KNOT THEORY AND QUANTUM INVARIANTS

ETHAN PESIKOFF

In these notes, we explore the connections between knot theory and quantum groups. Specifically, we develop a representation theory of the quantum group $U_q(\mathfrak{sl}(2))$ and employ it to develop the Jones-Conway Polynomial, but we also prove more general results and point towards broader theory. Section 1 introduces knot theory and derives the Jones polynomial in an elementary way. Sections 2 and 3 develop the relevant representation theory. Section 4 introduces R-matrices and the Yang Baxter equation, with a specific example drawn from the objects of section 3. Section 5 returns to topology and sets up a derivation of the Jones Conway polynomial, which is finally proven in section 7 after some background on category theory and tangles in section 6. Finally, section 8 suggests where one might go from here, seeing as how we develop and broader and more general theory in these notes than is used to simply derive the Jones Conway polynomial.

CONTENTS

1. Introduction to knots

1.1. Basic Definitions and Examples.

Definition 1.1. *A link L of m components is a subset of S^3 or $\mathbb{R}^3$ which consists of m disjoint, piecewise linear, simple closed curves. A link of one component is a knot.*

Remark. Rather than requiring piecewise linearity, we could instead mandate that L is a submanifold of the ambient space. Either definition avoids poorly behaved curves, but these more subtle topological considerations are not our focus.

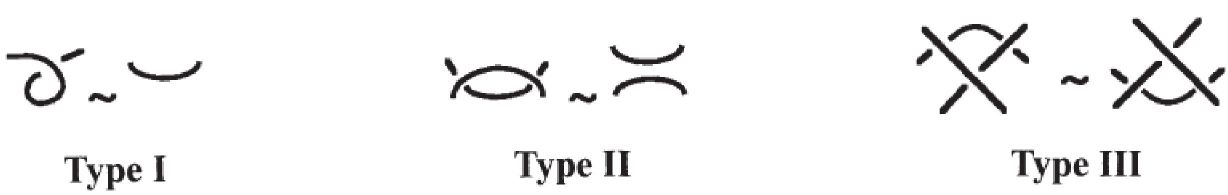
Definition 1.2. *Links L_1, L_2 are equivalent if there is an orientation-preserving piecewise-linear homeomorphism $h : \mathbb{R}^3 \to \mathbb{R}^3$ such that $h(L_1) = L_2$.*

Example 1. The following knots are equivalent, and their class is call the unknot, often denoted O.

FIGURE 1. Two equivalent knots.

From topology, we have that h is *isotopic* to the identity. That is, there is a map $H : \mathbb{R}^3 \times [0, 1] \to \mathbb{R}^3 \times [0, 1]$ st $H(x, 0) = (h(x), 0)$ and $H(x, 1) = (x, 1)$. We think informally of equivalent links as being the same. Often our future definitions will assume that when we say 'link' or 'knot', we are discussing up to equivalence.

Theorem 1.3. *Two links are equivalent iff they are related by a sequence of Reidemeister moves, which are defined as*



where the $\sim$ means equivalence of links. The figures above represent local differences, with the remainder of the link assumed to be the same (we continue this convention going forward).

We have followed standard practice defining the Reidemeister moves 'by picture', but in principal they can be defined much more concretely. We might do this by assigning explicit canonical forms of each side of each equivalence. Then a Reidemeister moves consists of locally replacing one concrete shape by another.

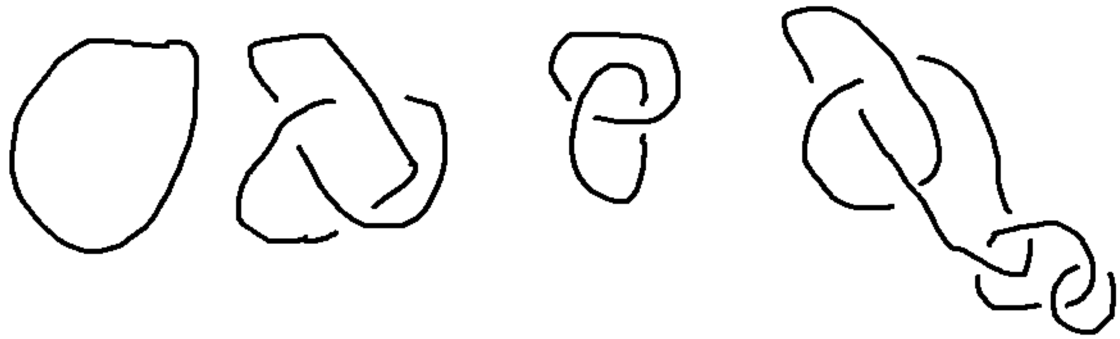


FIGURE 2. Some link diagrams. The first of the four above knots (from the left) is the unknot, and the second is a trefoil.

Classifying knots is very difficult, and even the classification of relatively simple knots is not complete. As such, we want some invariants that will help us distinguish between knots that aren't equivalent. By an invariant we mean some object assigned to a link which is the same for all links of its equivalence class.

The first attempt at an invariant comes from the *knot diagram*.

Definition 1.4. *A diagram of a link is a projection onto $\mathbb{R}^2$ such that there are no triple-intersections or tangencies, with orientations of crossings marked.*

In particular, all drawings of knots in these notes and elsewhere are more properly knot diagrams. It is also the case that every link has multiple diagrams (infinitely many in fact). For example, one may add to any valid diagram a 'twist' as in Figure 1.

Clearly, the diagram of a knot is not unique. Even the unknot (loop which is not knotted) has infinitely many distinct diagrams. However, since the number of crossings in a knot diagram is a non-negative integer, we can work with the following definition.

Definition 1.5. *The crossing number of a knot is the minimum number of crossings in any of its diagrams.*

Example 2. The crossing number of O the unknot is 0. In fact, O is the only know with crossing number 0.

This is certainly an invariant, but it has two main flaws: it is difficult to calculate and does not distinguish very well between knots. For example, there are already 21 knots with crossing number 8 and 253293 with crossing number 15.

There are lots of other knot-invariants. Some are combinatorial, using properties of the diagrams, and others are topological. For example, those of you who have seen some algebraic topology might appreciate that the fundamental group of the complement of a knot (called its *knot group*) may also be an interesting object of study. Also of interest is the *Seifert surface* of a knot K, a connected, compact oriented surface with K as its boundary. Interestingly, such a surface always exists, and we call the genus of that Seifert surface the genus of K, $g(K)$.

1.2. **The Jones Polynomial: A Combinatorial Approach.** The Jones Polynomial is an important and relatively simple knot invariant. In this section, we derive it combinatorially and give some examples. In a future section, we'll show how to derive something similar (and slightly more general) using the machinery of quantum groups.

Our first building block on the way to the Jones polynomial is the *Kauffman Bracket*:

Definition 1.6. *The Kauffman bracket is a function from unoriented link diagrams D in the oriented plane to Laurent polynomials with integer coefficients in variable A. It is defined by the following rules.*

(i) $\langle \bigcirc \rangle = 1$,

(ii) $\langle D \sqcup \bigcirc \rangle = (-A^{-2} - A^{2})\langle D \rangle$,

(iii) $\langle$  $\rangle = A\langle$  $\rangle + A^{-1}\langle$  $\rangle$.

We see that rule (iii) above allows us to reduce any link diagram to a collection of disjoint unknots, from which we can calculate $\langle D \rangle$ via (i) and (ii). Therefore, the three above rules are sufficient to define the Kauffman bracket on all links. It is left as an exercises that the three rules are consistent (and don't lead to multiple values being assigned to the same link). We now prove a couple of properties of the Kauffman Bracket.

First, notice that if we replace D by its reflection (switch the orientation of every crossing), then the effect on the Kauffman bracket is to interchange A and A^{-1}.

Lemma 1.7. *A type I Reidemeister move changes the Kauffman bracket in the following way:*
$\langle$  $\rangle = -A^{3}\langle$ $\rangle$, $\quad\langle$  $\rangle = -A^{-3}\langle$ $\rangle$.

Proof. .

$\langle$  $\rangle = A\langle$  $\rangle + A^{-1}\langle$  $\rangle$

$= (A(-A^{-2} - A^{2}) + A^{-1})\langle$  $\rangle$.

The other case follows similarly. ☐

Example 3. We calculate the bracket of the trefoil knot.

$\langle$  $\rangle = A\langle$  $\rangle + A^{-1}\langle$  $\rangle$

$= (-A^{4} - A^{-4})$.

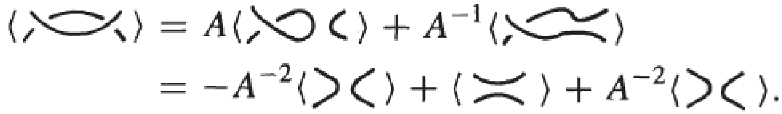
$\langle$  $\rangle = A\langle$  $\rangle + A^{-1}\langle$  $\rangle$

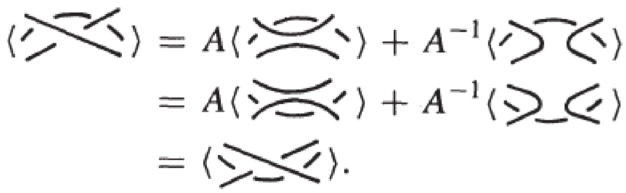
$= A(-A^{4} - A^{-4}) + A^{-7}$

$= (A^{-7} - A^{-3} - A^{5})$.

Lemma 1.8. *The Kauffman bracket is invariant under Reidemeister type II and III.*

Proof. .





□

Definition 1.9. *The writhe $w(D)$ of a diagram of an oriented link is the sum of the signs of the crossings, as defined by*

.

The writhe of a diagram is unchanged by type II and III Reidemeister moves, and changes predictably under a type I. This bring us to the following theorem.

Theorem 1.10. *We have for D a diagram of a link L that*

$$(-A)^{-3w(D)}\langle D\rangle$$

is an invariant of L.

Proof. By Lemma 1.8, the given expression is invariant under moves of types II, III. It is invariant under type I by Lemma 1.7 because the two factors $(-A)^{-3w(D)}$ and $\langle D\rangle$ change by inverse factors. Then since any two diagrams of two equivalent links are related by moves of type I, II, III, then the result follows. □

Finally, we arrive at the definition of the Jones Polynomial, which is an invariant of links and is more or less a re-dressing of the above invariant of Theorem 1.10.

Definition 1.11. *The Jones polynomial is given by*

$$V(L) = \left((-A)^{-3w(D)}\langle D\rangle\right)_{t^{1/2}=A^{-2}} \in \mathbb{Z}[t^{-1/2}, t^{1/2}],$$

where the subcript means we substitute $t^{1/2}$ for A^{-2}.

The Jones polynomials gives us a good tool to distinguish knots. But even though we have the Jones polynomial, there remains much about it to be understood. For example, we know that $V(O) = 1$ for O the unknot, but it is unknown whether there exists any other knot K with $V(K) = 1$. However, rather than pursuing this further we move onto another line of analysis.

2. Representations of $\mathfrak{sl}(2)$

2.1. **Lie Algebras.** Eventually, we want to look at the representations of the so-called quantum analogue of $\mathfrak{sl}(2)$, but to start, we review the representations of the classical $\mathfrak{sl}(2)$ itself. We work over an algebraically closed field k unless otherwise specified. For all practical purposes, feel free to think of $\mathbb{C}$. The proofs in the quantum case will be similar but more complex, and hence an understanding of this classical case proves pedagogically useful. Classically (as in John's lectures, for example), we analyze the representations of $\mathfrak{sl}(2)$ directly as maps from $\mathfrak{sl}(2)$ into the general linear group. Here we present an alternative approach in which we instead analyse the equivalent representations of an associative algebra built from $\mathfrak{sl}(2)$, called $U(\mathfrak{sl}(2))$. We recall the definitions of a Lie algebra and an enveloping algebra.

Definition 2.1. *(1) A Lie algebra L is a vector space with bilinear map $[\cdot,\cdot] : L \times L \to L$ such that for all $x, y \in L$ we have*

$$[x, y] = -[y, x], \quad [x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$$

(2) A sub-Lie algebra $L' \subseteq L$ is is a subspace closed under $[\cdot,\cdot]$.
(3) A morphism of Lie algebras is a linear map of vector spaces $f : L \to L'$ such that $f([x, y]) = [f(x), f(y)]$ for all $x, y \in L$.
(4) We say a Lie algebra is commutative if $[\cdot,\cdot]$ is the zero-map.

Example 4. Any field (viewed as a vector space over itself) is a commutative Lie algebra with the commutator as its bracket.

If A is any associative algebra, it is quick to show that A can also be made into a Lie algebra by letting the lie bracket be the commutator operator. That is, we let $[x, y] = xy - yx$. We call this Lie algebra $L(A)$.

Definition 2.2. *For any vector space V, the tensor algebra $T(V)$ is the algebra given by*

$$T(V) = \bigoplus_{n=1}^{\infty} V^{\otimes n}$$

with product given by

$$(v_1 \otimes \cdots \otimes v_n)(w_1 \otimes \cdots \otimes w_m) = v_1 \otimes \cdots \otimes v_n \otimes w_1 \otimes \cdots \otimes w_m,$$

extended linearly.

$T(V)$ is clearly associative. Let $I(L)$ be the ideal of $T(L)$ generated by all elements of the form $xy - yx - [x, y]$ for $x, y \in L$. We now give a universal property of U and i_L. To any Lie algebra, we assign an enveloping algebra $U(L)$ and a morphism $i_L : L \to L(U(L))$.

Definition 2.3. *Let $U(L) = T(L)/I(L)$. We define $i_L : L \to L(U(L))$ as the composition of inclusion of L into $T(L)$ followed by the quotient from $T(L)$ to $U(L)$, with the commutator Lie bracket applied to $U(L)$.*

Effectively, we have constructed an associative algebra whose product has commutator which is our original Lie bracket on L.

Theorem 2.4. *Let L be a Lie algebra and A an associative algebra and $f : L \to L(A)$ some morphism of Lie algebras. Then there exists a unique morphism of algebras $\varphi : U(L) \to A$ such that $\phi \circ i_L = f$.*

Proof. By definition of the tensor algebra, f extends to a morphism of algebras $\overline{f}$ from $T(L)$ to A defined by $\overline{f}(x_1 \dots x_n) = f(x_1) \dots f(x_n)$ for $x_i \in L$. Then the existence of ϕ follows since $\overline{f}(I(L)) = \{0\}$. In order to show this, it suffices to show that $\overline{f}(xy - yx - [x, y]) = 0$, which follows because

$$\overline{f}(xy - yx - [x, y]) = f(x)f(y) - f(y)f(x) - f([x, y]),$$

which is 0 since f is a morphism of Lie algebras.

Since L generates $T(L)$ as an algebra, it generates $U(L)$. Hence follows uniqueness. $\square$

The key observation is now that by definition, an L-module is a $U(L)$-module, which is the same as a morphism of algebras $\rho : U(L) \to \hom(V, V)$. Then by Theorem 2.4, it is equivalent to a morphism of Lie algebras $\rho : L \to \mathfrak{gl}(V)$. Then instead of looking for representations of L directly, we study representations of $U(L)$.

2.2. **The Lie Algebra $\mathfrak{sl}(2)$ and its Simple Representations.** We recall that $\mathfrak{gl}(2) = L(K_2(k))$ of 2x2 matrices is four-dimensional, with basis

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and that $\mathfrak{sl}(2)$ is the subspace of matrices of trace 0. We calculate that $[X, Y] = H, [H, X] = 2X, [H, Y] = -2Y$, with all other pairings equal to 0. Then

$$\mathfrak{gl}(2) = \mathfrak{sl}(2) \oplus kI,$$

which reduced investigation of $\mathfrak{gl}(2)$ to $\mathfrak{sl}(2)$.

Observe that $U := U(\mathfrak{sl}(2))$ is isomorphic to the algebra generated by X, Y, H with the above relations $[X, Y] = H, [H, X] = 2X, [H, Y] = -2Y$. By the Poincare-Birkhoff-Witt Theorem (which we do not prove here), $\{X^i Y^j H^k\}_{i,j,k\geq 0}$ forms a basis for U.

We now seek to study representations of U. That is, we want all finite-dimensional U-modules. We begin our search with a definition.

