THE LOCAL LANGLANDS CORRESPONDENCE FOR $\mathrm{GL}_n(K)$

CONTENTS

1. INTRODUCTION

1.1. **The purpose and structure of this essay.** In this essay, we provide an introduction to the statement of the local Langlands correspondence for $\mathrm{GL}_n(K)$, which we state below in Theorem 2.2. Very broadly, the correspondence is a bijective matching between certain representations of $\mathrm{GL}_n(K)$ for a local field K, and certain n-dimensional representations of the Weil group of K which preserves important structure of both sides. As we explain below in Section 2.2, the local correspondence is a natural generalization of local class field theory - in fact, the $n = 1$ case of the correspondence is precisely local class field theory. **The primary goal of this essay is the explain the statement of the local correspondence and to give some insight into the nature of the objects involved.**

Equally important is what this essay is not, since the theory underlying the local correspondence is broader and deeper than this essay can be. First, we do not present anything in the way of a proof of the correspondence. This essay is also not a complete introduction to the motivations and subtleties of all objects involved. Several technical lemmas are not proven here, and much broader theory goes unaddressed (for example, ℓ-adic representations and monodromy, or any extension to general reductive groups).

After stating the correspondence theorem and explaining how it generalizes local class field theory, we spend some time with the objects on each side of the correspondence. On the $\mathrm{GL}_n(K)$ side, we first define an *admissible representation* and give some properties. We err on the side of more detail at the start of this section in order to get the reader comfortable with the basic objects. We then give a statement and a broad sketch of the (quite lengthy) proof of the Bernstein-Zelevinsky classification, which classifies irreducible admissible representations in terms of building blocks called *supercuspidal representations*. Finally, we define the *Whittaker Model* and the L and ϵ-factors, which characterize the correspondence. We broadly follow Wedhorn's exposition in [**We**], except when discussing the Bernstein-Zelevinsky classification, which we draw moreso from Rodier's exposition in [**Ro**]. There are, of course, also many additional details, clarifications, examples, commentaries, changes in organization, etc... original to the author of this essay (as there are in all its section) or from other sources (cited when relevant).

On the Galois side of the correspondence, we follow [**We**] in defining Weil-Deligne representations of the Weil group as well as the L and ϵ-factors associated to objects on this side of the correspondence. Rather than attempting to fully grasp the general case, we follow John Tate's 1950 thesis in [**Ta1**] to understand the abelian case, which we can do much more explicitly (and which presents some very nice theory in its own right). Additional number theoretic review for this section can be found in [**Ta2**].

1.2. **Assumed prerequisites.** We assume some familiarity with local class field theory and with basic representation theory (roughly a full first course, notably including Frobenius reciprocity). Other areas like Fourier analysis and basic category theory are also useful, although we recall some definitions for the reader.

2. Motivation and statement

2.1. **Some notation.** Throughout the essay we maintain some standard notation.

- p is a prime number in $\mathbb{Z}$.
- K denotes a local field of mixed characteristic (a finite extension of Q_p).
- v_K is the discrete valuation associated to K.
- $\mathcal{O}_K$ is the ring of integers of K, with uniformizer π_K (sometimes abbreviated π). Unfortunately, π often also denotes a representation, but the difference is clear from context.
- k is the residue field of K, and q is its cardinality. For a field L, the residue field is denoted k_L.
- $|\cdot|_K$ is the absolute value of K, normalized in the standard way such that $|x|_K = q^{-v_K(x)}$.
- $\overline{K}$ denotes an algebraic closure of K, and $\overline{k}$ the accompanying algebraic closure of k.
- K^{un} denotes the maximal unramified extension of K, with residue $\overline{k}$.
- K^{ab} denotes the maximal abelian extension of K.
- W_K denotes the absolute Weil group of K.

All representations are assumed to be complex unless specified otherwise.

2.2. **Local class field theory.** We briefly recall local class field theory. For any finite extension L/K, $\mathrm{Gal}(k_L/k)$ is an extension of finite field and thus generated by the Frobenius map $x \mapsto x^q$. For L/K unramified, the natural surjective map res : $\mathrm{Gal}(L/K) \to \mathrm{Gal}(k_L/k)$ is an isomorphism, and hence $\mathrm{Gal}(L/K)$ is generated by $\mathrm{res}^{-1}\mathrm{Fr}$ (often also called Fr). Now letting L/K be any (possible infinite) Galois extension, the Weil group $W(L/K)$ is defined as $\mathrm{res}^{-1}(\langle \mathrm{Fr} \rangle)$. It follows that $W_K = \mathrm{Gal}(L/K)$ when the extension is finite. Furthermore, we endow W_K with a topology such that the inertia group $I_{L/K} \coloneqq \ker(\mathrm{res})$ is an open subgroup, where $I_{L/K}$ inherits its profinite subspace topology from $I_{L/K} \subset \mathrm{Gal}(L/K)$. We have that W_K is dense in $\mathrm{Gal}(L/K)$. Before we state local class field theory, recall that since $\mathrm{Gal}(K^{\mathrm{un}}/K) \simeq \hat{\mathbb{Z}}$ where $\hat{\mathbb{Z}}$ is the profinite integers, then K^{un}/K is an abelian extension, and so $K^{\mathrm{un}} \subseteq K^{\mathrm{ab}}$. Note also that $\mathrm{Gal}(K^{\mathrm{ab}}/K) = \mathrm{Gal}(\overline{K}/K)^{\mathrm{ab}}$.

Theorem 2.1 (Local Artin Reciprocity). *There is a unique topological isomorphism*

$$Art_K : K^* \to W(K^{ab}/K)$$

such that $Art_K(\pi)|_{K^{un}} = Fr_{K^{un}/K}$, *and for all finite extensions* L/K, $Art_K(N_{L/K}(L^*))|_L = \{1\}$, *where* $N_{L/K}$ *denotes the norm from* L *to* K. *Furthermore, for all finite extensions* L/K, Art_K *induces an isomorphism*

$$K^*/N_{L/K}(L^*) \simeq W(K^{ab}/K)/W(K^{ab}/L) \simeq Gal(L/K).$$

Recall that under Art_K, $\mathcal{O}_K^*$ maps onto the inertia group $I_{K^{\mathrm{ab}}/K} = I_K^{\mathrm{ab}}$, where I_K denotes $I_{\overline{K}/K}$. (We similarly let W_K denote $W_{\overline{K}/K}$.)

Reformulating Theorem 2.1 gives a natural path towards the local Langlands correspondence. Let $\mathcal{A}_1(K)$ denote the irreducible complex representations (π, V) of $K^* = \mathrm{GL}_1(K)$ such that the stabilizer of any $v \in V$ is open in K, up to isomorphism. As we see later on in Proposition 3.6, this implies that $\mathcal{A}_1(K)$ is one-dimensional. So we can think of $\mathcal{A}_1(K)$ as the set of homomorphisms from $K^* \to \mathbb{C}^*$ such that the preimage of any point is open; equivalently, $\mathcal{A}_1(K)$ is the set of continuous homomorphisms $K^* \to \mathbb{C}^*$ where $\mathbb{C}^*$ has the discrete topology.

Now let $\mathcal{G}_1(K)$ denote the set of continuous maps $W_K \to \mathbb{C}^* = \mathrm{GL}_1(\mathbb{C})$ for $\mathbb{C}^*$ with the Euclidean topology. We notice that such a map $W_K \mapsto GL_1(\mathbb{C})$ is continuous precisely when the restriction to I_K is continuous, simply by the definition of the topology on W_K. But recall that I_K is a profinite group (i.e., it is compact and totally disconnected); then its image under a continuous map is compact and totally disconnected set in $\mathbb{C}^*$. That is, I_K must have finite image, and so continuity of $W_K \to \mathbb{C}^*$ with respect to the Euclidean topology is precisely continuity with respect to the discrete topology. Then Artin Reciprocity provides a unique natural bijection between $\mathcal{A}_1(K)$ and $\mathcal{G}_1(K)$! In fact, we notice that the existence of such a bijection is equivalent to our original statement of local class field theory.

2.3. **The main statement of the local correspondence.** We now state the main theorem of this essay. Many definitions in the main theorem go unexplained for now, and it is the main purpose of this essay to define them and understand what they are. Above, we rather suggestively defined $\mathcal{A}_1(K)$ and $\mathcal{G}_1(K)$ using GL_1. We now let $\mathcal{A}_n$ denote all irreducible admissible representations of $\mathrm{GL}_n(K)$ up to isomorphism, and let $\mathcal{G}_n(K)$ denote Frobenius semisimple n-dimensional Weil-Deligne representations of W_K, also up to isomorphism.

Theorem 2.2 (Local Langlands Correspondence for $\mathrm{GL}_n(K)$)**.** *There is a unique collection of bijections $rec_n : \mathcal{A}_n(K) \to \mathcal{G}_n(K)$ such that*

(1) for $\pi \in \mathcal{A}_1(K)$, we have $rec_1(\pi) = \pi \circ Art_K^{-1}$;

(2) for $\pi_1 \in \mathcal{A}_{n_1}(K)$ and $\pi_2 \in \mathcal{A}_{n_2}(K)$, we have

$$L(\pi_1 \times \pi_2, s) = L(rec_{n_1}(\pi_1) \otimes rec_{n_1}(\pi_2), s)$$
$$\epsilon(\pi_1 \times \pi_2, s, \psi) = \epsilon(rec_{n_1}(\pi_1) \otimes rec_{n_1}(\pi_2), s, \psi),$$

where ψ is a nontrivial additive character of K;

(3) for $\pi \in \mathcal{A}_n(K)$ and $\chi \in \mathcal{A}_1(K)$, we have

$$rec_n(\pi\chi) = rec_n(\pi) \otimes rec_1(\chi);$$

(4) for $\pi \in \mathcal{A}_n(K)$ with central character ω_π, we have

$$\det \circ rec_n(\pi) = rec_1(\omega_\pi);$$

(5) for $\pi \in \mathcal{A}_n(K)$, we have $rec_n(\pi^\vee) = rec_n(\pi)^\vee$ where $(\cdot)^\vee$ denotes the contragradient.

Furthermore, rec_n is independent of choice of ψ.

In the course of this essay, we will see that each side of the correspondence comes with its own basic 'building block' objects. These are called *supercuspidal representations* on the GL_n side and *irreducible Weil-Deligne representations* on the Galois side. The following theorem tells us that these match up precisely under the correspondence.

Theorem 2.3. *Let π be an irreducible admissible representation of $GL_n(K)$ and let $\rho = (r, N)$ be the Weil-Deligne representation associated to π under Theorem 2.2. Then*

(1) the representation π is supercuspidal if and only if ρ is irreducible, and

(2) the representation π is essentially square integrable if and only if ρ is indecomposable.

