C}[u, v, w]/(uw - v^2)$, where $(u, v, w) = (z_1^2, z_1 z_2, z_2^2)$. Therefore, taking spectra gives $\mathbb{C}^2/\{\pm 1\} = \text{Spec}(\mathbb{C}[u, v, w]/(uw - v^2))$. In other words, the equation $uw = v^2$ in $\mathbb{CP}^3$ gives a local model for our singularity. The a neighborhood of the blow-up can be covered by two charts, corresponding to charts on the $\mathbb{CP}^1$ at the center of the blow-up. On one chart, let $t = z_2/z_1$. Then $dz_1 \wedge dz_2 = \frac{1}{2} du \wedge dt$, which is non-vanishing on this affine chart. Similarly on the other chart.

Finally, we show the Kummer surface is simply connected. Consider a real picture, where our Kummer surface is the real 4-torus T^4 quotiented by $(x, y, z, w) \sim (-x, -y, -z, -w)$. Then $\pi_1(T^4) = \pi_1((T^4)^\circ) = \mathbb{Z}^4$, where $(T^4)^\circ$ denotes T^4 with the fixed points of the action removed. Then $(T^4)^\circ \to (T^4)^\circ/\sim$ is a double cover, and we have the sequence

$$1 \to \mathbb{Z}^4 \to \pi_1((T^4)^\circ/\sim) \to \mathbb{Z}/2\mathbb{Z} \to 1.$$

Notice that conjugation of $\mathbb{Z}/2\mathbb{Z}$ on $\mathbb{Z}^4$ induces the inversion action $x \mapsto x^{-1}$. Notice also that gluing back in the cone on $\mathbb{RP}^3$ kills the generator of the $\mathbb{Z}/2\mathbb{Z}$. Therefore, for all x lying in the image of $\mathbb{Z}^4 \to \pi_1((T^4)^\circ/\sim)$, $x = x^{-1}$. Since $\mathbb{Z}^4$ has no torsion, the result follows.

Proposition 4.7. *The intersection form of the K3 surface is* $E_8(-1)^{\oplus 2} \oplus U^{\oplus 3}$.

Proof sketch. Since the K3 surface can be given the structure of a projective variety (like the Fermat quartic), the pullback of Fubini-Study gives it the structure of a Kahler manifold, which allows unlocks the tools of Hodge theory - great! We claim that $h^{2,0} = 1$. By hypothesis, there is some some $0 \neq \theta \in H^{2,0}$ non-vanishing. Suppose there is also τ. Then $\tau/\theta : M \to \mathbb{C}$ is a holomorphic function, and hence bounded and constant by compactness of M. Since $\pi_1 = 0$, then $b_1 = 0$, and so $h^{1,0} = h^{0,1} = 0$. It only remains to find $h^{1,1}$. Define the holomorphic Euler characteristic ξ_{hol} to be the alternating sum of $\sum(-1)^i h^{i,0}$. Then in our case, $\xi_{hol}(M) = 2$. Recall Noether's formula (which comes from Rieman-Roch, but its proof falls outside our scope); which states that

$$12\xi_{hol}(M) = \langle c_1^2(M), [M]\rangle + \chi(M).$$

Since here $c_1(M) = 0$, then $\chi(M) = 24$, which means $b_2 = 22$. Then $h^{1,1} = 20$.

Now to calculate the signature form Q: For this, recall the Hodge-Riemann bilinear relations. They give that Q is positive definite on $H^{0,2} \oplus H^{2,0}$ and negative definite on $H^{1,1}_{prim} = [\omega]^\perp$ for ω the Kahler form. Further, Q is positive definite on $\langle[\omega]\rangle$ because $Q([\omega], [\omega]) = \int_M [\omega]^2 > 0$ by Wirtinger's theorem. So by dimension, the signature of Q is $(3, 19)$.