Definition 2.5. *Let V be a U-module and λ a scalar. A vector $v \neq 0$ in V is said to be of highest weight $\lambda \in k$ if $Hv = \lambda v$. If, in addition, we have $Xv = 0$, then we say v is a highest weight vector of weight λ.*

Proposition 2.6. *Any non-zero finite-dimensional U-module V has a highest weight vector.*

Proof. Since k is algebraically closed and V is finite-dimensional, the operator H has eigenvector $w \neq 0$ with eigenvalue α. If $Xw = 0$, then w is a highest weight vector, and we are done. Else, consider the sequence $X^n w$. By a straightforward matrix computation, we have

$$H(X^n w) = (\alpha + 2n)(X^n w).$$

Then $(X^n w)_{n\geq 0}$ is a set of eigenvectors of H with distinct eigenvalues. Since V is finite-dimensional, there can only be finitely many such eigenvectors, and so there must be an n such that $X^{n+1}w = 0$. Then $X^n w$ is a highest weight vector. $\square$

The proof of the following lemma is another unenlightening but tedious computation.

Lemma 2.7. *Let v be a highest weight vector of weight λ. For $p \in \mathbb{N}$, set $v_p = \frac{1}{p!} Y^p v$. Then*

$$Hv_p = (\lambda - 2p)v_p, Xv_p = (\lambda - p + 1)v_{p-1}, Yv_p = (p + 1)v_{p+1}.$$

Finally, we state and prove the main classification theorems for simple U-modules.

Theorem 2.8. *(1) Let V be a finite-dimensional U-module generated by a highest weight vector v of weight λ.*

(a) We have $\lambda = \dim(V) - 1$.

(b) We have $v_p = 0$ for $p > \lambda$, and $\{v = v_0, v_1, \ldots, v_\lambda\}$ is a basis for V.

(c) The operator H acting on V is diagonalizable with $\lambda + 1$ distinct eigenvalues $\{\lambda, \lambda - 2 \ldots, -\lambda\}$.

(d) Any other highest weight vector in V is a scalar multiple of v and is of weight λ.

(e) The module V is simple.

(2) Any simple finite-dimensional U-module is generated by a highest weight vector. Two finite-dimensional U-modules generated by highest weight vectors of the same weight are isomorphic.

Proof. (1) By Lemma 2.7, $\{v_p\}_{p \geq 0}$ is a sequence of eigenvectors for H with distinct eigenvalues. Then as V is finite-dimensional, there is some n such that $v_n \neq 0 = v_{n+1}$. Then $v_m = 0$ for all $m > n$, and $v_m \neq 0$ for all $m \leq n$. We get $\lambda = n$ since $0 = Xv_{n+1} = (\lambda - n)v_n$. Since $\{v, v_1, \ldots, v_n\}$ have distinct eigenvalues, they are linearly independent. They also generate V as Lemma 2.7 shows that any element of V generated by v as a module is a linear combination in $\{v_i\}$. Then $\dim(V) = \lambda + 1$. Then we have shown (a) and (b), and (c) follows from Lemma 2.7.

To prove (d), let v' be another highest weight vector. It is then an eigenvector of H and thus a scalar multiple of some v_i. By Lemma 2.7, $Xv_i = 0$ iff $i = 0$, and hence $v' = cv$ for $c \in k$.

For (e), let V' be a non-zero U-submodule of V and let v' be a highest weight vector of V'. Then v' is also a highest weight vector for V. By (d), $v' = cv$, and so $v \in V'$. Then since v generates $V \supseteq V'$, we have $V = V'$. Then V is simple.

(2) Let v be a highest weight vector of V. If V is simple, then the submodule generated by v is equal to V. Then V is generated by a highest weight vector. If V, V' are generated by highest weight vectors v, v' of shared weight λ, then the linear map sending $v_i \to v'_i$ for all i is a module isomorphism.

$\square$

We call $V(n)$ the unique simple module of weight $n+1$. We denote the corresponding morphism of Lie algebras by $\rho(n) : \mathfrak{sl}(2) \to \mathfrak{gl}(n+1)$. We have in the basis $\{v_0, \ldots, v_n\}$ that

$$\rho(n)(X) = \begin{pmatrix} 0 & n & 0 & \cdots & 0 \\ 0 & 0 & n-1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

$$\rho(n)(Y) = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 2 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & n & 0 \end{pmatrix},$$

$$\rho(n)(H) = \begin{pmatrix} n & 0 & \cdots & 0 & 0 \\ 0 & n-2 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -n+2 & 0 \\ 0 & 0 & \cdots & 0 & -n \end{pmatrix}.$$

2.3. **Semisimplicity and Clebsch-Gordan.** Now armed with a classification of simple U-modules, we want to classify all U-modules as direct sums of simple modules. Semisimplicity of U refers to this result that any finite-dimensional U-module can be decomposed as a direct sum of simple modules.

We define the Casimir element to be $C = XY + YX + H^2/2$.

Lemma 2.9. *We have C lies in the center of U.*

Proof. It suffices to show Lie brackets with H, X, Y vanish. We have

$$\begin{aligned} [H, C] &= [H, X]Y + X[H, Y] + [H, Y]X + Y[H, X] + [H, H^2]/2 \\ &= 2XY - 2XY - 2YX + 2YX = 0. \\ [X, C] &= X[X, Y]Y + [X, Y]X + [X, H]H/2 + H[X, H]/2 \\ &= XH + HX - XH - HX = 0. \end{aligned}$$

The remaining case follows similarly. □

Lemma 2.10. *Any central element in U acts by a scalar on $V(n)$. The Casimir element acts by $(n^2 + 2n)/2$ for $n \neq 0$.*

Proof. Let Z be central in U. It commutes with H which decomposes $V(n)$ into a direct sum of one-dimensional eigenspaces. Then Z as an operator is diagonal with the same eigenvectors $\{v = v_0, v_1 \dots, v_n\}$ as H and corresponding eigenvalues α_i. Now

$$\alpha_{p+1} Y v_p = \alpha_{p+1}(p+1)v_{p+1} = (p+1)Z v_p = ZY v_p = YZ v_p = \alpha_p Y v_p.$$

So all α_i are equal.

Then in order to compute the scalar action of C, we only need to find Cv for the highest weight vector v. By Lemma 2.7 and centrality of C we have

$$Cv = XYv + YXv + H^2v/2 = nv + n^2v/2 = (n^2 + 2n)v/2.$$

□

Recall now that a U-module is semi-simple if it can be decomposed as the direct sum of simple U-modules. We now prove that all finite-dimensional U-modules are semi-simple by inductively breaking them down into smaller directly summed parts.

Theorem 2.11. *Any finite-dimensional U-module is semi-simple.*

Proof. It suffices to show that for any finite-dimensional U-module V and submodule $V' \subseteq V$, there exists V'' such that $V \simeq V' \oplus V''$. Let $L = \mathfrak{sl}(2)$.

First, suppose V' has codimension 1, and we induct on dimension. If $\dim(V') = 0$, then we take $V'' = V$. If $\dim(V') = 1$, then $V', V/V'$ are trivial one-dimensional representations. Then there is a basis $\{v_1 \in V', v_2\}$ of V such that $Lv_1 = 0$ and $Lv_2 \in V' = kv_1$. Then we have $[L, L]v_1 = [L, L]v_2 = 0$. Furthermore, we have that the action of L on V is trivial. Then we let V'' be any supplementary space to V' in V.

Now assume that the result holds for all V' of dimensions less than p for $p > 1$. First, suppose V' is not simple. Then let V_1 be a proper submodule of V, and $\pi : V \to V/V_1 = \overline{V}$ the canonical projection. Then $\pi(V') = \overline{V'}$ is a submodule of $\overline{V}$ of codimension 1 and dimension $< p$. By induction, $\overline{V} = \overline{V'} \oplus \overline{V''}$, which then implies that

$$V = V' \oplus \pi^{-1}(\overline{V''}).$$

Since $\dim(\overline{V''}) = 1$, then V_1 is a submodule of codimension 1 of $\pi^{-1}(\overline{V''})$. Therefore again by induction, we have for some V'' that $\pi^{-1}(\overline{V''}) = V_1 \oplus V''$. Then $V = V' + V_1 + V''$. As V_1 is contained in V', then $V = V' + V''$. As $\dim(V) = \dim(V') + \dim(V'')$, then $V = V' \oplus V''$.

Now suppose that V' is simple. Then C acts on V' be a non-zero scalar α by Lemma 2.10, and so C/α is the identity operator. Now V/V' is 1-dimensional and hence a trivial module. Then C sends V into V', so C/α is the projection from V to V', and it is a morphism of U-modules since it is in the center. Then $V'' = \ker\,(C/\alpha)$ is a supplementary module to V', and $V = V' \oplus V''$.

We now allow V' any codimension, and we reduce to the case of codimension 1. Let W (resp. W') be the subspace of all linear maps from V to V' whose restriction to V' is a homothety (resp. 0). We then see that W' is codimension 1 in W.

We endow $\hom(V, V')$ with a U-module structure. For $x \in L, v \in V, v' \in V'$, we let

$$(xf)(v) = xf(v) - f(xv).$$

We now check that W, W' are submodules of $\hom(V, V')$. For $f \in W$, let $\alpha \in k$ be the scalar such that $f(v) = \alpha v$ for all $v \in V'$. Then for all $x \in L$ we have

$$(xf)(v) = xf(v) - f(xv) = x(\alpha v) - \alpha(xv) = 0.$$

Similarly, W' is a submodule. Then $W = W' \oplus W''$ for some W''. Let f be a generator of W''. Then by definition, it acts on V' as a scalar $\alpha \neq 0$. Then f/α is a projector from V to V'. Since W'' is one dimensional, it is trivial. Then $xf = 0$ for all $x \in L$, and so $xf(v) - f(xv) = 0$ for all $v \in V$. Then f/α is a morphism of modules, and hence the result follows. □

Our final goal in this section is to investigate the tensor products of simple U-modules. There is a natural module structure on the tensor product $V \otimes V'$ for U-modules V, V', given by $x(v \otimes v') = xv \otimes v' + v \otimes xv'$ for $x \in U$. Thus $V \otimes V'$ is semi-simple by Theorem 2.11. By distributitivity of the tensor product over direct sums and by our classification of U-modules, it suffices to decompose $V(n) \otimes V(m)$ into simple modules. The decomposition, given in the following theorem, is called the Clebsch-Gordan formula.

Theorem 2.12. *For $n \geq m$, we have*

$$V(n) \otimes V(m) \simeq V(n+m) \oplus V(n+m-2) \oplus \cdots \oplus V(n-m).$$

Proof. It suffices to show that for all $0 \leq p \leq m$, $V(n) \otimes V(m)$ contains a highest weight vector of weight $n+m-2p$. If so, there is a non-zero isomorphism of modules from $V(n+m-2p)$ into $V(n) \otimes V(m)$, and this map is injective as the kernel must be a module and is hence 0 by simplicity. Since each of these modules has distinct highest weight their sum as submodules of $V(n) \otimes V(m)$ is direct. Thus, the RHS of the Clebsch-Gordan formula embeds into the left. Then to conclude, we show both sides have the same dimension. The dimension of the LHS is

$$\begin{aligned}\sum_{p=0}^{m}(n+m-2p+1) &= (n+1)(m+1) \\ &= \dim(V(n))\dim(V(m)) \\ &= \dim(V(n) \otimes V(m)).\end{aligned}$$

Then the theorem follows as a result of the following lemma, which shows the existence of such highest weight vectors. ☐

Lemma 2.13. *Let v be a highest weight vector of $V(n)$ and v' of $V(m)$. Define $v_p = \frac{1}{p!}Y^p v$ and $v'_p = \frac{1}{p!}Y^p v'$ for $p \geq 0$. Then*

$$\sum_{i=0}^{p}(-1)^i \frac{(m-p+i)!(n-i)!}{(m-p)!n!} v_i \otimes v'_{p-i}$$

is a highest weight vector of $V(n) \otimes V(m)$ of weight $n+m-2p$.

Proof. The proof is essentially a calculation. Set

$$\alpha_i = (-1)^i \frac{(m-p+i)!(n-i)!}{(m-p)!n!}$$

and

$$w = \sum_{i=0}^{p} \alpha_i v_i \otimes v'_{p-i}.$$

It suffices to check that $Xw = 0$ and $Hw = (n+m-2p)w$. The latter holds because the tensors $v_i \otimes v'_{p-i}$ are all of weight $n+m-2p$. By Lemma 2.7, we have

$$H(v_i \otimes v'_{p-i}) = H(v_i) \otimes v'_{p-i} + v_i \otimes H(v'_{p-i}) = (n+m-2p)v_i \otimes v'_{p-i}.$$

We also have by Lemma 2.7 that

$$\begin{aligned} Xw &= \sum_{i=0}^{p} \alpha_i X(v_i) \otimes v'_{p-i} + \sum_{i=0}^{p} \alpha_i v_i \otimes X(v'_{p-i}) \\ &= \sum_{i=0}^{p} \alpha_i (n-i+1) v_{i-1} \otimes v'_{p-i} + \sum_{i=0}^{p} \alpha_i (m-p+i+1) v_i \otimes v'_{p-i-1} \\ &= \sum_{i=1}^{p} \left(\alpha_i (n-i+1) + \alpha_{i-1}(m-i+1) \right) v_i \otimes v'_{p-i}. \end{aligned}$$

We now compute

$$\begin{aligned} &\alpha_i(n-i+1) + \alpha_{i-1}(m-i+1) \\ &= (-1)^i \frac{(m-p+i)!(n-i)!}{(m-p)!n!}(n-i+1) + (-1)^{i-1}\frac{(m-p+i-1)!(n-i+1)!}{(m-p)!n!}(n-p+i) \\ &= 0. \end{aligned}$$

□

3. $U_q(\mathfrak{sl}_2)$ and its Representations

3.1. **Simple Representations of U_q.** Our main algebraic object of study in these lectures is $U_q = U_q(\mathfrak{sl}_2)$. We assume going forward that q is not a root of unity.

For fixed q, we let

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}} = q^{n-1} + q^{n-3} + \cdots + q^{-n+1}.$$

We then have $[-n] = -[n]$ and $[m+n] = q^n[m] + q^{-m}[n]$. We define additional functions in the natural way, like $[n]! = [n][n-1]\ldots[1]$.

Definition 3.1. *We define U_q as the algebra generated by E, F, K, K^{-1} with*

$$\begin{aligned} &KK^{-1} = K^{-1}K = 1 \\ &KEK^{-1} = q^2 E, KFK^{-1} = q^{-2}F \\ &[E, F] = \frac{K - K^{-1}}{q - q^{-1}} \end{aligned}$$

The above definition is opaque, but it turns out to be natural. There is another equivalent presentation of U_q (not discussed here) such that setting $q = 1$ recovers our classical U. We thus say that U_q is the *q-deformation* or *quantum analogue* of U. Interestingly, there is no natural q-deformation of the Lie algebra $\mathfrak{sl}(2)$ itself, which is why we study enveloping algebras instead.

Going forward, let V be a finite-dimensional U_q-module. We let V^λ be the subspace generated by all eigenvectors of eigenvalue $\lambda \in k$. If this space is non-empty, then we say λ is a weight of V.

Lemma 3.2. *We have $EV^\lambda \subset V^{q^2\lambda}$ and $FV^\lambda \subset V^{q^{-2}\lambda}$.*

Proof. The proof is a straightforward computation. For $v \in V^\lambda$, we have

$$K(Ev) = q^2 E(Kv) = q^2 \lambda Ev \quad K(Fv) = q^{-2}F(Kv) = q^{-2}\lambda Fv.$$

□

Totally analogously to the classical case, we define highest weight vectors in V.

Definition 3.3. *An element $0 \neq v \in V$ is a highest weight vector of weight λ if $Ev = 0$ and $Kv = \lambda v$. Further, V is a highest weight module of weight λ if it is generated by such a highest weight vector.*

Lemma 3.4. *We have V contains a highest weight vector. Moreover, the endomorphisms induced by E and F on V are nilpotent.*

Proof. We show for E, and the same argument holds for F. Since k is algebraically closed and V finite-dimensional, K has an eigenvector w of weight α. If $Ew = 0$, we are done. Else consider the sequence $E^n w$ for non-negative integers n. By Lemma 3.2, these are eigenvectors with distinct eigenvalues, so they must eventually stabilize at 0 by finite-dimensionality of V. The last E^n which is non-zero is thus a highest weight vector.