3. The GL_n side

3.1. **Admissible representations.** We endow $\mathrm{GL}_n(K)$ with the natural product topology coming from n^2 copies of K. Observe that the sets $1 + \pi^m \mathcal{O}_K$ form a fundamental system of open and compact neighborhoods of the identity, where 1 is the identity matrix. The set is open as each point $x \in 1 + \pi^m \mathcal{O}_K$ has an open neighborhood which is the set of matrices which differ from x component-wise by an element of K of large valuation. The set is compact as $\mathcal{O}_K$ is compact. For all $m \geq 1$, $1 + \pi^m \mathcal{O}_K \subset \mathrm{GL}_n(\mathcal{O}_K)$, and $\mathrm{GL}_n(\mathcal{O}_K)$ is a maximal compact open subgroup.

Definition 3.1. *A representation $(\pi, V) : GL_n(K) \to V$ is called smooth if the stabilizer of each $v \in V$ is open in $GL_n(K)$. If furthermore the invariant space V^H for every open subgroup $H \subset G$ is finite dimensional, we say π is admissible. The space of admissible representations up to isomorphism is denoted $\mathcal{A}_n(K)$.*

The definition of smooth and admissible representations may seem opaque at first, but there are several good reasons to define them this way. One reason which is difficult to see now is that admissible representations satisfy the local Langlands correspondence of Theorem 2.2, but this is for now an unenlightening explanation. Let's give some additional motivation, keeping in mind that it will make more sense as we go along. First of all, we can see immediately that they have nice properties. For example, we have the following.

Definition 3.2. *For (π, V) a smooth representation of G, let V^* denote the standard dual space of V, and recall that it is naturally a G-module by $(g\lambda)(v) = \lambda(g^{-1}v)$. Let*

$$V^\vee = \{\lambda \in V^* | \mathit{Stab}_G(\lambda) \text{ is open}\},$$

which by construction is smooth. We call $V^\vee$ the contragradient, and we refer to $V^\vee$ with its module structure by $\pi^\vee$.

We notice that V and $(V^\vee)^\vee$ are isomorphic as G-modules under restriction of the standard module isomorphism $(V^*)^* \simeq V$. Observe that π is admissible if and only if $\pi^\vee$ is admissible. By a theorem of Gelfand and Kazhdan found in [**BZ1**] 7.3, we have that for π smooth and irreducible, $\pi^\vee$ is isomorphic to the representation $g \to \pi({}^t g^{-1})$ for $g \in G$. In general, this biduality is our first hint that admissible representations are worth considering.

In Section 3.2, we will see that the set of irreducible admissible representations has a surprisingly simple classification in terms of building blocks called supercuspidal representations. In the Langlands correspondence of Theorem 2.2, these supercuspidals correspond precisely to irreducible representations on the Galois side (see Theorem 2.3). Irreducible representations and the way that they underlie all representations is of course a very natural structure to consider, and the analogous system of supercuspidals and the admissible representations they generate is the natural analogue on the $\mathrm{GL}_n(K)$-side.

We will see some examples of admissible representations in Example 3.10, when we discuss induction of representations and the Steinberg representation of $\mathrm{GL}_2(K)$. For readers already familiar with normalized parabolic induction, notice that the induced representations from admissible representations are also admissible, more evidence that $\mathcal{A}_n(K)$ is a nice object. We discuss this more in Section 3.2, but for now we continue on to investigate the structure of admissible representations.

Notice that any open subgroup $H \subset \mathrm{GL}_n(K)$ contains a compact open subgroup since for $x \in H$, we can take $x + \pi_K^m \mathrm{GL}_n(\mathcal{O}_K)$ for large enough m. Then if π is smooth, the stabilizer of any $v \in V$ is open, and so in particular $v \in V^C$ for some open compact subgroup $C \subset G$. Conversely, if $V = \cup_C V^C$ over compact open subgroups $C \subset G$, then the stabilizer of each $v \in V$ is the union over all C such that $v \in V^C$, and hence π is smooth. That is, we have the following lemma.

Lemma 3.3. *A representation π is smooth if and only if $V = \bigcup_C V^C$. Furthermore, π is admissible if all V^C are finite dimensional.*

We have proven the first part, and the second follows easily from Definition 3.1.

It turns out that in order to grasp the structure of admissible representations, we should really examine a different object, called the Hecke algebra, which is easier to understand. Despite how different it looks, we see below that non-degenerate representations the Hecke algebra are actually equivalent as a category to the admissible representations of $\mathrm{GL}_n(K)$. Intuitively, this allows us to shift the complexity of our situation away from the multiple conditions of Definition 3.1 and into the Hecke algebra object itself, which we can then study with standard representation theoretic techniques. Additionally, the language of the Hecke algebra lends itself to cleaner definitions of some objects which appear in Theorem 2.2, particularly the central character ω_π.

To simplify notation, we let $G = \mathrm{GL}_n(K)$. The Hecke algebra $\mathcal{H}(G)$ has underlying vector space the collection of locally constant, compactly supported measures ϕ on G with complex coefficients, with product given by convolution. More concretely, if we pick a Haar measure on G (which we recall is unique up to scaling by a constant), then we can identify $\mathcal{H}(G)$ with the collection of locally constant, compactly supported functions $G \to \mathbb{C}$, with product

$$(f_1 * f_2)(h) = \int_G f_1(hg^{-1}) f_2(g) dg.$$

For C a compact open subgroup of G, we let $\mathcal{H}(G//C)$ be the subalgebra of $\mathcal{H}(G)$ of those measures which are both left and right invariant under C. Under the interpretation of $\mathcal{H}(G)$ as functions from G to $\mathbb{C}$, we identify $\mathcal{H}(G//C)$ as the set of maps on the double quotient $C\backslash G/C \to \mathbb{C}$ with finite support. Notice that for $\mathrm{vol}(C) \coloneqq \int_G 1_C dg$, we have

$$(f * 1_C)(h) = \int_G f(hg^{-1}) 1_C(g) dg = \mathrm{vol}(C) f(h),$$

and so $e_C \coloneqq \mathrm{vol}(C)^{-1} 1_C$ is a unit in $\mathcal{H}(G//C)$. Suppose that $C' \subset C$ are two open compact subgroups of G. Then $\mathcal{H}(G//C)$ is a $\mathbb{C}$-subalgebra of $\mathcal{H}(G//C')$. Note, however, that the unit elements e_C and $e_{C'}$ differ by a scale factor.

For any $\phi \in \mathcal{H}(G)$, there is some maximal collection of disjoint open sets $H_i \subset G$ on which ϕ is locally constant and non-zero. By compactness of the support of ϕ, this collection is finite. Each H_i has a stabilizer in G, which is open as the preimage of of H under the multiplication map $G \times H \to G$. We may take a compact open subgroup C_i of that stabilizer. Intersecting all C_i gives a compact subgroup C under which ϕ is invariant. Then $\phi \in \mathcal{H}(G//C)$. Then we have

$$\mathcal{H}(G) = \bigcup_C \mathcal{H}(G//C).$$

The above should remind us of the similar condition of Lemma 3.3 which characterizes smooth representations. In fact, if (π, V) is a smooth representation of G, then V becomes an $\mathcal{H}(G)$-module under the action

$$\pi(\phi)v = \int_G \pi(g) v d\phi \tag{1}$$

for $\phi \in \mathcal{H}(G)$. Notice that by our characterization of $\mathcal{H}(G//C)$, the integral boils down essentially to a finite sum and is hence well-defined. Since $V = \cup_C V^C$, then every $v \in V$ is equal to $\pi(e_C)v$ for some open compact subgroup $C \subset G$. Then $\mathcal{H}(G) \cdot V = V$, or equivalently V is non-degenerate as an $\mathcal{H}(G)$-module. In fact this action of $\mathcal{H}(G)$ on V provides a functor from smooth G-representations to non-degenerate $\mathcal{H}(G)$-modules.

We now sketch a proof that the above functor is actually an equivalence of categories, broadly following [**Ca**]. First, we notice that via Equation (1) which defines the functor, smooth representations of G and $\mathcal{H}(G)$-modules are in bijective correspondence. The following lemma tells us that subrepresentations also naturally correspond.

Lemma 3.4. *For any subspace $V' \subset V$, we have V' is stable under all operators $\pi(g)$ for $g \in G$ if and only if it is stable under all operators $\pi(f)$ for $f \in \mathcal{H}(G)$. Hence, irreducible G-representations correspond to simple $\mathcal{H}(G)$-modules.*

Proof. If V' is stabilized by all $g \in G$, then the integrand of Equation (1) lies in V' for all g, and so is the integral $\pi(\phi)v$ also lies in V'. Conversely, suppose there is $g \in G$ and $v \in V'$ such that $\pi(g)v \notin V'$. Then since π is smooth, the stabilizer H of v is open, and so translating this stabilizer, there is an open set (which can be taken to be compact) where for all $h \in H$, we have $\pi(h)v = w \notin V$. Now we can define a measure ϕ with support contained in this compact open H, and then $\pi(\phi)v$ is a member of the subspace $\langle w \rangle \not\subset V$. ☐

For (π, V) and (π', V'), let $u : V \to V'$ be a linear map. Then it follows easily that $\pi'(g)u = u\pi(g)$ for all $g \in G$ if and only if $\pi'(f)u = u\pi(f)$ for all $f \in \mathcal{H}(G)$. Finally, the equivalence of categories follows from a version of Schur's lemma, which is also interesting in its own right.

Lemma 3.5. *Let (π, V) be an irreducible smooth representation of G. Then every map of representations $u : V \to V$ (i.e., an intertwining map, a map such that $\pi(g)u = u\pi(g)$ for all g in G) is scalar.*

Proof. From our earlier discussion of the topology on G, we know that G has a countable basis consisting of translates of $1 + \pi_K^m \mathrm{GL}_n(\mathcal{O}_K)$. Then for all compact open subgroups $C \subset G$, we have $[G : K]$ is countable. Then $\mathcal{H}(G//C)$ has countable dimension over $\mathbb{C}$ because all $\phi \in \mathcal{H}(G//C)$ vanish outside of some finite set of cosets $C\backslash G/C$. Since there is a countable basis of neighborhoods of the identity, then $\mathcal{H}(G) = \cup_C \mathcal{H}(G//C)$ is also of countable dimension. For any $0 \neq v \in V$, the map

$$\mathcal{H}(G) \to V, \;\; f \mapsto \pi(f)v$$

is surjective, and so the dimension of V is also countable.

Now let A be the algebra of intertwining maps from V to V. Then the map $A \to V$ by $u \mapsto u(v)$ is injective because V is irreducible. Then we see that A also has countable dimension, but we also know that A is a division algebra over $\mathbb{C}$. We want to show $A = \mathbb{C}$ (hence the result follows), so assume for contradiction that this is not the case. Then A has a non-scalar operator x and hence a subfield isomorphic to $\mathbb{C}(x)$. But in $\mathbb{C}(x)$, the set $\{(x - \lambda)^{-1} | \lambda \in \mathbb{C}\}$ is both uncountable and linearly independent, which is a contradiction. ☐

The following two propositions constitute our first substantial step towards classifying irreducible admissible representations.