Wu's formula says that $w_2(TM) = S \cdot S \pmod 2$, which in particular states that Q is even if $w_2(M) = 0$. But since $c_1(M) = 0$, then of course its reduction mod 2 is also trivial, and so Q is even. The final ingredient is Milnors classification of lattices. Namely, if $L = (\mathbb{Z}^n, B)$ is integral non-degenerate and non-definite, then if L is odd it is diagonal with $\pm 1s$, and if even it is a sum of $E_8, E_8(-1), U$, where $U = (0, 1//1, 0)$. Then by combinatorics in the signature, the intersection form must be $Q = E_8(-1)^{\oplus 2} \oplus U^{\oplus 3}$. ∎

4.3. **Elliptic Surfaces.** K3 surfaces form part of a larger story of elliptically fibered complex surfaces, which we only touch on here. The most basic such surface, called the *rational elliptic surface* $E(1)$, is diffeomorphic to $\mathbb{CP}^2 \# 9\overline{\mathbb{CP}^2}$ and comes with a natural map $\pi : E(1) \to \mathbb{CP}^1$. (To see this map, consider a pencil of two generic cubics in $\mathbb{CP}^2$, and blow up at their nine intersection points.) The map π has a generic torus fiber with 12 singular nodal fibers. Pulling back along the map $\varphi_n : \mathbb{CP}^1 \to \mathbb{CP}^1$ via $z \mapsto z^n$ creates more elliptically fibered surfaces $E(n)$ with $12n$ singular fibers. All these surfaces are simply connected, and in fact $E(2)$ is a K3 surface.

5. NIELSEN REALIZATION FOR 4-MANIFOLDS

Now armed with some 4-manifold basics, we turn back to our guiding question: What do diffeomorphisms of 4-manifolds *look* like? We can try to take inspiration from the Nielsen-Thurston classification: for example, Is there some structure which is preserved in analogy with the reduction systems of reducible mapping classes? Recent results of Farb-Looijenga on elliptic surfaces show that in fact many mapping classes have representatives preserving the structure of an elliptic fibration, and more of their recent work specifically on K3s finds minimal-entropy representatives of certain mapping classes, in analogy with pseudo-Anosovs on surfaces. [FLb; FLa]

But here we focus on Nielsen realization: which finite groups of mapping classes lift to isomorphic groups of individual diffeomorphisms? Unlike the case of 2-manifolds, Nielsen realization sometimes holds and sometimes does not.

5.1. **Recent work: K3s and del Pezzos.** Farb-Looijenga [FL24] study the Nielsen realization probem for K3 surfaces in three versions: complex, metric, and smooth. To answer the metric problem, they use a Torelli theorem for K3 surfaces and analyse the Teichmuller space of Ricci-flat metrics. They show that a finite subgroup $G < \mathrm{Mod}(M)$ (for M a K3 surface) may be realized by isometries if and only if an associated lattice-invariant contains no (-2)-vectors, and they give a similar but more restrictive condition for complex realization. There results in the smooth case also have the following simple-to-state corollary.

Theorem 5.1. *[FL24, Cor 1.10] Let M be a K3 surface, and consider $[f] \in \mathrm{Mod}(M)$ the class of a Dehn twist f about a -2-sphere. Then $[f]$ has no finite order representative in $\mathrm{Diff}^+(M)$. In fact, f is not even topologically isotopic to a finite order diffeomorphism of M.*

In [Lee24; Lee23; LLR], the authors study del Pezzo surfaces and show that for cyclic groups in $\mathrm{Aut}(Q_{\text{del Pezzo}})$, the realization questions in the metric, complex, and smooth settings are often equivalent. They make heavy use of Coxeter theory to analyze the automorphisms of the associated intersection forms. They prove that odd prime order diffeomorphisms are realizable by complex automorphisms, for some choice of complex structure.