To show E is nilpotent, we show that E cannot have non-zero eigenvalues. If E has eigenvalue $\lambda \neq 0$, then $K^n v$ is an eigenvector with eigenvalue $q^{-2n}\lambda$, thus giving E infinitely many eigenvalues. This is a contradiction. □

Before we classify simple V, we give one last computational result, also analogous to the classical case, where the proof is a computation following from Lemma 3.2.

Lemma 3.5. *Let $v \in V$ be a highest weight vector of weight λ. Set $v_0 = v$ and $v_p = \frac{1}{[p]!}F^p v$ for $p > 0$. Then*

$$Kv_p = \lambda q^{-2p} v_p, Ev_p = \frac{q^{-(p-1)}\lambda - q^{p-1}\lambda{-1}}{q - q^{-1}} v_{p-1}, Fv_{p-1} = [p]v_p.$$

Now, we classify all simple V.

Theorem 3.6. *(1) Let V be a finite-dimensional U_q-module generated by a highest weight vector v of weight λ.*

(a) We have $\lambda = \epsilon q^n$ where $\epsilon = \pm 1$ and $\dim(V) = n + 1$.

(b) We have $v_p = 0$ for $p > n$, and $\{v = v_0, v_1, \ldots, v_\lambda\}$ is a basis for V.

(c) The operator K acting on V is diagonalizable with $n{+}1$ distinct eigenvalues $\{\epsilon q^n, \epsilon q^{n-2}, \ldots, \epsilon q^{-n}\}$.

(d) Any other highest weight vector in V is a scalar multiple of v and is of weight λ.

(e) The module V is simple.

(2) Any simple finite-dimensional U_q-module is generated by a highest weight vector. Two finite-dimensional U_q-modules generated by highest weight vectors of the same weight are isomorphic.

Proof. By Lemma 3.5, the sequence $\{v_p\}_{p\geq 0}$ is a sequence of eigenvectors for K with distinct eigenvalues. Since V is finite-dimensional, there is some n for which $v_n \neq 0$ but $v_{n+1} = 0$. Then Lemma 3.5 shows that $v_m = 0$ iff $m > n$, and also

$$0 = E_{v+1} = \frac{q^{-n}\lambda - q^n\lambda{-1}}{q - q^{-1}} v_n.$$

Then $q^{-n}\lambda = q^n\lambda{-1}$, and so $\lambda = \pm q^n$. The rest of the proof of (a)-(c) is identical to the classical case.

(d) Let v' be another highest weight vector. It is an eigenvector for the action of K, and so it is a scalar multiple of some v_i. But then $Ev_i = 0$ iff $i = 0$ by Lemma 3.5.

(e) Let V' be a non-zero U_q-submodule of V and v' a highest weight vector of V'. Then v' is also a highest weight vector of V. By (d), v' is then a multiple of v, and so $v \in V'$, and then $V \subseteq V'$ and V is simple.

The proof of (2) is as in the classical case of Theorem 2.8. $\square$

This theorem then implies that there is a single simple U_q-module of dimension $n+1$ and generated by highest weight vector of weight ϵq^n. We call it $V_{\epsilon,n}$. Following the same methodology as the classical case, we then have

$$\rho_{\varepsilon,n}(E) = \varepsilon \begin{pmatrix} 0 & [n] & 0 & \cdots & 0 \\ 0 & 0 & [n-1] & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

$$\rho_{\varepsilon,n}(F) = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & [2] & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & [n] & 0 \end{pmatrix},$$

$$\rho_{\varepsilon,n}(K) = \varepsilon \begin{pmatrix} q^n & 0 & \cdots & 0 & 0 \\ 0 & q^{n-2} & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & q^{-n+2} & 0 \\ 0 & 0 & \cdots & 0 & q^{-n} \end{pmatrix}.$$

As we can see, the simple representations of U_q look very similar to those of U, as matrices.

3.2. **Semi-Simplicity and Quantum Clebsch-Gordan.** We define the quantum Casimir element as

$$C_q = EF + \frac{q^{-1}K + qK^{-1}}{(q-q^{-1})^2} = FE + \frac{qK + q^{-1}K^{-1}}{(q-q^{-1})^2}.$$

Lemma 3.7. *We have C_q belongs to the center of U_q.*

Proof. It sufficed to check that C_q commutes with generators. Since $KEFK^{-1}$, C_q commutes with K. For E, we have

$$EC_q = EFE + \frac{qK + q^{-1}K^{-1}}{(q-q^{-1})^2}E = C_qE,$$

and similarly for F. $\square$

Let U_q^K be the subalgebra of U_q of elements commuting with K.

Lemma 3.8. *An element of U_q belongs to U_q^K iff it is of the form*

$$\sum_{i \geq 0} F^i P_i E^i,$$

where $P_i \in k[K, K^{-1}]$.

Proof. The result is a consequence of the observations that $\{F^iK^\ell E^j\}_{i,j\in\mathbb{N},\ell\in\mathbb{Z}}$ is a basis for U_q and $K(F^iK^\ell E^j)K^{-1} = q^{2(j-i)}F^iK^\ell E^j$. $\square$

Now we consider the left ideal $I = U_qE \cap U_q^K$ of U_q^K.

Lemma 3.9. *We have* $I = FU_q \cap U_q^K$ *and* $U_q^K = k[K, K^{-1}] \oplus I$.

Proof. Let $u = \sum_{i\geq 0} F^iP_iE^i$ be an element of U_q^K. If $u \in U_qE$, then $P_0 = 0$. Hence $u \in FU_q \cap U_q^K$ and conversely. Since this sum-representation of u is unique for any $u \in U_q^K$, then we get the desired direct sum. $\square$

Then we have that I is a two-sided ideal, and the projection map $\varphi : U_q^K \to k[K, K^{-1}]$ is an algebra morphism. It is called the Harsh-Chandra homomorphism and allows us to write the action of the center Z_q on a highest weight module.

Lemma 3.10. *Let V be a highest weight U_q-module with highest weight λ. Then for any $z \in Z_q$ and any $v \in V$, we have $zv = \phi(z)(\lambda)v$.*

Proof. Let v_0 be a highest weight generating vector of V. Then by the previous lemma, z may be written as

$$z = \varphi(z) + \sum_{i\geq 0} F^iP_iE^i.$$

Since $Ev_0 = 0$ and $Kv_0 = \lambda v_0$, then $zv_0 = \varphi(z)(\lambda)v_0$. Then for arbitrary $v = xv_0$ for $x \in U_q$, we have

$$zv = zxv_0 = xzv_0 = \varphi(z)(\lambda)xv_0 = \varphi(z)(\lambda)v.$$

$\square$

In particular, we have

$$\varphi(C_q) = \frac{qK - q^{-1}K^{-1}}{(q - q^{-1})^2}.$$

Then C_q acts on a highest weight module of highest weight λ as multiplication by

$$\frac{q\lambda + q^{-1}\lambda^{-1}}{(q - q^{-1})^2}.$$

We now prove a technical lemma, which will aid us in semisimplicity.

Lemma 3.11. *There exists an element C in the center of U_q acting by 0 on $V_{\epsilon,0}$ and by a non-zero scalar on $V_{\epsilon',n}$ for all $n > 0$ and $\epsilon' = \pm 1$.*

Proof. Let

$$C = C_q - \epsilon\frac{q + q^{-1}}{(q - q^{-1})^2}.$$

Then C acts on $V_{\epsilon,0}$ by 0 and on $V_{\epsilon',n}$ by

$$\epsilon'\frac{q^{n+1} + q^{-(n+1)}}{(q - q^{-1})^2} - \epsilon\frac{q + q^{-1}}{(q - q^{-1})^2}.$$

If this is 0, then

$$q^{2n+2} - \epsilon\epsilon' q^{n+2} - \epsilon\epsilon' q^n + 1 = 0,$$

and so equivalently

$$(q^{n+1} - \epsilon\epsilon')(q^n - \epsilon\epsilon') = 0,$$

which contradicts what q is not a root of unity. $\square$

Finally, we prove semi-simplicity.

Theorem 3.12. *When q is not a root of unity, any finite-dimensional U_q-module is semisimple.*

Proof. We follow the proof of Theorem 2.11. It suffices to show that for any finite-dimensional U_q-module V and submodule $V' \subseteq V$, there exists V'' such that $V \simeq V' \oplus V''$.

First, suppose V' has codimension 1, and we induct on dimension. If $\dim(V') = 0$, then we take $V'' = V$. If $\dim(V') = 1$, then $V', V/V'$ are simple one-dimensional modules of respective weights ϵ_1 and ϵ_2. If $\epsilon_1 \neq \epsilon_2$, then there is a basis $\{v_1, v_2\}$ of V in which K acts diagonally. Since Ev_i is an eigenvector of K with eigenvalue $\epsilon_i q^2 \neq \epsilon_j$, then $Ev_i = 0$ for $i = 1, 2$. Similarly, F acts trivially. Then V is the direct sum of kv_1 and kv_2.

Otherwise, there exists a basis $\{v_1, v_2\}$ of V such that $V' = kv_1$ and $Kv_1 = \epsilon v_1$ and $Kv_2 = \epsilon v_2 + \alpha v_1$. Again Ev_1 is an eigenvector of K with eigenvalue $\epsilon q^2 \neq \epsilon$, and so it is 0. We prove that Ev_2 is also 0. Letting $Ev_2 = \lambda v_1 + \mu v_2$, we have

$$\epsilon\lambda v_1 + \mu(\epsilon v_2 + \alpha v_1) = KEv_2 + q^2 EKv_2 = q^2 E(\epsilon v_2 + \alpha v_1) = \epsilon q^2(\lambda v_1 + \mu v_2).$$

Then $\mu\epsilon^2(q^2 - 1) = 0$ and $\lambda\epsilon(q^2 - 1) = \mu\alpha$. Then $\lambda = \mu = 0$. F similarly acts as 0 on V. As $[E, F]$ acts as 0 on V, then $K = K^{-1}$ on V. In particular, $K^{-1}v_2 = \epsilon v_2 - \alpha v_1$, and so $\alpha = -\alpha = 0$. Then here K is diagonalizable, and we reach the same conclusion.

Now, let $\dim(V') = p > 1$ and assume the assertion holds for dimension $< p$. Either V' is simple or not. If not, the same argument from the proof of Theorem 2.8 holds. If so, then the one-dimensional quotient module V/V' has weight $\epsilon = \pm 1$. Notice that C from Lemma 3.11 acts by 0 on V/V'. Then $CV \subset V'$. We also know that C acts by a non-zero scalar α, on V', so C/α is a projector of V onto V', and it is U_q-linear as C is central. Then letting $V'' = \ker\,(C/\alpha)$ does the job.

We now allow V' any codimension, and we reduce to the case of codimension 1. Let W (resp. W') be the subspace of all linear maps from V to V' whose restriction to V' is a homothety (resp. 0). We then see that W' is codimension 1 in W.

We endow $\hom(V, V')$ with its standard U_q-module structure (for a more detailed explanation of this structure, see chapter III.5 of [Kas95]). For $f \in W$, let $\alpha \in k$ be the scalar such that $f(v) = \alpha v$ for all $v \in V'$. Then for all $x \in U_q$ and $v \in V'$ we have

$$(xf)(v) = \sum_{(x)} x' f(S(x'')v) = \alpha \left(\sum_{(x)} x' S(x'') \right) v = \alpha\epsilon(x)v.$$

Here our notation means we let x, x'' range over all values of x, and S is the operator defined by

$$S(E) = -EK^{-1}, S(F) = -KF, S(K) = K^{-1}, S(K^{-1}) = K.$$

By the codimension 1 case, we can find W'' such that $W = W' \oplus W''$. Let f generate W'', and by definition it acts as a scalar $\alpha \neq 0$ on V'. Then f/α is a projector from V to V' and that $V'' := \ker f$ is a supplementary space to V'. To conclude, we only need to check that V'' is a

U_q-module. Since W'' is a one-dimensional submodule, it is simple of weight ± 1. Then for all $x \in U_q$, we have $xf = \pm\epsilon(x)f$. So for $v \in V''$,

$$K^{-1}f(Kv) = (K^{-1}f)(v) = \pm\epsilon(K^{-1})f(v) = 0,$$

and so $f(Kv) = 0$. Then $KV'' \subset V''$. Similarly, V'' is stable under K^{-1}. Also, for any $v \in V''$ and hence also for Kv, we have

$$\begin{aligned} 0 &= \pm\epsilon(E)f(Kv) = (Ef)(Kv) \\ &= f(S(E)Kv) + Ef(K^{-1}Kv) = -f(Ev) + Ef(v). \end{aligned}$$

Then $f(Ev) = 0$, and so V'' is stable under E. Similarly, it is stable under F. Then V'' is a submodule. ☐

Now, as in the classical case, we end this section with a Clebsch-Gordan formula analogous to Theorem 2.12. Since

$$V_{\epsilon,n} \simeq V_{\epsilon,0} \otimes V_{1,n} \simeq V_{1,n} \otimes V_{\epsilon,0},$$

we only need investigate $V_{1,n}$, which we denote by V_n.

Theorem 3.13. *Let $n \geq m$ be integers. Then*

$$V_n \otimes V_m \simeq V_{n+m} \oplus V_{n+m-2} \oplus \cdots \oplus V_{n-m}.$$

Just as in the classical case, it suffices to find that $V(n)\otimes V(m)$ contains a highest weight module of weight q^{n+m-2p} for any $0 \leq p \leq m$.

Lemma 3.14. *Let $v^{(n)}$ be a highest weight vector of weight q^n in V_n and $v^{(m)}$ similarly in V_m. Define $v_p^{(n)} = \frac{1}{[p]!}F^p v^{(n)}$ and $v_p^{(m)} = \frac{1}{[p]!}F^p v^{(m)}$ for $p \geq 0$. Then*

$$v^{(n+m-2p)} = \sum_{i=0}^{p} (-1)^i \frac{[m-p+i]![n-i]!}{[m-p]![n]!} q^{-i(m-2p+i+1)} v_i^{(n)} \otimes v_{p-i}^{(m)}$$

is a highest weight vector of weight q^{n+m-2p} in $V_n \otimes V_m$.

Proof. As in the classical case, this is essentially computation. It is clear that $v_i^{(p)} \otimes v_{p-i}^{(m)}$ has weight $q^{n-2i+m-2(p-i)} = q^{n+m-2p}$. By a lengthy but unenlightening computation, we find that $Ev^{(n+m-2p)} = 0$, which concludes the theorem. ☐

We know understand all U_q-modules and their tensor products, and we understand them now as well as we will ever need to in these notes.

4. R-Matrices and the Yang-Baxter Equation

4.1. **Basic Notions.** We now pivot to a seemingly unrelated field, but we later see that the following theory is the perfect setting in which to apply our knowledge of U_q-modules. The following definition is stated in full generality, although we will only use it in the case that $k = \mathbb{C}$ and V finite-dimensional.

Definition 4.1. *Let V be a vector space over a field k. A linear automorphism c of $V \otimes V$ is an R-matrix if it is a solution to the Yang-Baxter equation*

$$(c \otimes id_V)(id_V \otimes c)(c \otimes id_V) = (id_V \otimes c)(c \otimes id_V)(id_V \otimes c)$$

in the automorphism group of $V \otimes V \otimes V$.

Finding all R-matrices is a very difficult task, but we only need a small set of solutions here.

Example 5. Let $\tau_{V,V} \in \mathrm{Aut}(V \otimes V)$ be the flip operator which sends $\tau(v_1 \otimes v_2) = v_2 \otimes v_1$. Then $\tau_{V,V}$ satisfies the Yang-Baxter equation because $(12)(23)(12) = (23)(12)(23)$ in S_3. We notice that this looks suspiciously like the braid relation, which is also the type III Reidemeister move...

The following lemma is immediate from the definition and basic properties of tensor products.