Proposition 3.6. *Let (π, V) be an irreducible and smooth complex representation of G.*

(1) The representation π is admissible.
(2) if G is commutative, then V is one-dimensional.

Proof. We do not prove (1), as the proof is difficult and requires several extra definitions and techniques that are otherwise extraneous to this essay. A proof can be found in [**BZ1**], 3.25. Given (1), we may prove (2) assuming that π is admissible. For any compact open subgroup $C \subset G$, we then have V^C is finite dimensional. Then $V = V^C$ for any C such that $V^C \neq \{0\}$, and so V is finite-dimensional. But then since G is commutative, we recall from basic representation theory that any finite dimensional complex representation of G is one-dimensional. $\square$

The following surprising proposition is our last preliminary step towards classifying irreducible admissible representations.

Proposition 3.7. *Every irreducible smooth representation (π, V) of G is either one-dimensional or infinite-dimensional. If one-dimensional, π is of the form $\chi \circ \det$ where χ is a quasicharacter of K^* (a continuous homomorphism $K^* \to \mathbb{C}^*$).*

Proof. We first show that the kernel of a finite dimensional representation π of G is open. Recall that any open subgroup $H \subset G$ contains a compact open subgroup. Pick a basis $\{v_1, \ldots, v_m\}$ of V and consider open stabilizers U_i of v_i. The finite intersection $U = \cap_i U_i$ is also open open and is the kernel of π.

Notice that since ker π is a subgroup, it contains the identity. Since we have the fundamental system of opens $1 + \pi_K^m \mathcal{O}_K$, then every open neighborhood of the identity contains the set of upper triangular matrices with 1s on the diagonal and all other elements in $\pi_K^m \mathcal{O}_K$. So for large enough m, π acts trivially on this set. But then we see that π acts trivially on all of the upper triangular matrices with 1s on the diagonal, and hence also on their conjugates. We recall from linear algebra that the conjugates of this unipotent group is all of $\mathrm{SL}_n(K)$. Then π factors through $\mathrm{SL}_n(K)$ and is 1-dimensional by irreducibility. By that same factoring, π must factor through the determinant map. $\square$

We now define the central character, which we use later to define the L and ϵ-factors, and which appears independently in the local correspondence Theorem 2.2.

Definition 3.8. *Let Z denote the center of G; it is isomorphic to K^*. For $(\pi, V) \in \mathcal{A}_n(G)$, we define its central character $\omega_\pi : Z \to \mathbb{C}^*$ by $\omega_\pi(z) id_V = \pi(z)$. Note that ω_π exists by Lemma 3.5 and can be thought of as a quasicharacter of K^*.*

Before ending this section, we define one more useful notion, which allows us to make new representations from old ones.

Definition 3.9. *For χ a quasicharacter of G, let the twisted representation $\pi\chi$ be given by $\pi\chi(g) = \pi(g)\chi(g)$.*

3.2. **The Bernstein-Zelevinsky Classification.** We now understand the basic properties of admissible representations and a couple of nontrivial facts about them (eg., irreducible smooth representations are admissible), but classifying them completely seems a long way off. In this section, we outline the Bernstein-Zelevinsky classification, which decomposes each irreducible admissible representation into basic building blocks called *supercuspidal* representations. We outline the proof following Rodier [**Ro**], but for brevity and didactic clarity we do not fill in every detail of each argument. We sketch or omit proofs of several intermediate statements, and we also black-box some results from analysis towards the end of the proof.

The very basic plan is this: In Section 3.2.1, given a finite set of irreducible admissible representations π_i of $\mathrm{GL}_{n_i}(K)$, we can consider their tensor product representation, and then use normalized induction (defined below) to produce a representation of $\mathrm{GL}_n(K)$ for $n = \sum_i n_i$. Representations which cannot be nontrivially induced by smaller representations in this way are called supercuspidal. Section 3.2.2 then shows that all irreducible admissible representations arise by this induction method from supercuspidal representations in an essentially unique way. The classification theorems Theorem 3.31 and Theorem 3.33 of Section 3.2.4 essentially state that if π is a certain kind of representations derived by induction from what we call an interval (or from a set of intervals), then it has a unique irreducible subrepresentation and quotient representation. These two theorems further classify the irreducible admissible representations completely by showing that each irreducible admissible representations arises in an essentially unique way as such a subrepresentation and quotient representation.

Recall also that supercuspidal representations play another major role, as the set of basic objects in Theorem 2.2 which correspond to irreducible Weil-Deligne representations on the Galois side.

In this section, we take a character to be a smooth character (i.e., a smooth one-dimensional representation). We also say that a representation π_i is *composed* of the quotients induced by its composition series.

3.2.1. *Basic notions and constructions.* The fundamental tool we use for the classification is normalized induction of representations, so we remind ourselves of the definition. Here let G be any group. Let δ_G be the positive square root of its modulus, the value such that if dg is a left Haar measure on G, then δ_G^{-2} is a right Haar measure. Let H be a closed subgroup of G and π a smooth representation of H on V. Then G acts on the right on the space of functions f from G to V such that

- $f(hg) = \delta_G \delta_H^{-1}(h)\pi(h)f(g)$ for all $h \in H$ and $g \in H$, and
- f is invariant under an open subgroup of G.

The representation of G on this space of functions is this also smooth, and we call it in the induced representation, denoted $\widetilde{Ind}_H^G\pi$. The subspace of functions with compact support modulo H is a subrepresentation called the *compact induction* of π from H to G, and we denote it by $\mathrm{Ind}_H^G\pi$. Note that if G/H is compact, then $\widetilde{\mathrm{Ind}}_H^G(\pi) = \mathrm{Ind}_H^G(\pi)$.

In a special case (which is our main focus below), we can also give a slightly more concrete description of the induced representation. Consider a partition $\alpha = (n_1, \ldots, n_r)$ of $n = n_1 +$

$\cdots + n_r$, and let P_α denote the subgroup of $\mathrm{GL}_n(K)$ which consists of matrices of the form

$$\begin{pmatrix} A_1 & * & * & * \\ 0 & A_2 & * & * \\ 0 & 0 & \ddots & * \\ 0 & 0 & 0 & A_r \end{pmatrix}$$

for $A_i \in \mathrm{GL}_{n_i}(K)$. Let U_α denote its unipotent radical, the subgroup with identity blocks along the diagonal. If (π_i, V_i) is an admissible representation of $\mathrm{GL}_{n_i}(K)$, then $\pi_1 \otimes \ldots \otimes \pi_r$ is an admissible representation of $\mathrm{GL}_\alpha(K)$ on $V_1 \otimes \ldots \otimes V_r$. Induction (technically *normalized induction*) as defined above gives us a representation of $\mathrm{GL}_n(K)$ on the space

$$V = \{f : \mathrm{GL}_n(K) \to V_1 \otimes \ldots \otimes V_r\} | f \text{ smooth}, f(umg) = \delta_\alpha(m)(\pi_1 \otimes \ldots \otimes \pi_r)(m)f(g), \\ u \in U_\alpha(K), m \in \mathrm{GL}_\alpha(K), g \in \mathrm{GL}_n(K)\},$$

where a map f is smooth if the stabilizer

$$\{g \in \mathrm{GL}_n(K) | f(gh) = f(h) \text{ for all } h \in \mathrm{GL}_n(K)\}$$

is open. This condition is equivalent to the above condition that f is invariant under the action of some open subgroup (and hence some compact open subgroup) of $\mathrm{GL}_n(K)$ acting by right translation. Here we can describe $\delta_\alpha := \delta_{\mathrm{GL}_\alpha(K)}$ explicitly as

$$\delta_\alpha(m) = \sqrt{|\det(\mathrm{Ad}_{U_\alpha}(m))|}$$

for Ad the adjoint. Observe that the induced representation of G on V carries the natural action by right translation.

Example 3.10 (The Steinberg representation). We can use induction to cook up the Steinberg representation for $\mathrm{GL}_2(K)$ from two characters. In general, the representation induced from a pair of characters χ_1, χ_2 is denoted $V(\chi_1, \chi_2)$ and is called a *principal series representation.* It is simple to see that $V(\chi_1, \chi_2)$ is always admissible, and it is irreducible unless $\chi_1\chi_2^{-1} = |\cdot|^{\pm 1}$.

We call $V(|\cdot|^{1/2}, |\cdot|^{-1/2})$ the *Steinberg representation* and is denoted St. More generally, $V(\chi\,|\cdot|^{1/2}, \chi\,|\cdot|^{-1/2}) = St \otimes \chi$ is called a twisted Steinberg representation for any character χ of K^*.

In general for induced representations, recall that we have Frobenius reciprocity about the adjoint relationship between induction and restriction of representations. In our case, it states the following. More information can be found for example in [**Ca**], [**BZ1**], and [**Cas1.**]

Theorem 3.11. *For π, ρ smooth representations of H and G, respectively, we have*

(1) $(\mathit{Ind}_H^G\pi)^\vee \simeq \widetilde{\mathit{Ind}}_H^G(\pi^\vee)$*, and*

(2) $\hom_G(\rho, \widetilde{\mathit{Ind}}_H^G\pi) \simeq \hom_H(\rho|_H, \delta_G\delta_H^{-1} \otimes \pi)$.

We now construct the Jacquet functor, a useful tool going forward. Let P be a locally compact totally disconnected group which is the semidirect product of closed subgroups M and U, and where U is the union of its compact subgroups. Let ω be a character of U stable under the action of M, and let π a representation of P on V. Let $V_{U,\omega}$ be the quotient of V by vectors of the form $\pi(u)x - \omega(u)x$ for $u \in U$ and $x \in V$. Then π induces a representation $\pi_{U,\omega}$ on $E_{U,\omega}$, whose restriction to U is a multiple of ω. Let $r_{U,\omega}$ denote $(\delta_P \delta_M \otimes \pi_{U,\omega})$ restricted to M. The following properties are found in any of [**BZ2**], [**Ca**], and [**Cas1**].

Proposition 3.12. *The functor which takes* $\pi \mapsto r_{U,\omega} = (\delta_P \delta_M \otimes \pi_{U,\omega})|_M$ *is exact, and*

$$Hom_P(\pi, \rho \otimes \omega) = Hom_M(r_{U,\omega}(\pi), \delta_P \delta_m^{-1} \otimes \rho).$$

If ω is the trivial character of U, then we denote $r_{U,i} = r_U$.

Returning to the case of $\mathrm{GL}_n(K)$, we move to define supercuspidal representations. Denote $\mathrm{GL}_n(K)$ by G_n, and for α a partition of n let G_α denote the block-diagonal matrices (with block sizes specified by α). As above, let P_α denote the block-upper triangular matrices, and U_α the subgroup of P_α where the blocks along the diagonal are identity matrices. We call the groups P_α for varying α the *standard parabolic subgroups* of G_n, and recall that we have the Levi decomposition $P_\alpha = G_\alpha \cdot U_\alpha$. We say for two partitions α, β of n that α is *finer* than β if $G_\alpha \subset G_\beta$ (i.e., this aligns with the standard and intuitive notion that each element of β is is 'further broken up' into elements of α). For $\alpha = (n_1, \dots, n_r)$ and ρ_i smooth representations of G_{n_i}, we let

$$\rho_1 \times \cdots \times \rho_r := \mathrm{Ind}_{P_\alpha}^{G_n}(\rho_1 \otimes \ldots \otimes \rho_r \otimes 1_{U_\alpha}).$$

We are now ready for the main definition of this section.