5.2. $n\mathbb{CP}^2$**: past results and ongoing research.** My current project investigates Nielsen realization for the positive-definite manifold $n\mathbb{CP}^2$, the n-fold connect sum of copies of $\mathbb{CP}^2$, all with the standard orientation coming from their complex structure. The intersection form of $n\mathbb{CP}^2$ is given by the identity matrix of rank n, and so its automorphisms are given by the finite group

$$\mathrm{Aut}(Q_{n\mathbb{CP}^2}) \simeq O(n, \mathbb{Z}) \simeq (\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n.$$

Hambleton-Tanase [HT04] prove using equivariant Yang-Mills moduli spaces that any smoothly realizable odd-order topological mapping class can be realized by an equivariant connect sum of standard linear actions on the individual copies of $\mathbb{CP}^2$ and S^4 which form the connect sum structure. In effect, their result also implies an algorithm that enumerates all smoothly realizable odd-order cyclic subgroups of $O(n, \mathbb{Z})$ over all n. Baraglia [Bar] also proves non-realizability for some class of subgroups, using standard theory about fixed-point sets and finite group actions. He also shows that the exact sequence

$$1 \to T(n\mathbb{CP}^2) \to \mathrm{Mod}(n\mathbb{CP}^2) \to O(n, \mathbb{Z}) \to 1$$

splits for all $n \geq 1$, where $T(n\mathbb{CP}^2)$ denotes the Torelli group, the subgroup of $\mathrm{Mod}(M)$ which acts trivially on $H_2(M; \mathbb{Z})$. However, the Torelli group itself remains very poorly understood, even in small cases. For example, we are unaware even whether it is trivial.

Our current research attempts to say more about Nielsen realization for $n\mathbb{CP}^2$. From now on, let $G \leq O(n,\mathbb{Z})$ denote the subgroup isomorphic to $(\mathbb{Z}/2\mathbb{Z})^n$ which is generated by reflections $\{\sigma_i : 1 \leq i \leq n\}$ across the n standard basis vectors. For any $g \in G$, let $\ell(g)$ denote the minimal word-length of g, written in the generators $\{\sigma_i\}$.

Theorem 5.2. *(1) The map* $\mathrm{Mod}(n\mathbb{CP}^2) \to \mathrm{Aut}(Q_{n\mathbb{CP}^2}) = O(n,\mathbb{Z})$ *is surjective for all* n.

(2) All cyclic subgroups of the standard $S_n \leq O(n,\mathbb{Z})$ *are realized by diffeomorphisms for* $n < 9$. *For* $1 \leq n \leq 5$, *the entire subgroup* $S_n \leq O(n,\mathbb{Z})$ *may be realized. For* $n \geq 8$, *it cannot be.*

(3) All groups $H < G$ *with* $H \simeq (\mathbb{Z}/2\mathbb{Z})^2$ *are smoothly realizable. (In particular, all elements of* G *are realized by smooth involutions.)*

(4) For $n \geq 4$, *the map* $\mathrm{Diff}^+(n\mathbb{CP}^2) \to O(n,\mathbb{Z})$ *has no section. In particular, the normal subgroup* $G \simeq (\mathbb{Z}/2\mathbb{Z})^n < O(n,\mathbb{Z})$ *does not lift.*

Proof sketch. Existance results follows from connect-sum constructions. To find such obstructions to realization, we use a variety of tools including Smith theory, uniformization for finite group actions on $\mathbb{RP}^2$, and the Hirzebruch G-signature theorem. □

Remark. Hambleton-Tanase's results [HT04] give explicit non-realizable subgroups, but they do not give a realizability criterion for all cyclic subgroups. As such, the minimal n for which cyclic realization fails remains open.

A complete classification of subgroup-realizability for $4\mathbb{CP}^2$ is mostly complete at time of writing. The case of $5\mathbb{CP}^2$ is also now mostly classified. We have also shown additional new restrictions on which subgroups may be realized for large n.

Proposition 5.3. *Let* $H \leq G$ *with* $H \simeq (\mathbb{Z}/2\mathbb{Z})^m$ *for some fixed rank* $1 \leq m \leq n$. *Let* x *be a smooth involution representing* $[x] \in H$, *and assume* H *is realizable as some group* $\tilde{H} <$ $\mathrm{Diff}^+(n\mathbb{CP}^2)$.

(1) The action of $\tilde{H}$ *is not free. Furthermore, the fixed set* $Fix(x)$ *contains at least one of either an isolated point,* $S^2, T^2, S_2, \mathbb{RP}^2, N_2, N_3$, *or* N_4, *where* N_g *denotes the non-orientable surface of genus* g.