Lemma 4.2. *if c is a solution to the Yang-Baxter equation, then so are λc, c^{-1}, and $\tau_{V,V} \circ c \circ \tau_{V,V}$, for λ any non-zero scalar.*

4.2. **The case of V_1.** Our next task is to find all solutions when $V = V_1 = V_{1,1}$, the unique 2-dimensional simple module over U_q. Precisely, we want to find all U_q-automorphisms of $V_1 \otimes V_1$ which are R-matrices. Recall that if v_0 is a highest weight vector of V_1, then the set $\{v_0, v_1\}$ is a basis of V_1. By the Clebsch-Gordan formula, we have $V_1 \otimes V_1 \simeq V_2 \oplus V_0$, and that the vectors $w_0 = v_0 \otimes v_0$ and $t = v_0 \otimes v_1 - q^{-1} v_1 \otimes v_0$ are highest weight vectors of respective weights q^2 and 1. Then we extend $\{w_0, t\}$ to a basis of $V \otimes V$ with

$$w_1 = F w_0 = q^{-1} v_0 \otimes v_1 + v_1 \otimes v_0, w_2 = \frac{1}{[2]} F^2 w_0 = v_1 \otimes v_1.$$

Proposition 4.3. *Any U_q-linear automorphism φ of $V_1 \otimes V_1$ is diagonalizable and of the form $\varphi(w_i) = \lambda w_i$ ($i = 0, 1, 2$) and $\varphi(t) = \mu t$ for λ, μ non-zero scalars. Also, φ is an R-matrix iff*

$$(\lambda - \mu)(q\lambda + q^{-1}\mu)(q^{-1}\lambda + q\mu) = 0.$$

Proof. Since φ is U_q-linear, the image under φ of a highest weight vector is a highest weight vector of the same weight. Since w_0, t have different weights, there are λ, μ such that $\varphi(w_0) = \lambda w_0$ and $\varphi(t) = \mu t$. Then for $i = 1, 2$,

$$\varphi(w_i) = \frac{1}{[i]} \varphi(F^i w_0) = \frac{1}{[i]} F^i \varphi(w_0) = \lambda w_i.$$

For the second part of the proposition, we must perform a tedious computation. We let ϕ denote the matrix of φ with respect to the basis $\{v_0 \otimes v_0, v_0 \otimes v_1, v_1 \otimes v_0, v_1 \otimes v_1\}$, and it is given by

$$\Phi = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \alpha & \gamma & 0 \\ 0 & \gamma & \beta & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix},$$

where

$$\alpha = \frac{q^{-1}\lambda + q\mu}{[2]}, \beta = \frac{q\lambda + q^{-1}\mu}{[2]}, \gamma = \frac{\lambda - \mu}{[2]}.$$

Then the automorphisms $\varphi \otimes id$ and $id \otimes \varphi$ then can be expressed as 8×8 matrices called Φ_{12} and Φ_{23}, respectively, in the basis $\{v_i \otimes v_j \otimes v_k\}_{i,j,k \in \{0,1\}}$. We have

$$\Phi_{12} = \begin{pmatrix} \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha & 0 & \lambda & 0 & 0 \\ 0 & 0 & \gamma & 0 & \beta & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma & 0 & \beta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda \end{pmatrix},$$

$$\Phi_{23} = \begin{pmatrix} \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha & \gamma & 0 & 0 & 0 & 0 & 0 \\ 0 & \gamma & \beta & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha & \gamma & 0 \\ 0 & 0 & 0 & 0 & 0 & \gamma & \beta & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda \end{pmatrix}.$$

Now, we have $\Phi_{12}\Phi_{23}\Phi_{12} - \Phi_{23}\Phi_{12}\Phi_{23}$ is

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & K & -\alpha\beta\gamma & 0 & 0 & 0 & 0 & 0 \\ 0 & -\alpha\beta\gamma & L & 0 & \alpha\beta\gamma & 0 & 0 & 0 \\ 0 & 0 & 0 & -K & 0 & \alpha\beta\gamma & 0 & 0 \\ 0 & 0 & \alpha\beta\gamma & 0 & M & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha\beta\gamma & 0 & -L & \alpha\beta\gamma & 0 \\ 0 & 0 & 0 & 0 & 0 & \alpha\beta\gamma & -M & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where $K = \alpha((\lambda - \alpha)\lambda - \gamma^2), L = \alpha\beta(\alpha - \beta), M = \beta(\gamma^2 + \lambda(\beta\lambda))$. If K, L, M are multiples of $\alpha\beta\gamma$, then so is the above matrix. Then Φ is an R-matrix iff $\alpha\beta\gamma = 0$, which would complete the proof.

So finally, we show that K, L, M are multiples of $\alpha\beta\gamma$. This is simply true, as we have

$$\lambda - \alpha = q\gamma, \lambda - \beta = q^{-1}\gamma, q^{-1}\lambda - \gamma = q^{-1}\alpha, q\lambda - \gamma = q\beta, \beta - \alpha = (q - q^{-1})\gamma.$$

Then

$$K = q\alpha\beta\gamma, L = -(q - q^{-1})\alpha\beta\gamma, M = -q^{-1}\alpha\beta\gamma.$$

□

To summarize, the R-matrices of $V_1 \otimes V_1$ belong to the following three types, depending on λ.

(1) If $\lambda = \mu$, φ is a homothety.
(2) If $q\lambda + q^{-1}\mu = 0$, then

$$\Phi = q\lambda \begin{pmatrix} q^{-1} & 0 & 0 & 0 \\ 0 & q^{-1} - q & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q^{-1} \end{pmatrix}.$$

(3) If $q^{-1}\lambda + q\mu = 0$, then

$$\Phi = q^{-1}\lambda \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & q - q^{-1} & 0 \\ 0 & 0 & 0 & q \end{pmatrix}.$$

We see that the second and third cases above are equivalent up to change of basis and exchanging q for q^{-1}.

4.3. **Another important class of R-matrices.** Let V be a finite-dimensional vector space with basis $\{e_1, \dots, e_N\}$. For two invertible scalars p, q and any family $\{r_{ij}\}_{1\leq i,j\leq N}$ of scalars in k such that $r_{ii} = q$ and $r_{ij}r_{ji} = p$ when $i \neq j$, we let c be an automorphism of $V \otimes V$ defined by

$$\begin{aligned}
&c(e_i \otimes e_i) = qe_i \otimes e_i && \\
&c(e_i \otimes e_j) = r_{ji}e_i \otimes e_j && \text{if } i < j \\
&c(e_i \otimes e_j) = r_{ji}e_i \otimes e_j + (q - pq^{-1})e_i \otimes e_j && \text{if } i > j
\end{aligned}$$

.

Proposition 4.4. *The automorphism c solves the Yang-Baxter equation. Moreover, we have*

$$(c - qid_{V\otimes V})(c + pq^{-1}id_{V\otimes V}) = 0,$$

or equivalently $c^2 - (q - pq^{-1})c - pid_{V\otimes V} = 0$.

Proof. Let $(ijk) = e_i \otimes e_j \otimes e_k$, and let $[i > j]$ for 1 if $i > j$ and 0 else (an indicator function). Then

$$c(e_i \otimes e_j) = r_{ji}e_j \otimes e_i + [i > j]\beta e_i \otimes e_j$$

for $\beta = q - pq^{-1}$. With this notation, we have

$$\begin{aligned}
&(c \otimes id)(id \otimes c)(c \otimes id)((ijk)) \\
&= r_{ji}r_{ki}r_{kj}(kji) + r_{ji}r_{ki}[j > k]\beta(jki) \\
&+ r_{kj}r_{ki}[i > j]\beta(kij) + r_{kj}[i > j][j > k]\beta^2(ikj) \\
&+ r_{ji}([j > i][i > k] + [i > j][j > k])\beta^2(jik) \\
&+ (r_{ji}r_{ij}[i > k]\beta + [i > j][j > k]\beta^3)(ijk))
\end{aligned}$$

and

$$\begin{aligned}
&(c \otimes id)(id \otimes c)(c \otimes id)((ijk)) \\
&= r_{ji}r_{ki}r_{kj}(kji) + r_{ji}r_{ki}[j > k]\beta(jki) \\
&+ r_{kj}r_{ki}[i > j]\beta(kij) + r_{ji}[i > k][j > k]\beta^2(jik) \\
&+ r_{kj}([k > j][i > k] + [i > j][j > k])\beta^2(ikj) \\
&+ (r_{jk}r_{kj}[i > k]\beta + [i > j][j > k]\beta^3)(ijk)).
\end{aligned}$$

Considering relationships of the form

$$[i > j][i > k] = [i > j][j > k] + [i > k][k > j],$$

it becomes clear that the above expansions are equal.

The second claim is short. We compute the action of $c^2 - \beta c - pid_{V\otimes V}$ on any vector $e_i \otimes e_j$. If $i \neq j$, then we get 0, and also if $i = j$ by the value of β. $\square$

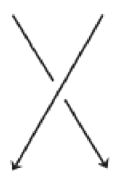
Notice that if $p = q^2$, we get c is a homothety. If $p = 1$ and $r_{ij} = 1$ for $i \neq j$, then c takes the matrix form of the third case in the previous section. Then this subsection is actually a generalization of the previous one.

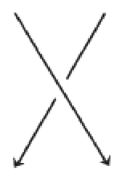
5. A RETURN TO TOPOLOGY

Now that we have some theory under our belts, we return to knot theory and the Jones-Conway polynomial, a slight generalization of the Jones polynomial that we derived in Section 1. This polynomial is more complicated than the Jones polynomial that we derived before by elementary methods. As such, we make use of more algebraic machinery to prove it. In fact, the machinery we have been developing (and will continue to develop) is just a special case of a much broader theory, phrased in more universal language, which applies to a broad class of objects beyond just U_q. The Jones-Conway polynomial is then just the tip of the iceberg of what this algebraic perspective can tell us about knots and three-manifolds, but we don't explore other applications in these notes.

5.1. **Definitions and Statement of the Main Theorem.**

Definition 5.1. *A triple (L_+, L_-, L_0) of oriented links in $\mathbb{R}^3$ is a Conway triple if they can be represented by link diagrams D_+, D_-, D_0 which are the same outside a small disk, and in which they are respectively isotopic to $X_+, X_-, \downarrow\downarrow$.*





Now we state the main theorem.

Theorem 5.2. *There exists a unique map $L \to P_L$ from the set of all oriented links to the ring $\Lambda := \mathbb{Z}[x, x^{-1}, y, y^{-1}]$ such that*

(1) if $L \sim L'$ then $P_L = P_{L'}$,
(2) the value of P on the unknot is 1, and
(3) for any Conway triple (L_+, L_-, L_0), we have

$$xP_{L_+} - x^{-1}P_{L_-} = yP_{L_0}.$$

We call this object the Jones-Conway polynomial, or the HOMFLY polynomial (after the initials of its creators).

Remark. We don't prove this here, but one can deduce that the Jones polynomial is given by $P_L(t^{-1}, t^{1/2} - t^{-1/2})$, a special case of the HOMFLY polynomial. (This can be found in most knot theory textbooks like [G B03]).

We let K denote the set of equivalence classes of links, $\Lambda[K]$ the free module generated by K, and Y the quotient of $\Lambda[K]$ by the ideal generated by elements $xL_+ - x^{-1}L_- - yL_0$, running over all Conway triples. The Λ-module Y is called the Skein module of $\mathbb{R}^3$.

Consider the following proposition.

Proposition 5.3. *Let $Q : \Lambda \to Y$ be the Λ-linear map sending 1 to the class $[O]$ of the unknot. Then Q is an isomorphism.*

Then the Skein module is free of rank 1, generated by $[O]$. Set

$$P_L = Q^{-1}([L]),$$

and then it is clear that P_L satisfies all conditions of Theorem 5.2. Then Proposition 5.3 implies Theorem 5.2. We show surjectivity and injectivity of Q.

5.2. **Surjectivity of Q.** This proof is purely topological and does not use our algebraic machinery. It is enough to check that Y is generated as a Λ-module by $[O]$.

Lemma 5.4. *As a Λ-module, Y is generated by the family $\{[O^{\otimes n}]\}_{n>0}$ of isotopy classes of trivial links.*

Proof. Let Y_m be the Λ-submodule generated by isotopy classes of links representable by link diagrams with $\leq m$ crossing points. Clearly, Y_m includes into Y_{m+1}, and Y is the union of all Y_m. Then it is enough to prove the statement for all Y_m. We induct on m. The $m = 0$ case holds by definition of trivial links. Suppose the assertion is true for integers $< m$, and let $[L]$ be the class of $L \in Y_m$ which may be represented with m crossings. Consider one such crossing, and consider the Conway triple (L_+, L_-, L_0) such that $L = L_+$ or $L = L_-$ and L_0 has one fewer crossing. By definition of Y as a quotient, we have $[L_+] = x^{-2}[L_-] \pmod{Y_{m-1}}$. In other words, change of crossings adjusts the isotopy class by an invertible scalar multiplication, $\pmod{Y_{m-1}}$. Since any link diagram may be changed into that of a trivial link by changing crossings, we know that $[L]$ belongs to the submodule generated by trivial links and Y_{m-1}. Then by induction, the result follows. □

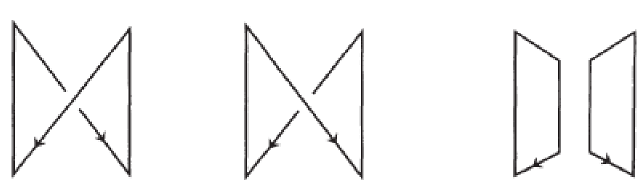
Lemma 5.5. *For any integer $n > 1$, we have*

$$[O^{\otimes n}] = \left(\frac{x - x^{-1}}{y}\right)^{n-1} [O].$$

Proof. The below figure implies that $(O^{\otimes n}, O^{\otimes n}, O^{\otimes n+1})$ is a Conway triple for all $n \geq 1$. By definition of Y, we then have

$$[O^{\otimes n+1}] = \frac{x - x^{-1}}{y}[O^{\otimes n}],$$

which concludes by induction.



□

These two lemmas then imply surjectivity.

5.3. **Injectivity of Q.** We will prove the following proposition later on, but for now we take it as given.

Proposition 5.6. *Let $q \neq 0$ not be a root of unity, and let m be an integer > 1. There exists a unique map $\phi_{m,q}$ from the set of oriented links to $\mathbb{C}$ such that*

(1) if $L \sim L'$, then $\phi_{m,q}(L) = \phi_{m,q}(L')$

(2) *the value of $\phi_{m,q}$ on the trivial knot is*

$$\phi_{m,q}(O) = \frac{q^m - q^{-m}}{q - q^{-1}} \neq 0,$$

and

(3) *when (L_+, L_-, L_0) is a Conway triple, we have*

$$q^m \phi_{m,q}(L_+) - q^{-m}\phi_{m,q}(L_-) = (q - q^{-1})\phi_{m,q}(L_0).$$

Let $\zeta_{q,m} : \Lambda \to \mathbb{C}$ be defined by $\zeta_{q,m}(x) = q^m$ and $\zeta_{q,m}(y) = q - q^{-1}$, which gives $\mathbb{C}$ a Λ-module structure. Then Proposition 5.6 implies that $\phi_{q,m}$ extends linearly to a unique Λ-linear map $\phi'_{m,q}$ from $\Lambda[K]$ to $\mathbb{C}$. Then for any Conway triple (L_+, L_-, L_0), we have

$$\begin{aligned}
&\phi'_{q,m}\left(x[L_+] - x^{-1}[L_-] - y[L_0]\right) \\
&= \zeta_{q,m}(x)\phi_{m,q}(L_+) - \zeta_{q,m}(x^{-1})\phi_{m,q}(L_-) - \zeta_{q,m}(y)\phi_{m,q}(L_0) \\
&= q^m\phi_{m,q}(L_+) - q^{-m}(x^{-1})\phi_{m,q}(L_-) - (q - q^{-1})\phi_{m,q}(L_0) \\
&= 0.
\end{aligned}$$

Then $\phi'_{m,q}$ factors through a unique Λ-linear map, denoted $\phi''_{m,q}$, from Y to $\mathbb{C}$ such that $\phi''_{m,q}([L]) = \phi_{m,q}([L])$ for any oriented link L.

Now let $f(x,y) \in \Lambda$ such that $Q(f(x,y)) = f(x,y)[O]$ vanishes in Y. Then

$$0 = \phi''_{m,q}(f(x,y)[O]) = f(q^m, q - q^{-1})\phi_{m,q}(O)$$

for any integer $m > 1$ and complex number q which is not a root of unity. Since $\phi_{m,q}(O) \neq 0$, we have $f(q^m, q - q^{-1}) = 0$. Since this is true for infinitely many complex numbers q, f is divisible by $y - (q - q^{-1})$. Since this also holds for infinitely many q, f must be 0. Then we have injectivity.