Definition 3.13. *A smooth representation of* G_n *is called supercuspidal (or in some sources cuspidal) if it is not expressable as* $\rho_1 \times \cdots \times \rho_r$ *for* $r > 1$.

That is, a supercuspidal representation is indecomposable with respect to this parabolic induction. Note that in some places, including [**Ro**], what we call supercuspidal is simply called cuspidal.

3.2.2. *Reducing the general case to supercuspidal representations.* Our first main goal is to show that any representation is induced as above by a set supercuspidal representations in an essentially unique way. After this, we will move on to classify the set of supercuspidal representations.

Let $\alpha = (n_i)$ and $\beta = (m_i)$ be partitions of n, with accompanying supercuspidal representations ρ_i of G_{n_i} and σ_i of Gm_i. We seek to study the space of intertwining operators $Hom_{G_n}(\rho_1 \times \cdots \times \rho_r, \sigma_1 \times \cdots \times \sigma_s)$. By Frobenius reciprocity Theorem 3.11, we have

$$Hom_{G_n}(\rho_1 \times \cdots \times \rho_r, \sigma_1 \times \cdots \times \sigma_s) \simeq Hom_{P_\beta}(\rho_1 \times \cdots \times \rho_r|_{P_\beta}, \delta_{P_\beta}^{-1} \sigma_1 \otimes \ldots \otimes \sigma_s \otimes 1_{U_\beta}),$$

where again P_β is the upper triangular block-upper-triangular matrices, blocked according to β, and U_β its subgroups where the diagonal blocks are the identity. This is in turn by the characteristic property of the Jacquet functor Proposition 3.12 isomorphic to

$$Hom_{G_\beta}(r_{U_\beta}(\rho_1 \times \cdots \times \rho_r), \sigma_1 \otimes \ldots \otimes \sigma_s).$$

Then the calculation of this space of intertwining operators essentially boils down to calculating $r_{U_\beta}(\rho_1 \times \cdots \times \rho_r)$.

Proposition 3.14. *The representation* $(\rho_1 \times \cdots \times \rho_r)$ *is composed of representations*

$$Ind_{P_{z(\alpha)}\cap G_\beta}^{G_\beta}\left(\rho_{z(1)}\otimes\ldots\otimes\rho_{z(r)}\otimes 1_{G_\beta\cap U_{z(\alpha)}}\right),$$

where z *is a permutation of* $1,\ldots,r$ *which is finer than* β *as a partition of* n *and that* $z(i) < z(i+1)$ *if* $n_{z(1)}+\cdots+n_{z(i)}$ *is not of the form* $m_1+\cdots+m_t$*.*

Proof. Note that $r_{U_\beta}(\rho_1\times\cdots\times\rho_r)$ depends only on the restriction of of ρ to P_β. Let $W_{\alpha,\beta}$ be the set of matrices of linear maps $(x_1,\ldots,x_n)\mapsto(x_{\overline{w}(1)},\ldots,x_{\overline{w}(n)})$ on K^n, where $\overline{w}$ is a permutation of $1,\ldots,n$ such that

- $\overline{w}(i+1)\geq\overline{w}(i)$ if i is not of the form $m_1+\cdots+m_t$, and
- where $\overline{w}^{-1}(i+1)\geq\overline{w}^{-1}(i)$ if i is not of the form $n_1+\cdots+n_t$.

The Bruhat decomposition (expressing a matrix as UPU' for U,U' upper triangular and P a permutation matrix) says G_n is the disjoint union of $P_\alpha wP_\beta$ for $w\in W_{\alpha,\beta}$. For $w\in W_{\alpha,\beta}$, let $I(w)$ be the space of functions f on $P_\alpha wP_\beta$ with values in the space of ρ such that

$$f(mug)=\delta_{P_\alpha}^{-1}(m)\rho(m)f(g)$$

for all $m\in G_\alpha, u\in U_\alpha, g\in P_\alpha wP_\beta$ which are invariant under the right action of P_β and whose support is compact modula P_α. Let π_w be the representation of P_β by translations of the space $I(w)$.

From the decomposition of G_n into the locally close double classes $P_\alpha wP_\beta$, we deduce that the restriction of ρ to P_β is composed of representations of π_w (in the composition series sense). Since the J functor is exact, then $r_{U_b}(\rho)$ is composed of representations $r_{U_b}(\pi_w)$. Note that an element of $I(w)$ is determined by its restriction to wP_β. Then we see that π_w is isomorphic to the induced representation

$$Ind_{w^{-1}P_\alpha w\cap P_\beta}^{P_\beta}\left(\delta_{w^{-1}P_\alpha w\cap P_\beta}^{-1}(\rho\circ int\ w)|_{w^{-1}M_\alpha w\cap P_\beta}\otimes 1_{w^{-1}U_\alpha w\cap P_\beta}\right).$$

Then we have

$$r_{U_b}(\pi_w)\simeq[Ind_{w^{-1}P_\alpha w\cap P_\beta}^{G_\beta}\left(r_{w^{-1}M_\alpha w\cap U_\beta}(\rho\circ int\ w)\otimes 1_{w^{-1}U_\alpha w\cap G_\beta}\right).$$

As $w^{-1}P_\alpha w\cap G_\beta$ is a standard parabolic subgroup of G_β and ρ is a cuspidal representation of G_β, then we see that this representation is 0 unless $w^{-1}G_\alpha w\subseteq G_\beta$. In this case, the action of w simply permutes the blocks of G_α, giving the result.

□

Example 3.15. Suppose the standard parabolic P_β is associated to P_α (that is, β is of the form $y(\alpha)$ for $y\in S_r$ the symmetric group). Then the representation $r_{U_\beta}(\rho_1\times\cdots\times\rho_r)$ is composed of representations $\rho_{z(1)}\otimes\ldots\otimes\rho_{z(r)}$ where z runs over all permutations in S_r where $n_{z(i)}=m_i$ for all i.

Corollary 3.16. *For* ω *an irreducible subquotient of* $\rho_1\times\cdots\times\rho(r)$*, there exists a parabolic subgroup* P_β *associated to* p_α *such that* $r_{U_\beta}(\omega)\neq 0$*.*

Then we see that ω embeds into a representation $\sigma_1 \times \cdots \times \sigma_s$ associated to a partition β for σ_i supercuspidal. Then $r_{U_\beta}(\omega) \neq 0$ and P_β is associated to P_α.

The following result from [**Ho**] is proven by quite different methods, and we do not prove it here. It and the the next result of Corollary 3.18 verify that representations induced from supercuspidals do indeed have finite length.

Proposition 3.17. *For ρ_i irreducible supercuspidal representations, $\rho_1 \times \cdots \times \rho_r$ is of finite length at most $r!$.*

Indeed, a composition series for $\rho_1 \times \cdots \times \rho_r$ induces one for $r_{U_\beta}(\rho_1 \times \cdots \times \rho_r)$ for each associated parabolic P_β associated to $P_{n_1,\ldots,n_r}$. The length of the composition series of $\rho_1 \times \cdots \times \rho_r$ is at most the sum of the lengths of the series for $r_{U_\beta}(\rho_1 \times \cdots \times \rho_r)$, which are given by the above Example 3.15.

Corollary 3.18. *Let $\rho_1, \ldots, \rho_r$ be representations of finite length of G_{n_i}. Then $\rho_1 \times \cdots \times \rho_r$ has finite length.*

Supposing the ρ are irreducible, we can embed ρ_i into a representation $\tau_1 \times \cdots \times \tau_s$ for τ_i irreducible and supercuspidal.

The following is a consequence of the preceding results (more details can be found in [**BZ2**], 2.9).

Proposition 3.19. *Let α, β be partitions of n with associated supercuspidal ρ_i, σ_i. The following are equivalent:*

(1) $r = s$ and we have for some $z \in S_r$ that $m_i = n_{z(i)}$ and $\rho_i = \sigma_{z(i)}$;
(2) $Hom_{G_n}(\rho_1 \times \cdots \times \rho_r, \sigma_1 \times \cdots \times \sigma_s) \neq 0$;
(3) $\rho_1 \times \cdots \times \rho_r$ and $\sigma_1 \times \cdots \times \sigma_s$ have equivalent Jordan-Holder decompositions;
(4) $\rho_1 \times \cdots \times \rho_r$ and $\sigma_1 \times \cdots \times \sigma_s$ have a common sub-quotient.

Proposition 3.19 has the following consequence: if π is a smooth irreducible representation of G_n, there is up to order a unique set of supercuspidal ρ_i of G_{n_i} such that π is a subquotient of $\rho_1 \times \cdots \times \rho_r$. We call the set $\{\rho_i\}$ the support of π.

3.2.3. *Restricting to P_n.* We have now by Proposition 3.19 sort of classified arbitrary irreducible admissible representations in terms of supercuspidals, but this classification is difficult to use. It does not allow us to easily compute the supercuspidal decomposition, nor does it tell us anything about the irreducible representation of given support. In this subsection, we build up some tools for a more concrete classification. For the next set of statements, we consider restricting representations to the subgroups P_n, where P_n is the subgroup of G_n with last row $(0, 0, \ldots, 0, 1)$. We have the following first proposition.

Proposition 3.20. *Let π be a representation of P_n. Then there exists $\pi = \pi_0 \supset \pi_1 \supset \cdots \supset \pi_n = \{0\}$ of subrepresentations such that the quotients π_{i-1}/π_i are of the form*

$$Ind_{G_{n-i}V_n^i}^{P_n} \left(\delta_{P_n}^{-1} r_{V_n^i, \omega_n^i}(\pi) \otimes \omega_n^i\right),$$

where V_n^i is matrices of P_n of the form

$$v = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 1 & v_{n-i} & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 & v_{n-i+1} & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 & v_{n-i+1} & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & \vdots & \dots & \ddots & v_{n-1} \\ 0 & 0 & 0 & \dots & 0 & 0 & \vdots & \dots & \dots & 1 \end{pmatrix}$$

with 1s along the diagonal and the upper left corner is id_{n-i}. Here, ω is a character of V_n^i defined by $\omega_n^i(v) = \omega(v_{n-i+1} \dots \omega(v_{n-1})$ for ω a nontrivial additive character of K.

Notice now that P_n is the semidirect product of G_{n-1} by V_n^1. Then by induction we can examine the representations we seek. In particular, we have the following corollary.

Corollary 3.21. *The irreducible representations of P_n are of the form*

$$\pi(\sigma, i) = Ind_{G_{n-i}V_n^i}^{P_n}\left(\sigma \otimes \omega_n^i\right)$$

where σ is a smooth irreducible representation of G_{n-i}.