(2) If the fixed set $Fix(x)$ *contains* k *isolated points, then* $m < 4 + v_2(k)$, *where* v_2 *denotes* 2*-adic valuation. If the fixed set* $Fix(x)$ *contains* k *components all diffeomorphic to the same one of* S^2, T^2, S_2, *or* N_g *for some* $g \leq 4$, *then* $m < c + v_2(k)$, *where* c *is the largest integer such that* $(\mathbb{Z}/2\mathbb{Z})^{c-1}$ *acts faithfully on the given surface type. (Note that* c *is some finite constant depending on the surface type. In particular,* $c < 10$ *in all cases.)*

Proof sketch. (1) Use the Hirzebruch G-signature theorem and Proposition 2.4 of [Edm89] to rule out free actions. For the second statement, note that if it were false, then again by [Edm89], all elements $x \in H$ would have $\ell(x) > 2m/3$, which is impossible.

(2) Let $\tilde{H}'$ denote some $(\mathbb{Z}/2\mathbb{Z})^{m-1} < \tilde{H}$ not containing x. Then $\tilde{H}'$ acts on the set of k copies of the specified type of fixed component. Bound the size of the largest stabilizer of this action from below.

□

Taking simple bounds and special cases of Proposition 5.3 gives the following theorem.

Theorem 5.4. *Let $H < G$ with $H \simeq (\mathbb{Z}/2\mathbb{Z})^m$, and suppose H is realized in* $\mathrm{Diff}^+(n\mathbb{CP}^2)$.

(1) We have $m < 10 + \log_2(n)$.
(2) If there is some $x \in H$ with $n - \ell(x) \equiv 1 \pmod 2$, then $m \leq 3$.
(3) If n is odd, then $m \leq 3$.

Notice that by combining Theorem 5.2 with Theorem 5.4, we have for n odd that the only unclassified realization problem inside G is the rank-3 case. We conjecture that all realizable mapping classes are realizable by equivariant connect sums. We also wonder whether the constant-bound holds for even n.

In the absence of absolute bounds or other more rigid structural criteria, we ask asymptotic questions. For example, using the stricter criterion of Proposition 5.3, does the proportion of realizable $(\mathbb{Z}/2\mathbb{Z})^m < G$ goes to 0 as m and n grow? Some numerical calculation suggests so.

Building on the work of [HT04], we wonder what proportion of odd-order elements of $S_n < O(n, \mathbb{Z})$ are realizable as n grows. In particular, as $n \to \infty$, does the proportion of realizable odd-order elements of S_n go to 0? If so, this may also imply that asymptotically, the proportion of realizable finite subgroups of S_n among all those containing an odd-order element also goes to 0 (that is, among all non 2-groups). We explore graph theoretic and partition theoretic approaches.

To analyze 2-groups in S_n, we use the following lemma.

Lemma 5.5. *Let n be odd, and suppose $H < S_n < O(n, \mathbb{Z})$ is a realizable 2-group with $|H| = 2^m$. Then H embeds into $SO(4)$, and in particular H has one of finitely many classified isomorphism types.*

Proof sketch. Since H is a 2-group, $Z(H)$ is nontrivial. In particular, it contains a nontrivial involution x. By [Edm89], any diffeomorphism realizing x must have an odd number of isolated fixed points. By a similar orbit-stabilizer argument to the proof of Proposition 5.3, one of these isolated fixed points is fixed by all of $C_H(x) = H$. Then $H \hookrightarrow SO(4)$. The isomorphism types of 2-groups in $SO(4)$ are completely classified as a finite list for fixed m. □

As n grows, we expect that the proportion of all 2-groups of S_n belonging to one of the above isomorphism types tends to 0. When n is even, we achieve similar but weaker restrictions.

Going forward, we aim to continue analyzing the small-n cases, develop asymptotic bounds, and look for larger, more structural constraints. For example, we wonder if other tools like a suitably compactified Teichmüller space or Seiberg-Witten theory can provide more general restrictions.

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