6. The Road to Theorem 5.6

6.1. **Some useful maps: (partial) transpose and (partial) trace.** This section is mostly devoted to some useful formalism rather than to any conceptual heavy-lifting. Assume all vector spaces mentioned are finite-dimensional. For vector spaces U, U', V, V', let $f : U \to U'$ and $g : V \to V'$ be linear maps, and define

$$\lambda : \hom(U, U') \otimes \hom(V, V') \to \hom(V \otimes U, U' \otimes V')$$

by

$$(\lambda(f \otimes g))\,(v \otimes u) = f(u) \otimes g(v).$$

Lemma 6.1. *The map λ is an isomorphism.*

Proof. We reduce λ to simple maps. We can write $U = \bigoplus_{i \in I} ku_i$ for a finite basis $\{u_i\}$ of U. Recall the two standard canonical isomorphisms

$$\hom(\bigoplus_{i \in I} U_i, V) \simeq \Pi_{i \in I} \hom(U_i, V)$$

and

$$(\bigoplus_{i \in I} U_i) \otimes V \simeq \bigoplus_{i \in I}(U_i \otimes V).$$

By these two isomorphisms, λ turns into a map from

$$(\Pi_i \hom(ku_i, U')) \otimes \hom(V, V')$$

to

$$\Pi_i \hom(V \otimes ku_i, U' \otimes V').$$

Since I is a finite set, we can replace Π_i by $\oplus_i$. Therefore, it remains to show that

$$\lambda : \hom(ku_i, U') \otimes \hom(V, V') \to \hom(V \otimes ku_i, U' \otimes V')$$

is an isomorphism in the case that $U = ku_i$. This amounts to checking that the map

$$\lambda' : U' \otimes \hom(V, V') \to \hom(V, U' \otimes V')$$

by

$$\lambda'(u' \otimes f)(v) = u' \otimes f(v)$$

is an isomorphism. We also know that for a basis $\{u'_i\}$ of U' we have

$$U' \otimes \hom(V, V') \simeq \oplus_{i \in I'} ku'_i \otimes \hom(V, V')$$

and

$$\hom(V, U' \otimes V') \simeq \Pi_{i \in I'} \hom(V, ku'_i \otimes V').$$

Then we can break up λ' into the direct product of the maps

$$\lambda' : ku'_i \otimes \hom(V, V') \to \hom(V, ku'_i \otimes V').$$

In this case, we have $\lambda'(u'_i \otimes f)(v) = u'_i \otimes f(v)$, which is clearly an isomorphism. Then this concludes the proof. ☐

Corollary 6.2. *The map* $\lambda_{U,V} : V \otimes U^* \to \hom(U, V)$ *given by*

$$\lambda_{U,V}(v \otimes \alpha)(u) = \alpha(u)v$$

for $u \in U$*,* $\alpha \in U'$*, and* $v \in V$ *is an isomorphism.*

Let V be a vector space with basis $\{v_i\}_i$, and let the dual basis $\{v^i\}_i$ denote the basis of V^*. Then we define the evaluation map on $V^* \otimes V$ by

$$ev_V(v^i \otimes v_j) = \delta_{ij}.$$

Now let $f : U \to V$ be a linear map. Using bases, we have

$$f_{u_j} = \sum_i f^i_j v_i$$

for scalars $(f^i_j)_{ij}$. Then

$$f = \lambda_{U,V}\left(\sum_{ij} f^i_j v_i \otimes u^j\right).$$

With $f = id_V$, we have

$$id_V = \lambda_{V,V}\left(\sum_i v_i \otimes v^i\right).$$

This then allows us to define the coevaluation map $\delta_V : k \to V \otimes V^*$ by $\delta_V(1) = \lambda^{-1}_{V,V}(id_V)$.

For vector spaces U, V, let $f : U \to V$, and recall that $f^* : V^* \to U^*$ is the transpose of f, defined as the linear map such that for all $\alpha \in V^*$ and $u \in U$, we have

$$\langle f^*(\alpha), u\rangle = \langle \alpha, f(u)\rangle.$$

We then notice by parsing definitions that the transpose is equal to the composition

$$V^* \xrightarrow{id_{V^*} \otimes \delta_U} V^* \otimes U \otimes U^* \xrightarrow{id_{V^*} \otimes f \otimes id_{U^*}} V^* \otimes V \otimes U \xrightarrow{ev_V \otimes id_{U^*}} U^*.$$

As a consequence, we can write

$$f^*(v^j) = \sum_i f_i^j u^i.$$

We generalize this notion to the partial transposes.

Definition 6.3. *Let $f^+ : X^* \otimes W \to V^* \otimes Y$ and $f^\times : V \otimes Y^* \to X \otimes W^*$ by*

$$f^+(x^i \otimes w_j) = \sum_{k,\ell} f_{kj}^{i\ell} v^k \otimes y_\ell$$

and

$$f^\times(v_i \otimes y^j) = \sum_{k,\ell} f_{i\ell}^{kj} x_k \otimes w^\ell,$$

where

$$f(v_i \otimes w_j) = \sum_{k,\ell} f_{ij}^{k\ell} x_k \otimes y_\ell.$$

A calculation shows that $(f^+)^\times = (f^\times)^+ = f^*$.

We also define the trace of a map by $tr : End(V) \to k$ via the composition

$$End(V) \xrightarrow{\lambda_{V,V}^{-1}} V \otimes V^* \xrightarrow{\tau_{V,V^*}} V^* \otimes V \xrightarrow{ev_V} k.$$

Definition 6.4. *With the above notation, we let*

$$tr_1(f)(v_j) = \sum_{i,\ell} f_{ij}^{i\ell} v_\ell$$

and

$$tr_2(f)(u_j) = \sum_{i,\ell} f_{ij}^{kj} u_k,$$

and these are called the partial traces of f.

Similarly to in the case of partial transposes, $tr_1(tr_2(f)) = tr_2(tr_1(f)) = tr(f)$.

6.2. **Tensor categories.** In this section, we assume a basic familiarity the category theory. If you have not seen category theory, we also give informal, intuitive exposition in this section and beyond, so that category theory is not necessary for a solid qualitative understanding. We give a rather terse exposition here, since the mechanics of tensor categories are not the main focus of these lecture notes. In fact, I would be surprised if much the following material really made it into the lecture, proper, at all.

Definition 6.5. *A tensor category is a category $\mathcal{C}$ equipped with a tensor product $\otimes$. Specifically, it is a tuple $(\mathcal{C}, \otimes, I, a, l, r)$ with a tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ with an object I the unit with respect to the $\otimes$ operation up to isomorphisms l, r (two isomorphisms for left and right tensor by I, respectively), and associative up to isomorphism, given by isomorphism a. More concretely, the behavior of a, r, l are given in the following two commutative diagrams.*

$$
\begin{tikzcd}
\Big(U \otimes (V \otimes W)\Big) \otimes X \arrow[dd, "a_{U,V\otimes W,X}"] & \Big((U \otimes V) \otimes W\Big) \otimes X \arrow[l, "a_{U,V,W}\otimes \mathrm{id}_X"'] \arrow[d, "a_{U\otimes V,W,X}"] \\
& (U \otimes V) \otimes (W \otimes X) \arrow[d, "a_{U,V,W\otimes X}"] \\
U \otimes \Big((V \otimes W) \otimes X\Big) \arrow[r, "\mathrm{id}_U \otimes a_{V,W,X}"] & U \otimes \Big(V \otimes (W \otimes X)\Big)
\end{tikzcd}
$$

$$
\begin{tikzcd}
(V \otimes I) \otimes W \arrow[rr, "a_{V,I,W}"] \arrow[dr, "r_V \otimes \mathrm{id}_W"'] & & V \otimes (I \otimes W) \arrow[dl, "\mathrm{id}_V \otimes l_W"] \\
& V \otimes W &
\end{tikzcd}
$$

If these isomorphisms a, l, r are all the identity, we say that $\mathcal{C}$ is a strict tensor category.

Definition 6.6. *Let $\mathcal{C} = (\mathcal{C}, \otimes, I, a, l, r)$ and $\mathcal{D} = (\mathcal{D}, \otimes I, a, l, r)$ be tensor categories. A tensor functor from $\mathcal{C}$ to $\mathcal{D}$ is a triple $(F, \varphi_0, \varphi_2)$ where $F : \mathcal{C} \to \mathcal{D}$ is a functor, φ_0 is an isomorphism from I to $F(I)$, and*

$$\varphi_2(U, V) : F(U) \otimes F(V) \to F(U \otimes V)$$

is a family of natural isomorphisms indexed by pairs (U, V) of objects in $\mathcal{C}$, obeying the natural compatibility relations. Concretely, we have the following diagrams.

$$
\begin{tikzcd}
\Big(F(U) \otimes F(V)\Big) \otimes F(W) \arrow[r, "a_{F(U),F(V),F(W)}"] \arrow[d, "\varphi_2(U,V)\otimes \mathrm{id}_{F(W)}"] & F(U) \otimes \Big(F(V) \otimes F(W)\Big) \arrow[d, "\mathrm{id}_{F(U)}\otimes \varphi_2(V,W)"] \\
F(U \otimes V) \otimes F(W) \arrow[d, "\varphi_2(U\otimes V,W)"] & F(U) \otimes F(V \otimes W) \arrow[d, "\varphi_2(U,V\otimes W)"] \\
F((U \otimes V) \otimes W) \arrow[r, "F(a_{U,V,W})"] & F(U \otimes (V \otimes W))
\end{tikzcd}
$$

$$
\begin{tikzcd}
I \otimes F(U) \arrow[r, "l_{F(U)}"] \arrow[d, "\varphi_0 \otimes \mathrm{id}_{F(U)}"] & F(U) \\
F(I) \otimes F(U) \arrow[r, "\varphi_2(I,U)"] & F(I \otimes U) \arrow[u, "F(l_U)"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
F(U) \otimes I \arrow[r, "r_{F(U)}"] \arrow[d, "\mathrm{id}_{F(U)} \otimes \varphi_0"] & F(U) \\
F(U) \otimes F(I) \arrow[r, "\varphi_2(U,I)"] & F(U \otimes I) \arrow[u, "F(r_U)"']
\end{tikzcd}
$$

If φ_0, φ_2 are identities of $\mathcal{D}$, our tensor functor is strict.

A natural transformation between two tensor functors $(F, \varphi_0, \varphi_2)$ and $(F', \varphi_0', \varphi_2')$ is a natural transformation between F, F' which is compatible with $\varphi_0, \varphi_2, \varphi_0', \varphi_2'$. Specifically, we have that the following diagrams commute.

$$\begin{tikzcd} & F(I) \arrow[dd, "\eta(I)"] \\ I \arrow[ur, "\varphi_0"] \arrow[dr, "\varphi_0'"'] & \\ & F'(I) \end{tikzcd} \quad and \quad \begin{tikzcd} F(U)\otimes F(V) \arrow[r, "\varphi_2(U,V)"] \arrow[d, "\eta(U)\otimes\eta(V)"] & F(U\otimes V) \arrow[d, "\eta(U\otimes V)"] \\ F'(U)\otimes F'(V) \arrow[r, "\varphi_2'(U,V)"] & F'(U\otimes V) \end{tikzcd}$$

A natural tensor transformation which is also an isomorphism is called a natural tensor isomorphism.

A tensor equivalence between tensor categories is a tensor functor $F : \mathcal{C} \to \mathcal{D}$ such that there is some $F' : \mathcal{D} \to \mathcal{C}$ such that FF' and $F'F$ are tensor isomorphic to the identities.

Basically, we can think informally of strict tensor categories as associative tensor categories, and we think of a (strict) tensor functor as a functor which respects the tensor product relations and axioms.

Strict tensor categories are a lot nicer to work with than general tensor categories. Given any tensor category $\mathcal{C}$, however, we may turn it into the strict tensor category $\mathcal{C}^{str}$. Given some tensor category $\mathcal{C} = (\mathcal{C}, \otimes, I, a, l, r)$, we let $\mathcal{S}$ be all finite sequences of $S = (V_1, \dots, V_k)$ of objects in $\mathcal{C}$, including the empty sequence $\varnothing$, which has length 0 by convention. We denote by $S * S'$ the appending of the two sequences S, S', and we let $S * \varnothing = \varnothing * S = S$. We define the map F inductively by

$$F(\varnothing) = I, F((V)) = V, F(S * (V)) = F(S) \otimes V.$$

Definition 6.7. *We define the category $\mathcal{C}^{str}$ as having objects $\mathcal{S}$ and let*

$$\hom_{\mathcal{C}^{str}}(S, S') = \hom_{\mathcal{C}}(F(S), F(S')).$$

In speech, we call $\mathcal{C}^{str}$ the *strictification* of $\mathcal{C}$. We spend the remainder of the subsection showing that $\mathcal{C}^{str}$ and $\mathcal{C}$ are in fact equivalent and that $\mathcal{C}^{str}$ is a strict tensor functor. If you are not comfortable with category theory, take the result as given - the proof is not relevant to what we'll do afterwards.

Lemma 6.8. *We have $\mathcal{C}^{str}$ is equivalent to $\mathcal{C}$.*

Proof. The above-defined map F on objects of $\mathcal{C}^{str}$ extends to a functor $F : \mathcal{C}^{str} \to \mathcal{C}$ which is the identity on morphisms and hence fully faithful. We also have that F is essentially surjective since any object in $\mathcal{C}$ is clearly isomorphic to the image under F of a sequence of length one. Since a functor is a category equivalence if it is essentially surjective and fully faithful, the lemma follows. Observe nonetheless that $G(V) = (V)$ defines the inverse equivalence of F explicitly. Clearly $FG = id_{\mathcal{C}}$, and $\theta : GF \to id_{\mathcal{C}^{str}}$ via the natural isomorphism

$$\theta(S) = id_{F(S)} : GF(S) \to S.$$

□

Now we define the tensor product on $\mathcal{C}^{str}$ by $S \otimes S' = S * S'$. This operation is clearly associative. Now we must define the tensor product of morphisms. We begin by constructing the natural isomorphism

$$\varphi(S, S') : F(S) \otimes F(S') \to F(S * S')$$

for any pair of objects $(S, S') \in (\mathcal{C}^{str})^2$. As with F above, we define φ inductively, beginning with $\varphi(\varnothing, S) = l_S$ and $\varphi(S, \varnothing) = r_S$. Next,

$$\varphi(S, (V)) = id_{F(S)\otimes V} : F(S) \otimes V \to F(S \otimes (V))$$

and

$$\varphi(S, S' * (V)) = (\varphi(S, S') \otimes id_V)^{-1} \circ a^{-1}_{F(S),F(S'),V}.$$

The following technical lemma is proven by induction and computation in [Kas95].

Lemma 6.9. *For $S, S', S'' \in \mathcal{C}^{str}$, we have*

$$\varphi(S, S' * S'') \circ (id_S \otimes \varphi(S', S'')) \circ a_{F(S),F(S'),F(S'')} = \varphi(S * S', S'') \circ (\varphi(S, S') \otimes id_{S''}).$$

Finally, we may define the tensor product of two morphisms $f : S \to T$ and $f' : S' \to T'$ in $\mathcal{C}^{str}$ by the commutative diagram

$$\begin{array}{ccc} F(S) \otimes F(S') & \xrightarrow{\varphi(S,S')} & F(S * S') \\ \downarrow{\scriptstyle f\otimes f'} & & \downarrow{\scriptstyle f*f'} \\ F(T) \otimes F(T') & \xrightarrow{\varphi(T,T')} & F(T * T') \end{array}.$$

Theorem 6.10. *With the above-defined tensor product, $\mathcal{C}^{str}$ is a strict tensor category, strict tensor equivalent to $\mathcal{C}$.*

Proof. Since the operation $*$ is a functor and strictly associative by construction, $\mathcal{C}^{str}$ is a strict tensor category.

We now claim that (F, id_I, φ) is a tensor functor from $\mathcal{C}^{str}$ to $\mathcal{C}$, where φ is the natural isomorphism discussed above. By Lemma 6.9, we see that the first of three commutative diagrams in the definition of a tensor functor Definition 6.6 is satisfied. The other two relations of Definition 6.6 are satisfied by our definitions of $\varphi(S, \varnothing)$ and $\varphi(\varnothing, S)$. The functor G from the proof of Lemma 6.8 is a strict tensor functor. Then θ from the same proof supplies the sought natural tensor isomorphism. ☐

6.3. **Tangles.** We think of a tangle as a sort of generalization of a link. It turns out that generalizing links to tangles is actually what allows us to present link invariants.