Example 3.22. Let π be the restriction of a supercuspial representation of G_n to P_n, and we have $r_{V_n^i,\omega_n^i}(\pi) = 0$ for $1 \leq i \leq n-1$ as ω_n^i is trivial on the unipotent radical of the parabolic $P_{n-i,i}$. On the other hand, Gelfand and Khazdan proved in [**GK**] that $r_{V_n^n,\omega_n^n}$ has dimension 1. Then π is irreducible and isomorphic to $\mathrm{Ind}_{V_n^n}^{P_n}\omega_n^n$. Note that V_n^n is the group of upper triangular unipotent matrices, and ω_n^n is an additive character 'in general position' of V_n^n.

Definition 3.23. *For π is smooth representation of P_n, we say that $r_{V_n^i,\omega_n^i}$ is the ith derivative of π for $1 \leq i \leq n$. It is a smooth representation of G_{n-i} and is notated as $\pi^{(i)}$. If π is a smooth representation of G_n, then the derivatives are defined as the derivatives of the restriction to P_n. Notice that $\pi^{(0)} = \pi$ and $\pi^{(n)} = 0$.*

We now consider restricting our induced representations to P_n. Let $n = n_1 + n_2$ with accompanying smooth representations ρ_i. We have information about the restriction of $\rho_1 \times \rho_2$ to P_n from the derivatives of $\rho_1 \times \rho_2$. Analogously to in Proposition 3.14, we have that $(\rho_1 \times \rho_2)^{(i)}$ is composed of the representations $\rho_1^{(j)} \times \rho_2^{(n-i-j)}$ for $0 \leq j \leq i-1$. We have the following proposition, whose proof reduces to Example 3.22.

Proposition 3.24. *The restriction to P_n of an irreducible smooth representation of G_n is of finite length.*

Notice that $(\rho_1, \rho_2) \mapsto \rho_1 \times \rho_2$ makes the Grothendieck group into a commutative graded ring R. If π is an irreducible smooth representation of G_n, let $\mathcal{D}\pi = \pi^{(0)} + \cdots + \pi^{(n)}$. This extends to a homomorphism of groups $\mathcal{D} : R \to R$. By our calculation of $(\rho_1 \times \rho_2)^{(i)}$, we have th following.

Proposition 3.25. *We have $\mathcal{D}$ is also a homomorphism of algebras.*

Example 3.26. For ρ_i irreducible and supercuspidal, Proposition 3.25 and Example 3.22 allow a calculation of $D(\rho_1 \times \cdots \times \rho_r)$. Specifically, we have that $(\rho_1 \times \cdots \times \rho_r)^{(i)} = \sum \rho_{i_1} \times \cdots \times \rho_{i_s}$ for $(i_1, dots, i_s)$ tuples of strictly increasing integers such that $n_{i_1} + \cdots + n_{i_s} = n_1 + \cdots + n_r - i$.

Now we state another crucial lemma, which allows us to decompose representations into supercuspidal pieces. For π is a representation of G_n and $s \in \mathbb{R}$, we let $\pi(s)$ denote the representation of G_n by

$$\pi(s)(g) = |\det g|^s \, \pi(g)$$

for $g \in G_n$.

Lemma 3.27. *If π is a smooth representation of G_n and s a real number, then let $\pi(s)$ be given by $\pi(s)(g) = |\det g|^s \, \pi(g)$. For τ a representation of P_n whose only nontrivial derivative is $\tau^{(n-m)} = \sigma$. If τ embeds into the restriction of π to P_n, then $\check{\sigma}(-1)$ is a quotient of $(\check{\pi})^{(n-m)}$.*

Proof. Let $Q_m = G_m V_n^{n-m}$. Note that in the decomposition of representations of P_n (see Proposition 3.20), the choice of character ω does not matter. So replacing it by $\overline{\omega}$ its complex conjugate (and similarly for ω_n^{n-m}), we get $\tau \simeq Ind_{Q_m}^{P_n} \left(\delta_{P_n}^{-1} \sigma \otimes \overline{\omega}_n^{n-m} \right)$. The injection $\tau \to \pi|_{P_n}$ induces by transposition a non-zero homomorphism $\check{\pi}_{P_n} \to \check{\tau}$:

$$0 \neq Hom_{P_n}(\check{\pi}_{P_n}, \check{\tau}) = Hom_{P_n} \left(\check{\pi}_{P_n}, \tilde{Ind}_{Q_m}^{P_m} \left(\delta_{P_n}^{+1} \check{\sigma} \otimes \omega_n^{n-m} \right) \right).$$

By Frobenius reciprocity Theorem 3.11, we have

$$0 \neq Hom_{Q_m} \left(\check{\pi}_{Q_m}, \delta_{P_n}^{+2} \delta_{Q_m}^{-1} \check{\sigma} \otimes \omega_n^{n-m} \right).$$

By definition of derivatives, this gives

$$0 \neq Hom_{G_n} \left((\check{\pi})^{(n-m)}, \delta_{P_n}^{2} \check{\sigma} \right).$$

□

From this lemma, we prove the following theorem, which steps towards a classification.

Theorem 3.28. *Let ρ_i be an irreducible and supercuspidal representation of G_{n_i}. Then if $\pi = \rho_1 \times \cdots \times \rho_r$ is reducible, there is i, j such that $n_i = n_j$ and $\rho_j = \rho_i(1)$.*

Proof. Let $n = n_1 + \cdots + n_r$. Example 3.26 shows that $\dim \pi^{(n)} = 1$. Then if π is reducible, it has irreducible component τ such that $\tau^{(n)} = 0$. Possibly changing the order of ρ_i, we may assume that τ is a subrepresentation of π. Let σ be an irreducible subrepresentation of $\tau|_{P_n}$. Then there is an integer $m < n$ such that $\sigma^{(m)} \neq 0$. Then $\sigma' = \sigma^{(m)}$ is an irreducible subrepresentation of $\pi^{(m)}$. By Lemma 3.27, we have that $\check{\sigma}'(-1)$ is a quotient of $(\check{\pi})^{(m)}$. Then by Example 3.22, Frobenius reciprocity Theorem 3.11, and Proposition 3.19, we conclude the result.

□

The results we have amassed until now allow us to study representations of the form $\pi = \rho \times \rho(1) \times \cdots \times \rho(r-1)$ for ρ an irreducible supercuspidal representation of G_ℓ. Let α be the partition $(\ell, \ldots, \ell)$ where $r\ell = n$.

Proposition 3.29. *Let $\pi = \rho \times \cdots \times \rho(r-1)$ for ρ an irreducible supercuspidal representation of G_ℓ, and let α be the partition $(\ell, \ldots, \ell)$ of n. Then*

(1) π admits a unique irreducible subrepresentation π_0, and

$$r_{U_\alpha}(\pi_0) = \rho \otimes \ldots \otimes \rho(r-1).$$

The restriction of π_0 to P_n is irreducible, and $\pi_0^{(n-\ell)}$ is the unique irreducible subrepresentation of $\rho \times \cdots \times \rho(r-2)$.

(2) π has a unique irreducible quotient π_1, and

$$r_{U_\alpha}(\pi_1) = \rho(r-1) \otimes \cdots \otimes \rho.$$

Further, π_1 is essentially square integrable (i.e., the coefficients of $\pi_1 \otimes \chi$ are square integrable modulo the center of G_n for some character $\chi$of G_n).

Proof. We give a sketch of the proof. For $r = 2$, this follows easily from what we have done. Else, notice that $r_{U_\alpha}(\pi)$ is composed of all the distinct irreducible representations of G_α. Then the quotients π_i is a Jordan-Holder series of π are determined by $r_{U_\alpha}(\pi)$. Only one of the π_i is a subrepresentation of π: it is the one such that $\rho \otimes \ldots \otimes \rho(r-1)$ is a component of $r_{U_\alpha}(\pi_i)$. To calculate $r_{U_\alpha}(\pi_0)$, we see that π_0 embeds into $\rho \times \cdots \times \rho(j-1) \times \sigma \times \rho(j+2) \times \cdots \times \rho(r)$ for σ an irreducible subspace of $\rho(j) \times \rho(j+1)$. We do the same for π_1. To see that π_1 is essentially square integrable, we use Harsh-Chandra's theorem from [**Cas1**]. $\square$

3.2.4. *Stating the classification.* Now finally, we are nearly ready to state the classification, which is stated in terms of 'intervals' that we define now.

Definition 3.30. *A segment is a set of representations of G_n of the form*

$$\Delta = \{\rho, \rho(1), \ldots, \rho(r-1)\}$$

and is often denoted $\Delta[\rho, \rho(r-1)]$. For $\Delta_1 = [\rho_1, \rho_1']$ and $\Delta_2 = [\rho_2, \rho_2']$, we say they are linked if $\Delta_1 \not\subset \Delta_2$, $\Delta_2 \not\subset \Delta_1$, and $\Delta_1 \cup \Delta_2$ is a segment. If Δ_1 and Δ_2 are linked, we say Δ_1 precedes Δ_2 if $\rho_2 = \rho_1(m)$ for some $m \in \mathbb{Z}_{\geq 0}$.

For a segment $\Delta = [\rho, \rho(r-1)]$, let $Z(\Delta)$ (resp., $L(\Delta)$) denote the unique irreducible subrepresentation (resp., quotient) of $\rho\times, \ldots, \times\rho(r-1)$. Let $\mathcal{S}$ denote the collection of all segments.

We can finally state a first classification theorem, which classified irreducible admissible representations in terms of subrepresentations from intervals.

Theorem 3.31. *Let $\Delta_1, \ldots, \Delta_r$ be segments such that for $i < j$, Δ_i does not precede Δ_j.*

(1) The representation $Z(\Delta_1) \times \cdots \times Z(\Delta_r)$ has a unique irreducible subrepresentation denoted $Z(\Delta_1, \ldots, \Delta_r)$, and

(2) the set of $(\Delta_i)_i$ determines $Z(\Delta_1, \ldots, \Delta_r)$ up to reordering the Δ_i.

(3) All smooth irreducible representations of G_n (and hence all irreducible admissible representations) are of this form.

The first part of $Theorem$ 3.31 follows from the following result of [**Ze**], 6.2.

Proposition 3.32. *Let $\Delta_1, \dots, \Delta_r$ be segments such that for $i < j$, Δ_i does not precede Δ_j. If $\pi = Z(\Delta_1) \times \cdots \times Z(\Delta_r)$, then $\pi|_{P_n}$ admits a unique irreducible subrepresentation.*

Proof. We give a sketch. Let m be the largest integer such that $\pi^{(m)} \neq 0$. To prove the proposition, we first show that if σ is an irreducible representation of $\pi|_{P_n}$, then $\sigma^{(m)} \neq 0$. The calculation of $\mathcal{D}\pi$ proves that $\pi^{(m)} = Z(\Delta_1^-) \times \cdots \times Z(\Delta_r^-)$, where Δ_i^- is the segment $[\rho_i, \rho_i'(-1)]$ where $\Delta_i = [\rho_i, \rho_i']$. Then by induction, $\sigma^{(m)}$ is the unique irreducible subrepresentation of $\pi^{(m)}$, namely $Z(\Delta_1^-, \dots, \Delta_r^-)$.

□

We now have one classification theorem, but recall that we would also like to classify the irreducible admissible representations in terms of *quotients* from intervals, analogously to Theorem 3.31, which classifies in terms of subrepresentations. In other words, we seek the following result.

Theorem 3.33. *Let $\Delta_1, \dots, \Delta_r$ be segments such that for $i < j$, Δ_i does not precede Δ_j.*

(1) The representation $L(\Delta_1) \times \cdots \times L(\Delta_r)$ has a unique irreducible subrepresentation denoted $L(\Delta_1, \dots, \Delta_r)$, and
(2) the set of $(\Delta_i)_i$ determines $L(\Delta_1, \dots, \Delta_r)$ up to reordering the Δ_i.
(3) All smooth irreducible representations of G_n (and hence all irreducible admissible representations) are of this form.

3.2.5. *Analytic tools for the proof of Theorem 3.33.* The proof of Theorem 3.33 requires even more machinery, and we do not prove it here. We do however, give three intermediate results which provide a road map to the proof. This section is not strictly necessary for the reader; however it introduces a few definitions and techniques which serve important roles in this area and which we don't introduce elsewhere in the essay. It also explains some terminology used in Theorem 2.3.

For $v \in V$ and $\lambda \in V\check{}$ and (π, V) smooth, let $c_{v,\lambda} : G \to \mathbb{C}$ by $g \mapsto \lambda(\pi(g)v)$. Let $\mathrm{GL}_n(K)^0 = \{g \in \mathrm{GL}_n(K) | \, |\det g|_K = 1\}$.

Definition 3.34. *Let r be a positive real number, and let (π, V) be a smooth representation. We say (π, V) is essentially L^r if for all $v \in V$ and $\lambda \in V\check{}$, c_v, λ is L^r on G^0, i.e.,*

$$\int_{G^0} |c_{v,\lambda}|^r \, dg$$

exists. An admissible representation is L^r if it is essentially L^r and it has a unitary central character.

The proof of the following can be found in [**Si**], 2.5.

Proposition 3.35. *If an admissible representation π is L^r, then π is $L^{r'}$ for all $r' \geq r$.*

Definition 3.36. *A representation is essentially square-integrable (resp. essentially tempered) if it is in L^2 (resp. $L^{2+\epsilon}$ for all $\epsilon > 0$). We have similar definitions without 'essentially'.*

Observe that an (essentially) square integrable representation is then also (essentially) tempered. It is also true that any supercuspidal representation is essentially square integrable. More specifically, we have the following results, about which can be found in [**Ze**] 9.3, [**Ja**] section 3, and [**Ze**] 9.7.

Proposition 3.37. *Every square-integrable representation of G_n is of the form $L(\Delta)$ for a segment $[\rho, \rho(r)]$ where ρ is an irreducible supercuspidal representation such that $\rho(r/2)$ is unitary.*

Proposition 3.38. *Every irreducible admissible tempered representation of G_n is of the form $L(\Delta_1) \times \cdots \times L(\Delta_r)$ for Δ_i as in Proposition 3.37.*

Proposition 3.39. *Let $\Delta_1, \ldots, \Delta_r$ be segments. Then $L(\Delta_1) \times \cdots \times L(\Delta_r)$ is irreducible if and only if no two segments are linked.*

The omitted proof of Theorem 3.33 follows from the preceding three propositions.

3.3. **Generic representations and the Whittaker model.** Fix some non-trivial additive quasicharacter ψ of K. Let $n(\psi)$ be the exponent of ψ, the largest integer such that $\psi(\pi_K^{-n}\mathcal{O}_K) = 1$. Furthermore, define a character of θ_ψ of the unipotent upper-triangular matrices $U_n(K)$ by

$$\theta_\phi((u_{ij})) = \psi(u_{12} + \cdots + u_{n-1,n}).$$

Definition 3.40. *We say that a representation (π, V) of $GL_n(K)$ is generic if the space*

$$\hom_{U_n(K)}(\pi|_{U_n(K)}, \theta_\psi)$$

of module homomorphisms is non-empty.

Zelevinsky proved the following theorem from the Bernstein Zelevinsky classification. Note the similarity to Proposition 3.39.

Theorem 3.41. *An irreucible representation $L(\Delta_1, \ldots, \Delta_r)$ is generic if and only if no two Δ_i are linked.*

Then we see the following corollary.

Corollary 3.42. *Every essentially tempered representation is generic. In particular, every supercuspidal representation is generic.*

For (π, V) generic, we can define a Whittaker model. We choose some non-zero $\lambda \in \hom_{U_n(K)}(\pi|_{U_n(K)}, \theta_\psi)$, and we define a map

$$V \to \{f : GL_n(K) \to \mathbb{C} \mid f(ug) = \theta(u)f(g) \text{ for all } g \in GL_n(K), u \in U_n(K)\}$$

by $v \mapsto (g \mapsto \lambda(\pi(g)v))$. For $GL_n(K)$ acting on the right by translation, this is an injective map of $GL_n(K)$-modules, called the Whittaker model $\mathcal{W}(\pi, \psi)$. This Whittaker model effectively allows us to realize the original representation π on a set of functions on the original group $\mathrm{GL}_n(K)$, rather than on the group itself.

3.4. **Defining the $L$and ϵ-factors.** We first define the $L$and ϵ-factors for pairs of supercuspidal representations (π, π') and then use the Bernstein Zelevinsky classification to define them for general pairs. Even though we are now well-equiped to make these definitions, their intuition may still feel opaque. For some more motivation, see for example [**JPPS**], [**We**], and their respective citations.

Fix some non-trivial unitary additive character ψ of K, and let π, π' be supercuspidal representations of $\mathrm{GL}_n(K)$ and $\mathrm{GL}_{n'}(K)$, respectively. (In fact, all the following definitions would work just as well for any generic representations.)

First, consider the case of $n = n'$. Let $\mathcal{S}(K^n)$ denote the set of locally constant functions $\phi : K^n \to \mathbb{C}$ with compact support. Pick some $W \in \mathcal{W}(\pi, \psi)$, $W' \in \mathcal{W}(\pi', \overline{\psi})$, and $\phi \in \mathcal{S}(K^n)$, and let

$$Z(W, W', \phi, s) = \int_{U_n(K)\backslash \mathrm{GL}_n(K)} W(g)W'(g)\phi((0, 0, \ldots, 0, 1)g) \left|\det g\right|^s dg,$$

where dg is a $\mathrm{GL}_n(K)$-invariant measure on $U_n(K)\backslash\mathrm{GL}_n(K)$. This integral is absolutely convergent for large enough Re (s), and it evaluates to a rational function of q^{-s}. More details can be found in [**JPPS**], which also defines the L and ϵ-factors. Consider the set

$$\{Z(W, W', \phi, s) | W \in \mathcal{W}(\pi, \psi), W' \in \mathcal{W}(\pi', \psi), \phi \in \mathcal{S}(K^n)\},$$

which generates an ideal in the PID $\mathbb{C}[q^s, q^{-s}]$. The generator of this ideal is by definition the L-factor denoted $L(\pi \times \pi', s)$.

We define $\epsilon(\pi \times \pi', s, \psi)$ by

$$\frac{Z(\tilde{W}, \tilde{W}', \hat{\phi}, 1 - s)}{L(\pi^\vee \times \pi'^\vee, 1 - s)} = \omega_{\pi'}(-1)^n \epsilon(\pi \times \pi', \psi, s)\frac{Z(W, W', \phi, s)}{L(\pi \times \pi', s)}.$$

We define $\tilde{W}$ by $\tilde{W}(g) = W(w_n\, {}^t g^{-1})$ and $w_n \in \mathrm{GL}_n(K)$ is the permutation matrix sending i to $n+1-i$. This is an element of $\mathcal{W}(\pi^\vee, \overline{\psi})$ by our commentary after Definition 3.2. We define $\tilde{W}' \in \mathcal{W}(\pi'^\vee, \psi)$ similarly. By $\hat{\phi}$ above we mean the Fourier transform, i.e.,

$$\hat{\phi}(x) = \int_{K^n} \phi(y)\psi({}^t yx)dy$$

for $x \in K^n$.

Now consider the case of $n' < n$. For $W \in \mathcal{W}(\pi, \psi)$, $W' \in \mathcal{W}(\pi', \overline{\psi})$, and $j \in \{0, 1, \ldots, n - n' - 1\}$, let

$$Z(W, W'.j, s) = \int_{U_{n'}(K)\backslash\mathrm{GL}_{n'}(K)} \int_{M_{j\times n'}(K)} W\left(\begin{pmatrix} g & 0 & 0 \\ x & I_j & 0 \\ 0 & 0 & I_{n-n'-j} \end{pmatrix}\right) W'(g) \left|\det g\right|^{s-(n-n')/2} dxdg$$

for dg a $\mathrm{GL}_{n'}(K)$-invariant measure on $U_{n'}(K)\backslash\mathrm{GL}_{n'}(K)$ and dx a Haar measure on the space of $(j\times n')$-matrices with coefficients in K. Like in the $n = n'$ case, this is absolutely convergent for Re (s) large and evaluates to a rational function in q^{-s}. The generator of the ideal generated by these functions is again defined to be $L(\pi \times \pi', s)$, and we define $\epsilon(\pi \times \pi', s, \psi)$ by

$$\frac{Z(w_{n,n'}\tilde{W}, \tilde{W}', n - n' - 1 - j, 1 - s)}{L(\pi^\vee \times \pi'^\vee, 1 - s)} = \omega_{\pi'}(-1)^{n-1}\epsilon(\pi \times \pi', \psi, s)\frac{Z(W, W', \phi, s)}{L(\pi \times \pi', s)},$$

where $w_{n,n'}$ is the matrix

$$\begin{pmatrix} I_{n'} & 0 \\ 0, w_{n-n'} & \end{pmatrix} \in \mathrm{GL}_n(K).$$

For $n > n'$, we define

$$L(\pi \times \pi', s) = L(\pi' \times \pi, s), \epsilon(\pi \times \pi', s, \psi) \qquad = \epsilon(\pi' \times \pi, s, \psi).$$

In all cases, $L(\pi \times \pi', s)$ is independent of the choice of ψ (hence ψ is omitted from the notation), and $\epsilon(\pi' \times \pi, s, \psi)$ is of the form cq^{-fs} for $c \in \mathbb{C}^*$ and $f \in \mathbb{Z}$.

Having completed the supercuspidal case, we move on to the general case. We use the following relations to define the L and ϵ-factors in general.