Let $[n] = \{1, 2, \ldots, n\}$ for $n > 0$. (Note that while this notation resembles $[n]_q$ in the quantum context, the two are wholly unrelated.) Let $[0]$ be the empty set and I the interval $[0, 1]$.

Definition 6.11. *Let k, ℓ be nonnegative integers. A tangle L of type (k, ℓ) is the union of a finite number of pairwise disjoint simple oriented polygonal arcs in $X = \mathbb{R}^2 \times I$ such that the boundary ∂L of L satisfying the condition*

$$\partial L = L \cap (\mathbb{R}^2 \times \{0, 1\}) = ([k] \times \{0\} \times \{0\}) \cup ([\ell] \times \{0\} \times \{1\}).$$

In other words, the tangle intersects the boundary planes of X transversally. Note that a link is a tangle of type $(0, 0)$.

All results which we discussed about isotopy and equivalence of knots and links at the beginning of these notes (triangle moves, Reidemeister moves, etc...) carry over to the case of tangles by similar proofs. A tangle diagram is similarly analogous to a knot or link diagram.

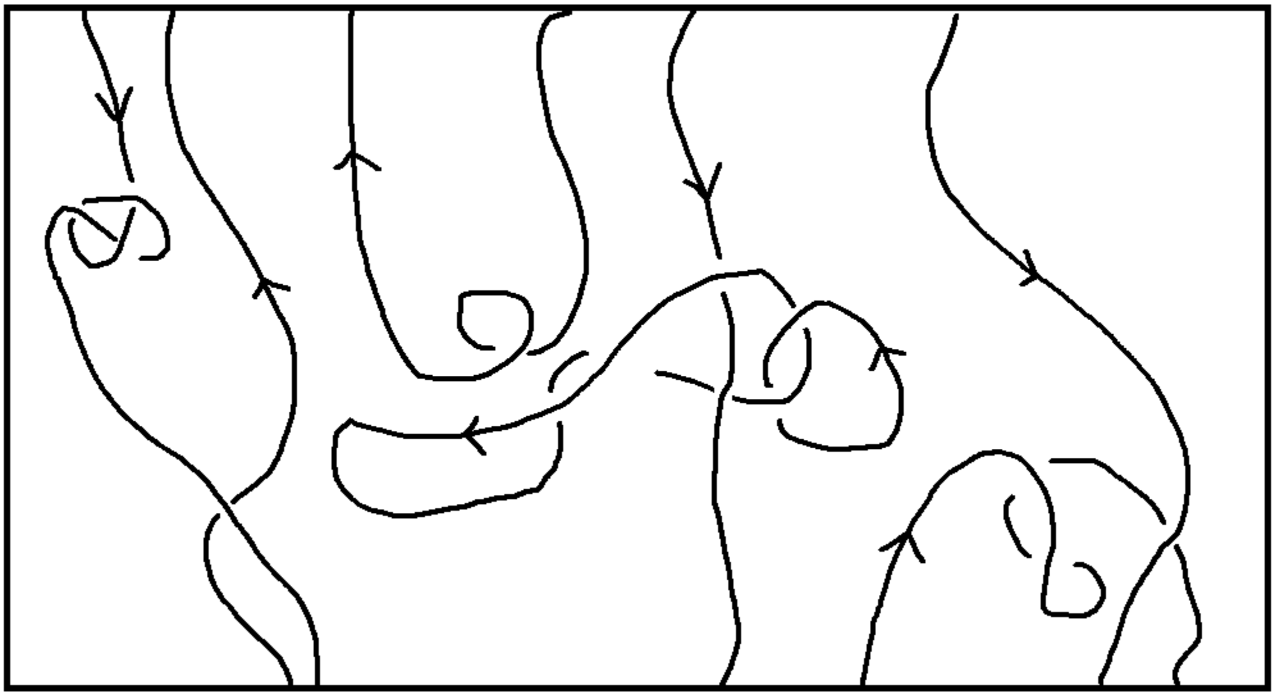


FIGURE 3. A tangle diagram of a tangle L type $(6,6)$ with $s(L) = (-,+,+,-,+,+)$ and $b(L) = (+,-,-,+,+,+)$.

Given a tangle L of type $(k.\ell)$, we define sequences $s(L)$ and $b(L)$ of $+,-$ signs. We let

$$s(L) = (\epsilon_1, \ldots, \epsilon_k), \;\; b(L) = (\eta_1, \ldots, \eta_\ell),$$

where $\epsilon_i = +$ (resp. $\eta_i = +$) iff the point $(i,0,0)$ (resp. $(i,0,1)$) is an endpoint (resp. origin) of L. Else, we set these coordinates to $-$.

We define the tangle $\uparrow$ up to equivalence by $s(\uparrow) = (-), b(\uparrow) = (-)$. Similarly, we have $\downarrow$ such that $s(\uparrow) = (+), b(\uparrow) = (+)$. We similarly define in the intuitive ways $X_\pm$ from Figure 5.1 as tangles, as well as $\overrightarrow{\cap}$, $\overleftarrow{\cap}$, $\overrightarrow{\cup}$, $\overleftarrow{\cup}$, where arrow direction denotes orientation.

As a special case of tangle, we have a braid.

Definition 6.12. *A braid is is tangle L of type (k,k) for some $k \in \mathbb{Z}_{\geq 0}$ such that $s(L) = b(L) = (+,+,\ldots,+)$. We obtain a link called the closure of a braid by connecting the ith top endpoint with the ith bottom endpoint of the braid.*

From braids one may derive the well-known *braid group* B_n which is a central object of algebraic study.

6.4. **Tangles as a strict tensor category.** We codify the set of tangles as a strict tensor category in a striking but intuitive way. Let $\mathcal{T}$ denote the *tangle category*. The objects of $\mathcal{T}$ are finite sequences of $\pm$ signs (including the empty sequence), and morphisms are tangles, up to equivalence. We let the morphisms of $\mathcal{T}$ be isotopy classes of oriented tangles. The source and target of L are $s(L)$ and $b(L)$, respectively (as from Definition 6.11). The identity elements are defined to be of the form $\downarrow\downarrow \ldots \downarrow$, and these morphisms clearly satisfy the identity axioms. The composition of morphisms is defined in the natural sense as concatenations of tangles. That is, $L \circ L'$ is the tangle formed by placing L on top of L'. It should be clear that with this structure, $\mathcal{T}$ is a category.

Next, we equip $\mathcal{T}$ with a tensor product. Given two sequences in $\mathcal{T}$ $\epsilon = (\epsilon_1, \dots, \epsilon_k)$ and $\epsilon' = (\epsilon_{k+1}, \dots, \epsilon_\ell)$, we let

$$\epsilon \otimes \epsilon' = (\epsilon_1, \dots, \epsilon_k, \epsilon_{k+1}, \dots, \epsilon_\ell).$$

We also set $\varnothing \otimes \epsilon = \epsilon = \epsilon \otimes \varnothing$. We clearly have associativity of the tensor product. On morphisms, if L, L' are isotopy classes of oriented tangles, then $L \otimes L'$ is the isotopy class of the tangle obtained by placing L' to the right of L as in Figure 6.4.

$$\boxed{L} \otimes \boxed{L'} = \boxed{L}\ \boxed{L'}$$

By construction, we then have the following proposition.

Proposition 6.13. *We have $\mathcal{T}$ is a strict tensor category in which the unit I is the empty set $\varnothing$.*

Observe that **endomorphisms in $\mathcal{T}$ are precisely isotopy classes of oriented links.** We now present a presentation of $\mathcal{T}$. The proof is tedious but an easy exercise in geometric manipulation and the use of Reidemeister moves.

Proposition 6.14. *We have that $\mathcal{T}$ is generated by elements*

$$X_\pm, \overrightarrow{\cap}, \overleftarrow{\cap}, \overrightarrow{\cup}, \overleftarrow{\cup}$$

under the relations

$$(\downarrow \cap) \circ (\cup \downarrow) = \downarrow = (\overleftarrow{\cap} \downarrow) \circ (\downarrow \overleftarrow{\cup}),$$

$$(\uparrow \overleftarrow{\cap}) \circ (\overleftarrow{\cup} \uparrow) = \uparrow = (\cap \uparrow) \circ (\uparrow \cup),$$

$$(\cap \uparrow\uparrow) \circ (\uparrow \cap \downarrow\uparrow\uparrow) \circ (\uparrow\uparrow X_\pm \uparrow\uparrow) \circ (\uparrow\uparrow\downarrow \cup \uparrow) \circ (\uparrow\uparrow \cup)$$

$$= (\uparrow\uparrow \overleftarrow{\cap}) \circ (\uparrow\uparrow\downarrow \overleftarrow{\cap} \uparrow) \circ (\uparrow\uparrow X_\pm \uparrow\uparrow) \circ (\uparrow \overleftarrow{\cup} \downarrow\uparrow\uparrow) \circ (\overleftarrow{\cup} \uparrow\uparrow),$$

$$X_+ \circ X_- = X_- \circ X_+ = \downarrow\downarrow,$$

$$(X_+ \downarrow) \circ (\downarrow X_+) \circ (X_+ \downarrow) = (\downarrow X_+) \circ (X_+ \downarrow) \circ (\downarrow X_+),$$

$$(\downarrow \overleftarrow{\cap}) \circ (X_\pm \uparrow) \circ (\downarrow \cup) = \downarrow,$$

$$(\cap \downarrow\uparrow) \circ (\uparrow X_\mp \uparrow) \circ (\uparrow\downarrow \cup) \circ (\uparrow\downarrow \overleftarrow{\cap}) \circ (\uparrow X_\pm \uparrow) \circ (\overleftarrow{\cup} \downarrow\uparrow) = \downarrow\uparrow,$$

$$(\uparrow\downarrow \overleftarrow{\cap}) \circ (\uparrow X_\pm \uparrow) \circ (\overleftarrow{\cup} \downarrow\uparrow) \circ (\cap \downarrow\uparrow) \circ (\uparrow X_\mp \uparrow) \circ (\uparrow\downarrow \cup) = \uparrow\downarrow .$$

A proof may be found in [Kas95].

6.5. **Representations of $\mathcal{T}$.** Let $Vect_f(k)$ be the category of finite-dimensional vector spaces over a field K, with morphisms given by linear maps. Recall the strictification construction of Definition 6.7. When we apply this to $Vect_f(k)$, we call the result $\mathcal{V}$.

Definition 6.15. *A representation of $\mathcal{T}$ is a strict tensor functor from $\mathcal{T}$ to $\mathcal{V}$.*

At first, this definition may seem opaque and irrelevant to our goals. In fact, representations of $\mathcal{T}$ are the central ingredient in the proof of Theorem 5.6. Recall that any link L is an endomorphism of the unit $\varnothing$ in $\mathcal{T}$. Then the image $F(L)$ of L under some representation functor F is a k-linear endomorphism of the unit of $\mathcal{V}$. The unit is the ground field, k. In other words, $F(L)$ is a linear map which is multiplication by a scalar which depends only on the isotopy class of L.

The following definition is a little but obtuse, but we'll see that it turns out to be the right concept.

Definition 6.16. *Let V be a finite-dimensional vector space. An enhanced R-matrix V is a pair (c, μ) where c is an automorphism of $V \otimes V$ which satisfies the Yang Baxter equation, and μ is an automorphism of V satisfying*

$$c(\mu \otimes \mu) = (\mu \otimes \mu)c,$$
$$tr_2(c^{\pm 1}(id_V \otimes \mu)) = id_V,$$
$$(\tau c^{\mp 1})^+(id_{V^*} \otimes \mu)(c^{\pm}\tau)^+(id_{V^*} \otimes \mu^{-1}) = id_{V^* \otimes V}.$$

Theorem 6.17. *Given an enhanced R-matrix (c, μ) on a finite-dimensional vector space V, there exists a unique strict tensor functor F from $\mathcal{T}$ to $\mathcal{V}$ such that $F((+)) = V$, $F((-)) = V^*$, and*

$$F(X_+) = c \quad F(\overrightarrow{\cup}) = \delta_V \quad F(\overleftarrow{\cup}) = (id_{V^*} \otimes \mu^{-1})\delta_{V^*}.$$

Then we necessarily also have

$$F(X_-) = c^{-1} \quad F(\overrightarrow{\cap}) = ev_V \quad F(\overleftarrow{\cap}) = ev_{V^*}(\mu \otimes id_{V^*}).$$

Proof. Let F be a strict tensor functor from $\mathcal{T}$ to $\mathcal{V}$. Set $F((+)) = V$ and $F((-)) = W$, and

$$F(\cup) = b : k \to V \otimes W, \quad F(\overleftarrow{\cup}) = b' : k \to W \otimes V,$$

$$F(\cap) = d : W \otimes V \to k, \quad F(\overleftarrow{\cap}) = d' : V \otimes W \to k,$$

$$F(X_+) = c = c^+, \quad F(X_-) = c^- : V \otimes V \to V \otimes V.$$

By the presentation of Proposition 6.14, we have

$$(id_V \otimes d)(b \otimes id_V) = id_V = (d' \otimes id_V)(id_V \otimes b')$$

$$(\mathrm{id}_W \otimes d')(b' \otimes \mathrm{id}_W) = \mathrm{id}_W = (d \otimes \mathrm{id}_W)(\mathrm{id}_W \otimes b),$$

$$\begin{aligned} &(d \otimes \mathrm{id}_{W \otimes W})(\mathrm{id}_W \otimes d \otimes \mathrm{id}_{V \otimes W \otimes W})(\mathrm{id}_{W \otimes W} \otimes c^{\pm} \otimes \mathrm{id}_{W \otimes W}) \\ &\qquad (\mathrm{id}_{W \otimes W \otimes V} \otimes b \otimes \mathrm{id}_W)(\mathrm{id}_{W \otimes W} \otimes b) \\ = \quad &(\mathrm{id}_{W \otimes W} \otimes d')(\mathrm{id}_{W \otimes W \otimes V} \otimes d' \otimes \mathrm{id}_W)(\mathrm{id}_{W \otimes W} \otimes c^{\pm} \otimes \mathrm{id}_{W \otimes W}) \\ &\qquad (\mathrm{id}_W \otimes b' \otimes \mathrm{id}_{V \otimes W \otimes W})(b' \otimes \mathrm{id}_{W \otimes W}), \end{aligned}$$

$$c^+c^- = c^-c^+ = \mathrm{id}_{V \otimes V},$$

$$(c \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes c)(c \otimes \mathrm{id}_V) = (\mathrm{id}_V \otimes c)(c \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes c),$$

$$(\mathrm{id}_V \otimes d')(c^{\pm} \otimes \mathrm{id}_W)(\mathrm{id}_V \otimes b) = \mathrm{id}_V,$$

$$gh = \mathrm{id}_{V \otimes W}, \quad \text{and} \quad hg = \mathrm{id}_{W \otimes V}$$

where $g : W \otimes V \to V \otimes W$ and $h : V \otimes W \to W \otimes V$ are defined as

$$g = (d \otimes \mathrm{id}_{V\otimes W})(\mathrm{id}_W \otimes c^{\mp} \otimes \mathrm{id}_W)(\mathrm{id}_{W\otimes V} \otimes b)$$

$$h = (\mathrm{id}_{W\otimes V} \otimes d')(\mathrm{id}_W \otimes c^{\pm} \otimes \mathrm{id}_W)(b' \otimes \mathrm{id}_{V\otimes W}).$$

The data $(V, W, b, b', d, d', c, c^-)$ satisfying the above relations are called a *representation data* for $\mathcal{T}$. We postpone the remainder of this proof until after a couple of lemmas. $\square$

Let V, W have bases $\{v_1, \dots, v_m\}$ and $\{w_1, \dots, w_n\}$ with corresponding dual bases $\{v^1, \dots, v^m\}$ and $\{w^1, \dots, w^n\}$. Also, let $b : k \to V\otimes W$, $b' : k \to W\otimes V$, $d : W\otimes V \to k$, and $d' : V\otimes W \to k$. Define matrices B, B', D, D' by

$$b(1) = \sum_{i,j} B_{ij} v_i \otimes w_j,$$

$$b'(1) = \sum_{i,j} B'_{ij} w_i \otimes v_j,$$

$$d(w_i \otimes v_j) = D_{ij},$$

$$d(v_i \otimes w_j) = D'_{ij}.$$

The proof of the following lemma is a simple calculation.