(1) We have $L(\pi \times \pi', s) = L(\pi' \times \pi, s)$ and $\epsilon(\pi \times \pi', s, \psi) = \epsilon(\pi' \times \pi, s, \psi)$ for all choices of parameters.
(2) For π of the form $L(\Delta_1, \ldots, \Delta_r)$ as in the Bernstein Zelevinsky classification, then for arbitrary π', we have

$$L(\pi \times \pi', s) = \prod_{i=1}^{r} L(L(\Delta(\Delta_i) \times \pi', s)$$
$$\epsilon(\pi \times \pi', \psi, s) = \prod_{i=1}^{r} L(L(\Delta(\Delta_i) \times \pi', \psi, s).$$

Note the two different uses of the notation L to refer to either the unique irreducible subquotient associated to set of intervals from the Bernstein-Zelevinsky classification, and on the other hand to the L-factor.
(3) For π of the form $L(\Delta), \Delta = [\rho, \rho(r-1)]$ and $\pi' = L(\Delta'), \Delta' = [\rho', \rho'(r-1)]$ with $r' \geq r$, then

$$L(\pi \times \pi', s) = \prod_{i=1}^{r} L(\rho \times \rho', s + r + r' - 1)$$
$$\epsilon(\pi \times \pi', \psi, s) = \prod_{i=1}^{r} \left(\left(\prod_{j=0}^{r+r'-2i} \epsilon(\rho \times \rho', \psi, s + i + j - 1) \right) \left(\prod_{j=0}^{r+r'-2i-1} \frac{L(\rho^\vee \times \rho'^\vee, 1 - s - i - j)}{L(\rho \times \rho', s + i + j - 1)} \right) \right).$$

For the L and ϵ-factors $L(\pi, s)$ and $\epsilon(\pi, \psi, s)$ associated to a single representation π, we simply take $\pi' = 1$ the trivial representation.

4. The Galois side

4.1. **Some definitions.** Recall that W_K denotes the Weil group of K, and that via the Artin map of local class field theory, we can define a valuation on $w \in W_K$ by $|w| = \left|\mathrm{Art}_K^{-1}(w)\right|_K$.

Definition 4.1. *We say a quasicharacter of W_K is unramified if it is trivial on the inertia group I_K. We say a quasichacracter $\omega : K^* \to \mathbb{C}^*$ for K a local field is unramified if it is trivial on $\mathcal{O}_K^*$. A quasicharacter of K is unramified if it is trivial on $\mathcal{O}_K$.*

Recall that under the Artin map, $\mathcal{O}_K*$ bijects with I_K, and so our notions of unramified for quasicharacters of W_K and K^* are naturally compatible.

Let ω be a quasicharacter of K^* Since $\mathcal{O}_K^*$ is compact, $\omega(\mathcal{O}_K^*)$ must also be compact. If $\omega(x) \notin S^1$ for some $x \in \mathcal{O}_K^*$, then by multipicativity of ω, the sequences $\omega(x^i)$ for $i \in \mathbb{Z}_{\geq 1}$ either approach $0 \notin \mathbb{C}^*$ or else have unbounded norm, contradicting compactness of $\omega(\mathcal{O}_K^*)$ in either case. Then $\mathcal{O}_K^*$ must be sent to the unit circle under ω. So as $K^* = \mathcal{O}_K \times \mathbb{Z}$, the choice of $\omega(\pi)$ and che choice of a character $\omega' : \mathcal{O}_K^* \to \mathbb{C}^*$ determines ω uniquely. Then $\omega(x) = \omega'(x)\,|(|\,x)^s$ for some $s \in \mathbb{C}$ and a character ω'.

Definition 4.2. *The exponent of a quasicharacter ω is* Re (s) *for this s as above.*

As a corollary, we also see hat ω is unramified if and only if it is trivial on $\mathcal{O}_K^*$, if and only if $\omega = |x|^s$. We call this character ω_s.

We now recall that a Dirichlet character mod m is a character of $(\mathbb{Z}/m\mathbb{Z})^*$ for $m \in \mathbb{Z}$, and such a Dirichlet character χ mod m is called primitive if it is not the pullback of a character of $(\mathbb{Z}/n\mathbb{Z})^*$ for $n|m$ and $n \neq m$.

Recall that $\mathcal{O}_K* \simeq \varprojlim_n (\mathcal{O}_K/(\pi)^n)^*$. Then a character of $\mathcal{O}_K*$ is the pullback of a character of some $(\mathcal{O}_K/(\pi)^n)^*$. The smallest such n for a given character ω is called the *conductor* of ω. The conductor of a character of K^* is defined to be the conductor of its restriction to $\mathcal{O}_K*$. So a quasicharacter of $K*$ by three pieces of information: an integer conductor n, a character mod n, and the image of some uniformizer.

Since W_K is dense in Gal$(\overline{K}/K)$, then the representations of Gal$(\overline{K}/K)$ restrict a full subcategory of the representations of W_K. Such a representation of W_K is called *Galois type*. Recall that since the open subgroups of W_K are precisely W_L for finite extensions L/K, and their intersection is ker $\Phi = 1$ for $\Phi = (x \to x^{-q})$ the geometric Frobenius. Therefore, a representation is Galois type if and only if its image is finite.

Conversely, we have the following proposition, which tells us that representations of Galois type suffice to understand general representations. We follow the proof of [**Zh**], 3.8, but the result can also be found in [**De2**], 4.10.

Proposition 4.3. *Every irreducible representation r of W_k is of the form $r = r' \otimes \omega_s$ for some $s \in \mathbb{C}$ and r' of Galois type.*

Proof. Notice that every representation of W_K is trivial on a finite-index subgroup J of I_K. As I_K/J is finite, then Φ^n acts trivially on I_k/J by conjugation for some $n > 0$. Then Φ^n is central in W_K/J. We see now that each power π^m of Φ^n has exactly one eigenvalue a_m if the representation is irreducible. Thus, each irreducible representation has a type specified by

$$(a_m) \in \varinjlim_{n|m} \{\chi_m, \phi_{n,m}\}_m$$

for $\chi_m = \mathbb{C}^*$ and $\phi_{m,n} = x^{m/n}$. For all $s \in \mathbb{C}$, the representations of type s each form an abelian category, and the direct sum over all such categories is Rep(W_k), the representations of W_K. The Galois representations are simply representations of type $s = 1$. So tensoring with ω_s gives an isomorphism Rep$(G_K) \to \text{Rep}_s(W_K)$, the representations of type s. $\square$

We observe that a representation of Galois type is then irreducible if and only if it is irreducible as a representation of Gal$(\overline{K}/K)$. A representation ρ is also Galois type if and only if the image of its determinant $\det \circ \rho$ is finite in $\mathbb{C}^*$.

As above in Theorem 3.11, we have a usual statement of Frobenius reciprocity.

Proposition 4.4. *For L/K a finite extension in $\overline{K}$, we get the canonical injection $W_L \to W_K$ with finite cokernel. So the restriction and induction maps between representations of W_L and W_K are adjoint.*

We now define the main actor of this section, the Weil-Deligne representation.

Definition 4.5. *A Weil-Deligne representation of W_K is a pair (r, N) of a representation $(r, V)r$ of W_K and a nilpotent $\mathbb{C}$-linear endomorphism of V such that*

$$r(\gamma)N = |\gamma|_K Nr(\gamma)$$

for $\gamma \in W_K$. We say (r, N) is Frobenius semisimple if r is semisimple.

We can note a few observations about Weil-Deligne representations.

(1) First, let $\gamma \in W_K$ be the element corresponding to π_K under the Artin map. Then N and qN are conjugate, and so N has all eiganvalues vanishing, and is thus automatically nilpotent.
(2) Second, the kernal of N is stable under W_K, and hence $N = 0$ if (r, N) is irreducible. Then the irreducible Weil-Deligne representations are precisely the irreducible continuous represenetations of W_K.
(3) Third, observe that the topology on the image $\mathbb{C}^n$ does not effect the defining property of a Weil-Deligne representations. Then we should feel free not to view them as complex representations but rather as representations over $\overline{\mathbb{Q}_\ell}$ for a prime number ℓ since $\overline{\mathbb{Q}_\ell}$ and $\mathbb{C}$ are isomorphic as fields. Indeed, such so-called $\ell-$adic representations give natural motivation for Wel-Deligne representations, and they also open the door towards a broader theory. We do not in this essay develop that theory or its major results like Grothendieck's ℓ-adic monodromy, but information can be found in [**FO**] but also in many other places.

Before, moving on to define L and ϵ-factors, we define the tensor product of Weil-Deligne representations. Let $\rho_i = (r_i, N_i)$ for $i = 1, 2$ be Weil-Deligne representations on complex vector spaces V_i. Their tensor product on $V_1 \otimes V_2$ is a Weil-Deligne representation given by

$$r(w)(v_1 \otimes v_2) = r_1(w)v_1 \otimes r_2(w)v_2 \text{ and } N(v_1 \otimes v_2) = N_1 v_1 \otimes v_2 + v_1 \otimes N_2 v_2$$

for $w \in W_K$ and $v_i \in V_i$.

4.2. **L and ϵ-factors.** We first define the L-factor. Let $\rho = ((r, V), N)$ be a Frobenius semisimple Weil-Deligne representation. Let V_N be the kernel of N and $V_N^{I_K}$ the space of $v \in V_N$ which are fixed by the action of the inertia group I_K. Then we let

$$L(\rho, s) = \det(1 - q^{-s}\Phi_{V_N^{I_K}})^{-1}$$

for Φ the geometric Frobenius in W_K. The reader may notice that this looks a lot like a multiplicative term found in certain L-functions. We shed some light on this in Section 4.3 and in particular Section 4.3.4.

To define the ϵ-factor is a little bit trickier. Fix a nontrivial additive character ψ of K and let n be its exponent (in the sense of Section 3.3). Let dx be an additive Haar measure on K. We first assume that V is one-dimensional, and hence r is some quasicharacter

$$r : W_K^{\text{ab}} \to \mathbb{C}^*.$$

For r unramified (i.e., $r = \omega_s$ for some $s \in \mathbb{C}$), we set

$$\epsilon(r, \psi, dx) = r(w)q^{n(\psi)}\text{vol}_{dx}(\mathcal{O}_K) = q^{n(\psi)(1-s)}\text{vol}_{dx}(\mathcal{O}_K)$$

for $w \in W_K$ an element with valuation $n(\psi)$.

For r ramified, let $f(r)$ be its conductor, and let $c \in \mathbb{C}$ be an element with valuation $n(\psi) + f(r)$. Then we set the ϵ-factor as follows (with some motivation for a special case found below in Section 4.3):

$$\epsilon(r, \psi, dx) = \int_{c^{-1}\mathcal{O}_K^*} r^{-1}(\text{Art}_K(X))\psi(x)dx. \tag{2}$$

This completes our definitions $\epsilon(r, \psi, dx)$ for V one-dimensional. In the general case, we rely on the following existence theorem of Deligne. Before we state the theorem, we remind ourselves that the Grothendieck group of of representations of W_K, denoted Groth(Rep(W_K)) is the group whose elements are isomorphism classes of representations and whose operation is direct sum. To form this group, we must include artificial inverses of representations, and we call these *virtual representations*. Since direct sum is additive in exact sequences, we say that these virtual representations can take on negative dimension in the natural way. (Note that we may perform this construction in any abelian category, not just in Rep(W_K).) The following theorem is proven in [**De2**].