Lemma 6.18. *With the above definitions, we have*

$$(id_V \otimes d)(b \otimes id_V) = id_V \iff BD = 1,$$

$$(d \otimes id_V)(id_V \otimes b) = id_W \iff DB = 1,$$

$$(d' \otimes id_V)(id_V \otimes b') = id_V \iff D'B' = 1,$$

$$(id_V \otimes d')(b' \otimes id_V) = id_W \iff B'D' = 1.$$

Next, we define linear maps $\alpha : W^* \to V$ and $\beta : V^* \to W$ by

$$\alpha(w^j) = \sum_i B_{ij} v_i, \quad \beta(v^j) = \sum_i B'_{ij} w_i.$$

The following is another lemma whose proof is unenlightening computation.

Lemma 6.19. *Assume B, B', D, D' are invertible (and hence that α, β are isomorphisms. Let f be an endomorphism of $V \otimes V$. Then we have*

$$\begin{aligned}(d\otimes \mathrm{id}_{W\otimes W})(\mathrm{id}_W\otimes d\otimes \mathrm{id}_{V\otimes W\otimes W})(\mathrm{id}_{W\otimes W}\otimes f\otimes \mathrm{id}_{W\otimes W})&\\ (\mathrm{id}_{W\otimes W\otimes V}\otimes b\otimes \mathrm{id}_W)(\mathrm{id}_{W\otimes W}\otimes b)&\\ = \quad \tau(\alpha^*\otimes\alpha^*)f^*(\alpha^{-1}\otimes\alpha^{-1})^*\tau,&\end{aligned}$$

$$\begin{aligned}(\mathrm{id}_{W\otimes W}\otimes d')(\mathrm{id}_{W\otimes W\otimes V}\otimes d'\otimes \mathrm{id}_W)(\mathrm{id}_{W\otimes W}\otimes f\otimes \mathrm{id}_{W\otimes W})&\\ (\mathrm{id}_W\otimes b'\otimes \mathrm{id}_{V\otimes W\otimes W})(b'\otimes \mathrm{id}_{W\otimes W})&\\ = \quad \tau(\beta\otimes\beta)f^*(\beta^{-1}\otimes\beta^{-1})\tau,&\end{aligned}$$

$$(\mathrm{id}_V\otimes d')(f\otimes \mathrm{id}_W)(\mathrm{id}_V\otimes b) = \mathrm{tr}_2\Big(f\,(\mathrm{id}_V\otimes\mu)\Big)$$

where $\mu=\alpha(\beta^{-1})^*$,

$$(d\otimes \mathrm{id}_{V\otimes W})(\mathrm{id}_W\otimes f\otimes \mathrm{id}_W)(\mathrm{id}_{W\otimes V}\otimes b) = (\mathrm{id}\otimes\alpha^*)(\tau_{V,V}f)^\times\tau_{V^*,V}(\alpha^{-1}\otimes \mathrm{id})^*,$$

$$(\mathrm{id}_{W\otimes V}\otimes d')(\mathrm{id}_W\otimes f\otimes \mathrm{id}_W)(b'\otimes \mathrm{id}_{V\otimes W}) = (\beta\otimes \mathrm{id}_V)\tau_{V,V^*}(f\tau_{V,V})^\times(\mathrm{id}_V\otimes\beta^{-1}).$$

Proof of Theorem 6.17. Let (c,μ) be an enhanced R-matrix on V. Let

$$b=\delta_V,\;\; b'=(id_{V^*}\otimes\mu^{-1})\delta_{V^*},\;\; d=ev_V,\;\; d'=ev_{V^*}(\mu\otimes id_{V^*}),\;\; c^-=c^{-1}.$$

We show (V,W,b,b',d,d',c,c^-) is a valid representation data.

Set $W=V^*$, $\alpha=id_V$, and $\beta=(\mu^*)^{-1}$. Pick a basis and dual basis $\{v_1,\dots,v_m\}$ and $\{v^1,\dots,v^m\}$, and define a matrix B' by

$$\beta(v^j)=\sum_i B'_{ij}v^i.$$

Since μ and β are isomorphisms, the matrix B' is invertible. Let $D'=B'^{-1}$. By definition b,b',d,d'

$$b(1)=\sum_i v_i\otimes v^i\;\; b'(1)=\sum_{i,j}B'_{ij}v^i\otimes v_j$$

$$d(v^i\otimes v_j)=\delta_{ij}\;\; d'(v_i\otimes v^j)=D'_{ij}.$$

We now prove the relations of a representation data. The first two follow because B,D are the identity.

The third relation follows since the first implies that $f(\mu\otimes\mu)=(\mu\otimes\mu)f$ for $f=c^\pm$. Then taking transposes gives $(\mu^*\otimes\mu^*)f^*=f^*(\mu^*\otimes\mu^*)$, which is like

$$\tau f^*\tau=\tau(\mu^*\otimes\mu^*)^{-1}f^*(\mu^*\otimes\mu^*)\tau.$$

Then the third relation follows by the first two and the definitions of α,β.

The fourth relation holds by definition of c^-, and the fifth as c is an R-matrix. The sixth follows from the second and third.

The final relation reduces by the fourth and fifth to

$$\big((\beta\otimes id_V)\tau_{V,V^*}(c^\pm\tau_{V,V})^\times(id_V\otimes\beta^{-1})\big)\big((id\otimes\alpha^*)(\tau_{V,V}c^\mp)^\times\tau_{V^*,V}(\alpha^{-1}\otimes id_{V^*})\big)=id_{V^*\otimes V}.$$

By filling in the definitions of α and β, we get

$$((\mu^*)^{-1} \otimes id_V)\tau_{V,V^*}(c^{\pm}\tau_{V,V})^{\times}(id_V \otimes \mu^*)(\tau_{V,V}c^{\mp})^{\times}\tau_{V^*,V} = id_{V^*\otimes V},$$

which in turn is equivalent to

$$(id_V \otimes (\mu^*)^{-1})(c^{\pm}\tau_{V,V}) \times (id_V \otimes \mu^*)(\tau_{V,V}c^{\mp})^{\times} = id_{V\otimes V^*}.$$

By taking transposes and recalling that $(f^{\times})^{+} = (f^{+})^{\times} = f^*$, we see that this is equivalent to the third relation. $\square$

Now let (c, μ) be an enhanced R-matrix, and let F be the associated unique strict tensor functor from $\mathcal{T}$ to $\mathcal{V}$. If L is some link in $\mathcal{T}$, then F takes L to an endomorphism of the ground field k. That is, $F(L)$ is a scalar in k. This is precisely the key insight that allows us to establish a scalar-invariant for all links (indexed by $q \in \mathbb{C}$). There is only one last ingredient we need before we finally prove our main theorem. We state the following corollary without proof. It is a consequence of a result of Artin, with more details in [Kas95] and many other places.

Proposition 6.20. *Let $c \in Aut(V \otimes V)$ be a solution to the Yang-Baxter equation. Then, for an $n > 0$, there is a unique group morphism $\rho_n^c : B_n \to Aut(V^{\otimes n})$ such that $\rho_n^c(\sigma_i) = c_i$ for $i = 1, \dots, n-1$.*

Proposition 6.21. *For a braid σ and its closure $\tilde{\sigma}$, we have*

$$F(\sigma) = \rho_n^c(\sigma), \;\; F(\tilde{\sigma}) = tr(\mu^{\otimes n} \circ \rho_n^c(\sigma)).$$

Proof. It suffices to show the first statement on generators σ_i of B_n. We have $\sigma_i = 1_{i-1} \otimes X_+ \otimes 1_{n-i-1}$ in $\mathcal{T}$. Then applying F gives

$$F(\sigma_i) = id_{V^{\otimes(i-1)}} \otimes c \otimes id_{V^{\otimes(n-i-1)}} = \rho_n^c(\sigma).$$

For the second statement, we express $\tilde{\sigma}$ as

$$\tilde{\sigma} = \overleftarrow{\cap}_n \circ (\sigma \uparrow \dots \uparrow) \circ \overrightarrow{\cup}_n$$

where

$$\overleftarrow{\cap}_n = \overleftarrow{\cap} \circ (\downarrow \overleftarrow{\cap} \uparrow) \circ \dots \circ (\downarrow \dots \downarrow \overleftarrow{\cap} \uparrow \dots \uparrow)$$

and

$$\overrightarrow{\cup}_n = (\downarrow \dots \downarrow \overrightarrow{\cup} \uparrow \dots \uparrow) \circ \dots \circ (\downarrow \overrightarrow{\cup} \uparrow) \circ \cup.$$

Then

$$F(\tilde{\sigma}) = F(\overleftarrow{\cap}_n) \circ (F(\sigma) \otimes id_{V^{*\otimes n}}) \circ F(\overrightarrow{\cup}_n).$$

Then the relations of Theorem 6.17 give that

$$F(\overrightarrow{\cup}_n) = \delta_{V^{\otimes n}} \;\; F(\overleftarrow{\cap}_n) = ev_{V^{*\otimes n}} \circ (\mu^{\otimes n} \otimes id_{V^{*\otimes n}}).$$

Then

$$F(\tilde{\sigma}) = ev_{V^{*\otimes n}} \circ \left((\mu^{\otimes n} \circ F(\sigma)) \otimes id_{V^{*\otimes n}} \right) \delta_{V^{\otimes n}},$$

which is the trace of $\mu^{\otimes n} \circ F(\sigma)$. $\square$

7. THE PROOF OF THEOREM 5.6

We are finally ready! Let V_m be a vector space of dimension m over k, with basis $\{v_1, \ldots, v_m\}$. Define c_m acting on $V_m \otimes V_m$ by

$$\begin{aligned} c_m(v_i \otimes v_j) &= \lambda_m q v_i \otimes v_i \quad i = j \\ &= \lambda_m v_j \otimes v_i \quad i < j \\ &= \lambda_m v_j \otimes v_i + \lambda_m (q - q^{-1}) v_i \otimes v_j \quad i > j, \end{aligned}$$

where v_j is a non-zero scalar. This is a special case of the R-matrix in Subsection 4.3, from which we see that c_m is an R-matrix satisfying additionally that

$$\lambda_m^{-1} c_m - \lambda_m c_m^{-1} = (q - q^{-1}) id_{V_m \otimes V_m}.$$

Define the automorphism μ_m acting on V_m by $\mu_m(v_i) = \lambda_m^{-1} q^{-2i+1} v_i$. Then

$$tr(\mu_m) = \frac{1}{\lambda_m q} \frac{q^m - q^{-m}}{q - q^{-1}}.$$

Lemma 7.1. *for $\lambda_m = q^{-m}$, the pair (c_m, μ_m) is an enhanced R-matrix.*

Proof. We prove the three relations of Definition 6.16. The first is clearly true, and the third is just a computation. We prove the second. By computation, we have

$$\begin{aligned} tr_2(c_m(id \otimes \mu_m))(v_i) &= (q^{-2(i-1)} + (1 - q^{-2} \sum_{j<i} q^{-2(j-1)}) v_i \\ &= (q^{-2(i-1)} + 1 - q^{-2(i-1)}) v_i = v_i. \end{aligned}$$

Then $tr_2(c_m(id \otimes \mu_m)) = id_{V_m}$. The same holds when we replace c by its inverse as

$$\begin{aligned} tr_2(c_m^{-1}(id \otimes \mu_m)) &= \lambda_m^{-2} tr_2(c_m(id \otimes \mu_m)) - \lambda_m^{-1}(q - q^{-1}) tr_2(id \otimes \mu_m) \\ &= \lambda_m^{-2} \left(1 - \lambda_m (q - q^{-1}) tr(\mu_m)\right) id_{V^m} \\ &= \lambda_m^{-2} \left(1 - q^{-m}(q - q^{-1})\right) id_{V^m} \\ &= id_{V_m}. \end{aligned}$$

□

Then we achieve the following corollary, where the value on the unknot comes from Proposition 6.21.

Corollary 7.2. *Let $\lambda_m = q^{-m}$. Then there is a unique strict tensor functor $F_{m,q}$ from $\mathcal{T}$ to $\mathcal{V}$ such that $F_{m,q}((+)) = V_m$, $F_{m,q}((-)) = V_m^*$, and*

$$F_{m,q}(\overrightarrow{\cup})(1) = \sum_{i=1}^{m} v_i \otimes v^i, F_{m,q}(\overleftarrow{\cup})(1) = \sum_{i=1}^{m} q^{2i-1-m} v^i \otimes v_i,$$

and $F_{m,q}(X_+) = c_m$. We also have that

$$q^m F_{m,q}(X_+) - q^{-m} F_{m,q}(X_-) = (q - q^{-1}) F_{m,q}(\downarrow\downarrow),$$

and the value of $F_{m,q}$ on the unknot is $tr(\mu_m) = (q^m - q^{-m})/(q - q^{-1})$.

Let F denote the restriction $F_{m,q}$ to oriented links, endomorphisms of the identity in $\mathcal{T}$. Recall that $F(L)$ is then a scalar. By definition of a Conway triple (L_+, L_-, L_0), there exist tangles L_i for $1 \leq i \leq 4$ such that

$$L_+ = L_1 \circ (L_2 \otimes X_+ \otimes L_3) \circ L_4, \quad L_- = L_1 \circ (L_2 \otimes X_- \otimes L_3) \circ L_4,$$

and $L_0 = L_1 \circ (L_2 \otimes \downarrow\downarrow \otimes L_3) \circ L_4$. Since $F_{m,q}$ is a tensor functor, we have

$$q^m F(L_+) - q^{-m} F(L_-) - (q - q^{-1}) F(L_0) = F_{m,q}(L_1) \left(F_{m,q}(L_2) \otimes S \otimes F_{m,q}(L_3) \right) F_{m,q}(L_4),$$

where

$$S = q^m F_{m,q}(X_+) - q^{-m} F_{m,q}(X_-) - (q - q^{-1}) F_{m,q}(\downarrow\downarrow).$$

Since the latter vanishes by our corollary, we have that

$$q^m F(L_+) - q^{-m} F(L_-) = (q - q^{-1}) F(L_0).$$

Then the composition of $\phi_{m,q}$ of F with a diffeomorphism of $\mathbb{R}^3$ onto $\mathbb{R}^2 \times (0,1)$ produces a complex-valued map on links in $\mathbb{R}^3$. Hence we have proven the main theorem of 5.6.

8. Towards a More General Theory

Our theory so far has been very 'hands on'. That is, we have proven precisely what we needed to reach the Jones polynomial existence proof. However, this same machinery has much greater power. A fuller explication may be found in [Kas95; Oht02]. We give enough definitions and results that the reader may begin to appreciate the more general picture.

8.1. Bialgebras and Hopf algebras.

We recall that a *coalgebra* is a structure whose properties are dual to those of an algebra. Specifically, we begin by presenting an alternative definition of an algebra.

Definition 8.1. *An algebra is a triple (A, μ, η) where A is a vector space and $\mu : A \otimes A \to A$ and $\eta : k \to A$ are linear maps satisfying the following commutative diagram axioms (called associativity and unit, respectively).*

$$\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "\mu \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes \mu"] & A \otimes A \arrow[d, "\mu"] \\
A \otimes A \arrow[r, "\mu"] & A
\end{tikzcd}$$

$$\begin{tikzcd}
k \otimes A \arrow[r, "\eta \otimes \mathrm{id}"] \arrow[dr, "\cong"'] & A \otimes A \arrow[d, "\mu"] & A \otimes k \arrow[l, "\mathrm{id} \otimes \eta"'] \arrow[dl, "\cong"] \\
& A &
\end{tikzcd}$$

Furthermore, commutativity is expressed as the following commutative diagram, where $\tau_{A,A}$ refers to the map $A \otimes A \to A \otimes A$ which acts via $(a, a') \mapsto (a', a)$ for $a, a' \in A$.

$$\begin{tikzcd}
A \otimes A \arrow[rr, "\tau_{A,A}"] \arrow[dr, "\mu"'] & & A \otimes A \arrow[dl, "\mu"] \\
& A &
\end{tikzcd}.$$

A morphism of algebras is a linear map $F : (A, \mu, \eta) \to (A', \mu', \eta')$ such that $\mu' \circ (f \otimes f) = f \circ \mu$ and $f \circ \eta = \eta'$.

In a similar vein, we define a coalgebra.

Definition 8.2. *A coalgebra is a triple (C, Δ, ϵ) where C is a vector space and $\Delta : C\omega C \otimes C$ and $\eta : C \to k$ are linear maps satisfying the following commutative diagram axioms (called coassociativity and counit, respectively).*

$$\begin{tikzcd}
C \arrow[r, "\Delta"] \arrow[d, "\Delta"] & C \otimes C \arrow[d, "\mathrm{id}\otimes\Delta"] \\
C \otimes C \arrow[r, "\Delta\otimes\mathrm{id}"] & C \otimes C \otimes C
\end{tikzcd}$$

$$\begin{tikzcd}
k \otimes C & C \otimes C \arrow[l, "\varepsilon\otimes\mathrm{id}"'] \arrow[r, "\mathrm{id}\otimes\varepsilon"] & C \otimes k \\
& C \arrow[ul, "\cong"] \arrow[u, "\Delta"] \arrow[ur, "\cong"'] &
\end{tikzcd}$$

Furthermore, cocommutativity is expressed as the following commutative diagram, where $\tau_{C,C}$ refers to the map $C \otimes C \to C \otimes C$ which acts via $(c, c') \mapsto (c', c)$ for $c, c' \in C$.

$$\begin{tikzcd}
& C \arrow[dl, "\Delta"] \arrow[dr, "\Delta"] & \\
C \otimes C \arrow[rr, "\tau_{C,C}"] & & C \otimes C
\end{tikzcd}.$$

A morphism of coalgebras is a linear map $F : (C, \Delta, \epsilon) \to (C', \Delta', \epsilon')$ such that $(f \otimes f) \circ \Delta = \Delta' \circ f$ and $\epsilon = \epsilon' \circ f$.

A bialgebra is a vector space endowed simultaneously with algebra and coalgebra structures, such that either of the two following equivalent conditions is met.

(1) The maps μ, η are coalgebra morphisms.
(2) The maps Δ, ϵ are algebra morphisms.

Equivalence comes easily by writing the commutative diagrams associated with each and noting that they are the same.

Given a bialgebra CH, we define the convolution operator $\star$ on linear maps f, g from H to itself. Let $f \star g$ be the convolution

$$H \xrightarrow{\Delta} H \otimes H \xrightarrow{f \otimes g} H \otimes H \xrightarrow{\mu} H.$$

Definition 8.3. *Let $(H, \mu, \eta, \Delta, \epsilon)$ be a bialgebra. An endomorphism S of H is called an antipode if $S \star id_H = id_H \star S = \eta \circ \epsilon$. If a bialgebra has an antipode, it is called a Hopf algebra.*

We might ask whether a Hopf algebra can have multiple antipodes - the answer is no. If S, S' are entipodes, then

$$S = S \star (\eta\epsilon) = S \star (id_H \star S') = (S \star id_H) \star S' = (\eta\epsilon) \star S' = S'.$$

We know that U_q is an algebra in the earlier defined way. We also equip it with maps Δ, ϵ such that

$$\begin{gathered} \Delta(E) = 1 \otimes E + E \otimes K, \ \ \Delta(F) = K^{-1} \otimes F + F \otimes 1, \\ \Delta(K) = K \otimes K, \ \ \Delta(K^{-1}) = K^{-1} \otimes K^{-1}, \\ \epsilon(E) = \epsilon(F) = 0, \ \ \epsilon(K) = \epsilon(K^{-1}) = 1. \end{gathered}$$

We also let

$$S(E) = -EK^{-1}, \ \ S(F) = -KF, \ \ S(K) = K^{-1} \ \ S(K^{-1}) = K.$$

Proposition 8.4. *With the above maps Δ, ϵ, U_q is a Hopf algebra.*

Proof. First, we show that Δ defined an algebra morphism from U_q to $U_q \otimes U_q$. It suffices to show the following relations

$$\begin{gathered} \Delta(K)\Delta(K^{-1}) = \Delta(K^{-1})\Delta(K) = 1, \\ \Delta(K)\Delta(E)\Delta(K^{-1}) = q^2\Delta(E), \\ \Delta(K)\Delta(F)\Delta(K^{-1}) = q^{-2}\Delta(F), \\ [\Delta(E), \Delta(F)] = \frac{\Delta(K) - \Delta(K^{-1})}{q - q^{-1}}. \end{gathered}$$

The first is clear by definition of K. For the second we have

$$\begin{aligned} \Delta(K)\Delta(E)\Delta(K^{-1}) &= (K \otimes K)(1 \otimes E + E \otimes K)(K^{-1} \otimes K^{-1}) \\ &= 1 \otimes KEK^{-1} + KEK^{-1} \otimes K \\ &= q^2(1 \otimes E + E \otimes K) \\ &= q^2\Delta(E). \end{aligned}$$

The third relation follows similarly. For the final, we have

$$\begin{aligned} [\Delta(E), \Delta(F)] &= (1 \otimes E + E \otimes K)(K^{-1} \otimes F + F \otimes 1) - (K^{-1} \otimes F + F \otimes 1)(1 \otimes E + E \otimes K) \\ &= K^{-1} \otimes EF + F \otimes E + EK^{-1} \otimes KF + EF \otimes K - K^{-1} \otimes FE \\ &\quad - K^{-1}E \otimes FK - F \otimes E - FE \otimes K \\ &= K^{-1} \otimes [E, F] + [E, F] \otimes K \\ &= \frac{K^{-1} \otimes (K - K^{-1}) + (K - K^{-1}) \otimes K}{q - q^{-1}} \\ &= \frac{\Delta(K) - \Delta(K^{-1})}{q - q^{-1}}. \end{aligned}$$

Next, we show that Δ is coassociative. As usual, it suffices to show on generators, but since there are four that is many calculations. We give a sample, noting that they are all similar; we have

$$(\Delta \otimes id)\Delta(E) = 1 \otimes 1 \otimes E + 1 \otimes E \otimes K + E \otimes K \otimes K = (id \otimes \Delta)\Delta(E).$$

It is easy to check even more simply than above that ϵ defines an algebra morphism from U_q to k satisfying the counit axiom diagram.

For a proof that the given S is an antipode, see [Kas95]. The proof - while not especially complicated - would require further development of the theory of Hopf algebras than makes sense to do here. □

Interestingly, we see that U_q is a Hopf algebra which is neither commutative or cocommutative.

Definition 8.5. *Let $(H, \mu, \eta, \Delta, \epsilon)$ be a bialgebra. We call is quasi-cocommutative if there is an invertible element $R \in H \otimes H$ such that for all $x \in H$, we have*

$$\Delta^{op}(x) = R\Delta(x)R^{-1},$$

where $\Delta^{op} = \tau_{H,H} \circ \Delta$. We call R a universal R-matrix on H.

We sometimes abbreviate $\tau_{H,H}$ as τ when context is clear.

Furthermore, we define the following notation. Suppose that $R = \sum_i x_i^{(1)} \otimes x_i^{(2)} \otimes x_i^{(3)}$. Then let R_{ab} for $a, b \in \{1, 2, 3\}$ be equal to $\sum_i y_i^{(1)} \otimes y_i^{(2)} \otimes y_i^{(3)}$ where $y_i^{k_j} = x_i$ for any $j \leq p$ and $y_i^{(k)} = 1$ otherwise. For example, if $R = \sum_i s_i \otimes t_i$, then

$$R_{31} = \sum_i t_i \otimes 1 \otimes s_i.$$

Definition 8.6. *A quasi-cocommutative bialgebra $(H, \mu, \eta, \Delta, \epsilon, R)$ is braided if*

$$(\Delta \otimes id_H)(R) = R_{13}R_{23}, \;\; (id_H \otimes \Delta)(R) = R_{13}R_{12}.$$

We say a quasi-cocommutative Hopf algebra is braided if the same relations are satisfied.

We have the following central result about braided bialgebras.

Theorem 8.7. *Let $(H, \mu, \eta, \Delta, \epsilon, R)$ be a braided bialgebra. Then we have*

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12},$$

$$(\epsilon \otimes id_H)(R) = 1 = (id_H \otimes \epsilon)(R).$$

Furthermore given an invertible antipode S, we have

$$(S \otimes id_H)(R) = R^{-1} = (id_H \otimes S^{-1})(R), \;\; (S \otimes S)(R) = R.$$

Proof. Let $R = \sum_i s_i \otimes t_i$. Then the above relations which we seek to prove are equivalent to

$$\sum_{i,j,k} s_k s_j \otimes t_k s_i \otimes t_j t_i = \sum_{i,j,k} s_j s_i \otimes s_k t_i \otimes t_k t_j,$$

$$\sum_i \epsilon(s_i) t_i = \sum_i s_i \epsilon(t_i) = 1,$$

$$R^{-1} = \sum_i S(s_i) \otimes t_i = \sum_i s_i \otimes S^{-1}(t_i).$$

The first relation of Definition 8.6 imply that

$$\begin{aligned}
R_{12}R_{13}R_{23} &= R_{12}(\Delta \otimes id)(R) \\
&= (\Delta^{op} \otimes id)(R)R_{12} \\
&= (\tau \otimes id)(\Delta \otimes id)(R)R_{12} \\
&= (\tau \otimes id)(R_{13}R_{23})R_{12} \\
&= R_{23}R_{13}R_{12}.
\end{aligned}$$

Since $(\epsilon \otimes id)\Delta = id$, we then have

$$R = (\epsilon \otimes id \otimes id)(\Delta \otimes id)(R) = (\epsilon \otimes id \otimes id)(R_{13}R_{23}) = (\epsilon \otimes id)(R)\epsilon(1)R.$$

Since $\epsilon(1) = 1$ and R is invertible, then we have $(\epsilon \otimes id)(R) = 1$. Similarly, we have $(id \otimes \epsilon)(R) = 1$.

Now suppose that S is an invertible antipode. We know then that

$$(\mu \otimes id)(S \otimes id \otimes id)(\Delta \otimes id)(R) = (\epsilon \otimes id)(R) = 1.$$

Then

$$1 = (\mu \otimes id)(S \otimes id \otimes id)(R_{13}R_{23}) = (S \otimes id)(R)S(1)R.$$

As $S(1) = 1$, then $(S \otimes id)(R) = R^{-1}$.

If we now instead consider the braided Hopf algebra $(H, \mu, \eta, \Delta^{op}, \epsilon, S^{-1}, S, \tau(R))$. The proof that this is indeed a braided Hopf algebra follows from definitions. Then by the same logic as above, we get

$$(S^{-1} \otimes id)(\tau(R)) = \tau(R)^{-1},$$

which is equivalent to $(id \otimes S^{-1})(R) = R^{-1}$. Then finally, we have

$$\begin{aligned}
(S \otimes S)(R) &= (id \otimes S)(S \otimes id)(R) \\
&= (id \otimes S)(R^{-1}) \\
&= (id \otimes S)(id \otimes S^{-1})(R) \\
&= (id \otimes id)(R) \\
&= R.
\end{aligned}$$

□

But what's the point of all of the structure? The key is that *Universal R-matrices generate R-matrices.* More specifically, we show the existence of a solution to the Yang-Baxter equation on every module V of a braided bialgebra $(H, \mu, \eta, \Delta, \epsilon, R)$. We define $\check{R} = \tau R$. For $R = \sum_i s_i \otimes t_i$, we have

$$\check{R}(v \otimes v') = \sum_i t_i v' \otimes s_i v.$$

By Theorem 8.7, $\check{R}$ is an isomorphism with inverse given by

$$\check{R}^{-1}(v \otimes v') = R^{-1}(v' \otimes v)$$

for $v, v' \in V$.

Proposition 8.8. *We have that $\check{R}$ is an isomorphism of H-modules, and $\check{R}$ solves the Yang Baxter equation over V.*

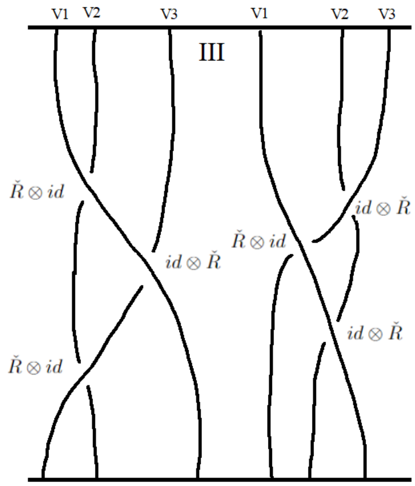


FIGURE 4. Since $\check{R}$ satisfies Yang-Baxter, the following two compositions are equal, which matches the topological equivalence of the associated braids.

Proof. Since we know $\check{R}$ is invertible, we need to show it is H-linear. For $x \in H$, we have

$$\begin{aligned}\check{R}(x(v\otimes v')) &= \tau(R\Delta(x)(v\otimes v'))\\ &= \tau(\Delta^{op}(x)R(v\otimes v'))\\ &= \Delta(x)\tau(R(v\otimes v'))\\ &= x(\check{R}(v\otimes v'))\end{aligned}$$

Now we prove that $\check{R}$ is an R-matrix. In sum notation we have that

$$(\check{R}\otimes id)(id\otimes\check{R})(\check{R}\otimes id)(v\otimes v'\otimes v'') = \sum_{i,j,k} t_k t_j v'' \otimes s_k t_i v' \otimes s_j s_i v.$$

We also have

$$(id\otimes\check{R})(\check{R}\otimes id)(id\otimes\check{R})(v\otimes v'\otimes v'') = \sum_{i,j,k} t_j t_i v'' \otimes t_k s_i v' \otimes s_k s_j v.$$

The two right hand sides are equal in lieu of the beginning of the proof of Theorem 8.7. ☐

Conversely, it can be shown that any R-matrix arises in this way from a braided bialgebra, via what is known as the FRT construction after its creators Fadeev, Reshetikhin, and Takhtadjian. The universal R-matrices thus give us operators $\check{R}$ which satisfy a braid relation, which also may be considered a Reidemseister type III move (a heuristic illustration is provided below). A Reidemeister type II move corresponds to peforming $\check{R}$ and then $\check{R}$ inverse, vs performing no operation at all, and hence also provides an equivalence. A type I move is similarly seen to be a topological invariant in this framework. We thus understand now in more conceptual terms how our machinery provides us with topological data.

In earlier sections, we worked hard to find useful R-matrices on the representation V_1 of U_q, and we then used these Yang Baxter solutions for the Jones-Conway polynomial. Now, we see that all such R-matrices are actually derived easily from the *universal* R-matrices of U_q. This is certainly a more elegant, universal, and generalizable approach, but it is nonetheless still difficult. In particular, it requires us to construct the universal R-matrices for U_q. Doing so falls beyond the scope of these notes. However, armed with these powerful tools, one may significantly improve the knot-theoretic results. Firstly, one reaches new invariants by using different R-matrices, following much the same line of thought as we have. But further, tools of higher-dimensional representations lead us to understand commensurably higher-dimensional object such as 2-knots (which lie naturally in $\mathbb{R}^4$ rather than $\mathbb{R}^3$). Via other constructions than the tangle representation, we may also learn about the structures of other low-dimensional manifolds, which are otherwise finicky topological objects that evade simple analysis.

9. Conclusion

I hope you enjoyed these lecture notes (which admittedly surpass the accompanying lectures in detail and scope)! I know they are long, but hopefully you got something out of them and have engaged with them at a level of detail that you find worthwhile and fun. There's lots more material to be found in my sources below, should you choose to continue learning. The books [G B03] and [Lic97] focus on knot theory, while [Kas95] gives an overview of quantum groups and their applications to knots and [Oht02] does the same with more emphasis on 3-manifolds. I thank Professor Igor Frenkel for advising me and aiding me with this material.

References

[G B03] H. Zieschang G. Burde. *Knots*. Graduate Texts in Mathematics. Springer International Publishing, 2003.

[Kas95] K. Kassel. *Quantum Groups*. de Gruyter Studies in Mathematics. de Gruyter International Publishing, 1995.

[Lic97] W. B. R. Lickorish. *An Introduction to Knot Theory*. Graduate Texts in Mathematics. Springer International Publishing, 1997.

[Oht02] T. Ohtsuki. *Quantum Invariants: A Study of Knots, 3-Manifolds, and Their Sets*. World Scientific, 2002.