Theorem 4.6. *There exists a unique function ϵ which associates to K a nontrivial additive character ψ of K, an additive Haar measure dx on K, and a representation r of W_K a non-zero complex number $\epsilon(r, \psi, dx) \in \mathbb{C}^*$ such that the following three conditions hold.*

(1) If r is one-dimensional $\epsilon(r, \psi, dx)$ is the factor defined above.

(2) We have $\epsilon(\cdot, \psi, dx)$ is multiplicative in exact sequences of representations of W_K. Hence we get a homomorphism

$$\epsilon(\cdot, \psi, dx) : \textit{Groth}(\textit{Rep}(W_K)) \to \mathbb{C}^*.$$

(3) For any tower of finite field extensions $L'/L/K$ and every additive Haar measures $\mu_{L'}$ on L' and μ_L on L, we have

$$\epsilon(\textit{Ind}_{L'/L}[r'], \psi \circ \textit{Tr}_{L/K}, \mu_L) = \epsilon([r'], \psi \circ \textit{Tr}_{L'/K}, \mu_{L'})$$

when $\dim([r]) = 0$, where $[r]$ denotes the class of r in Groth(Rep(W_K))

We observe that $\epsilon(\chi, \psi, \alpha dx) = \alpha\epsilon(\chi, \psi, dx)$ for χ a representation of dimension 1, for any $\alpha > 0$. Then by inductivity, we have $\epsilon(r, \psi, \alpha dx) = \alpha^{\dim r}\epsilon(\chi, \psi, dx)$. So if $\dim([r]) = 0$, then our choice of Haar measure dx does not matter. Now finally, we define the ϵ in the correspondence Theorem 2.2.

Definition 4.7. *For a Weil-Deligne representation $\rho = (r, N)$, we let*

$$\epsilon(\rho, \psi, s) = \epsilon(\omega_s r, \psi, dx)\det(\Phi|_{V^{I_K}/V_N^{I_K}}$$

where dx is the Haar measure which is self-dual under the Fourier transform

$$\hat{f}(y) = \int f(x)\psi(xy)dx.$$

Remark. Having now defined all the objects in Theorem 2.2. We saw in Theorem 2.3 that supercuspidal representations and irreducible Weil-Deligne representations correspond, and also that weaker properties of essential square-integrability and indecomposability align, too. Furthermore, notice that on each side of the correspondence, we have a way of building larger representations from smaller ones:

(1) On the $\mathrm{GL}_n(K)$-side, we discussed at length the normalized induction that turns supercuspidal representations into all admissible representations.
(2) On the Galois side, we use tensor products.

Wonderfully, these two construction methods also coincide! Loosely speaking, the induced representation from a pair of admissible representations corresponds to the tensor product of the associated Weil-Deligne representations. So even though we do not construct the correspondence in this essay, we already see a glimmer of how it works.

4.3. **Tate's thesis.** Until now in this section, we have examined the Galois side in full generality. While of course this allows us to state the correspondence, the details and intuition of the L and ϵ factors in particular may still feel out of reach. Rather than diving into a broad motivation in the general situation, we now present a sort of 'case study'. We discuss a portion John Tate's 1950 thesis [**Ta1**], which defines the L and ϵ-factors from Section 4.2 in a more elementary and concrete way for the abelian case, and which also gives us additional insights into their behavior. In this section, the abelian assumption lets us deal with objects that are all defined very explicitly from one another in a way that seems totally alien to what we have done until now. However, the L and ϵ-factors we define below match exactly with those of Section 4.2. So a main goal of this section is to build up some intuition for those abstruse definition, a way of conceptualizing them that is - while not general - intuitively much more approachable. We sometimes also follow exposition of [**De1**] rather than Tate's original presentation.

Let G be any abelian and locally compact topological group (we will want to think about $G = W_K$ if this group is abelian). We remark that any quasicharacter of G is a character if G is compact. This is because of $|\omega(x)| > 1$, then the sequence $(\omega(x^i))_i$ does not converge in K^*. If conversely $|\omega(x)| < 1$, consider $\omega(x^{-1})$ instead. We record a useful and easy lemma.

Lemma 4.8. *If G is a compact abelian group and ω is a nontrivial character of G and dx is a Haar measure on G, then $\int_G \omega(x)dx = 0$.*

Proof. Since ω is nontrivial, pick some $a \in G$ such that $\omega(a) \neq 1$. Then

$$\omega(a)\int_G \omega(x)dx = \int_G \omega(ax)dx = \int_G \omega(x)d(a^{-1}x) = \int_G \omega(x)dx$$

by translation invariance of the Haar measure. Then $\int_G \omega(x)dx = 0$. ∎

4.3.1. *Some Fourier analysis.* We remind the reader of some basic main results from the theory of Fourier analysis on abelian groups, but since the subject is not the main focus of the essay, we do not provide proofs.

Definition 4.9. *The Pontryagin dual of G, denoted $\hat{G}$ is the group of characters on G, which we topologize with the compact-open topology (i.e., where a subbase of the topology is indexed by pairs (U, K) for U open in G and K compact in $\mathbb{C}$, and each element of the subbase is given by the sets characters ω with $\omega(U) \subseteq K$).*

Theorem 4.10. *The groups G and $\hat{\hat{G}}$ are naturally isomorphic.*

Definition 4.11. *For $f \in L^2(G) \cup L^1(G)$, let $\hat{f} : \hat{G} \to \mathbb{C}$ by $\hat{f}(\xi) = \int_G f(y)\xi(y)dy$.*

Theorem 4.12. *The Fourier transform extends by continuity to give an isomorphism $L^2(G) \to L^2(\hat{G})$. Moreover, we have a unique Haar measure on $\hat{G}$ such that $\hat{\hat{f}}(x) = f(-x)$ for all $x \in G$. We say this measure is dual to dx.*

4.3.2. *Deriving the the L-factors.* We now let $S(K)$ be the $\mathbb{C}$-vector space of compactly supported and locally constant functions $K \to \mathbb{C}$, topologized as the direct limit of its finite dimensional subspaces. According to this topology, any linear map from $S(K)$ is continuous. (In the global cases of $K = \mathbb{R}, \mathbb{C}$ instead of being a local field, the analogous class of function is the Schwarz functions, which intuitively are the *rapidly decaying functions* to $\mathbb{C}$.) In particular, a linear map $S(K) \to \mathbb{C}$ is called a *tempered distribution*. For example, the point mass δ_0 which acts by $\langle \delta_0, f\rangle = f(0)$ is a tempered distribution. We denote the space of all tempered distributions by $S(K)'$, and we denote $\lambda(f)$ by $\langle \lambda, f\rangle$ for $\lambda \in S(K)'$ and $f \in S(K)$.

Definition 4.13. *We let K^* act on $S(K)'$ by $\langle a\lambda, f\rangle = \langle \lambda, f(xa^{-1})\rangle$. We say λ is an ω-eigendistribution if $a\lambda = \lambda(a)\lambda$. The space of such distributions is denoted $S'(\omega)$. We define ω-eigendistributions of $K*$ similarly, as linear maps $\lambda : C_c^\infty \to \mathbb{C}$ such that$\langle \lambda, f(xa^{-1})\rangle = \langle \omega(a)\lambda, f(x)\rangle$ for all $a \in K^*$. Here C_c^∞ denotes the subspace of $S(K)$ of functions supported away from 0.*

We assemble a few smaller facts before proving a longer proposition. First, we notice that we have the exact sequence

$$0 \to C_c^\infty(K^*) \to S(K) \to \mathbb{C} \to 0,$$

where the map $S(K) \to \mathbb{C}$ is evaluation at 0. Then $S(K) = C_c^\infty(K^*) \oplus \mathbb{C}f$, for any $0 \neq f \in S(K)$. Secondly, we see that δ_0 spans all distributions support only at 0, and is an ω-eigendistribution if and only if ω is trivial. Third, any ω-eigendistribution on $S'(\omega)$ restrictas to an eigendistribution of K^* since the action of ω is point-wise. Finally, we have the following lemma.

Lemma 4.14. *The ω-eigendistributions of $K*$ are spanned by $\omega(x)d^*x$, where d^*x is a Haar measure on $K*$.*

Proof. We first let λ, σ be two ω-eigendistributions of K'. Then for $f \in S(K')$,

$$\frac{\omega(a)\lambda(f(x))}{\omega(a)\sigma(f(x))} = \frac{\lambda(f(x))}{\sigma(f(x))} = \frac{\lambda(f(a^{-1}x))}{\sigma(f(a^{-1}x))}.$$

Since this holds for all $a, x \in K^*$ then in general $\lambda(f)$ and $\sigma(f)$ are equal up to a scalar multiple. As this holds for all f, then λ and σ are also linearly dependent. Finally, we see that

$$\omega(a)\int_G f(x)\omega(x)d^*x = \int_G f(x)\omega(ax)d^*x = \int_G f(a^{-1}x)\omega(x)d^*(a^{-1}x) = \int_G f(a^{-1}x)\omega(x)d^*x,$$

and so $\omega(x)d^*x$ is indeed an ω-eigendistribution. $\square$

Now we use these collected facts to prove the following proposition.

Proposition 4.15. *For any quasicharacter ω, $S'(\omega)$ is one-dimensional.*

Proof. First, suppose that ω is ramified. Let $\lambda \in S'(\omega)$. Then we can pick $c \in \mathbb{C}$ such that on $C_c^\infty(K^*)$, λ and $c\omega(x)d^*x$ coincide by our lemma. We show that this determines where λ sends any function. Define $f^0 \in S(K)$ to be the indicator function of $\mathcal{O}_K$, and let $a \in \mathcal{O}_K^*$ such that ω is nontrivial on a. Then since λ is an ω-eigendistribution,

$$\langle \lambda, f^0 \rangle = \omega(a)\langle \lambda, f^0 \rangle,$$

from which it follows that $\langle \lambda, f^0 \rangle = 0$. Then $\langle \lambda, f^0(x\pi^{-n}) \rangle = 0$ for all $n \geq 0$. Since $f \in S(K)$ is constant on a neighborhood of 0, the difference between f and a function supported away from 0 is precisely such an indicator function $f^0(x\pi^{-n})$. Then

$$\langle \lambda, f \rangle = \langle \lambda, f - f(0)f^0(x\pi^{-n}) \rangle = c\int_{K^*-(\pi)^n} f(x)\omega(x)d^*x,$$

where the last equality follows by our lemma. So then $S'(\omega)$ is at most one-dimensional.

It remains to find a nontrivial element. Let $z_0(\omega)$ by

$$\langle z_0(\omega), f \rangle = \int_{K^*-(\pi)^n} f(x)\omega(x)d^*x.$$

By invariance of d^*x under multiplicative translation, the RHS is independent of n, if n is large enough that f is constant on $(\pi)^n$.

Now we suppose that ω is unramified but not trivial. The proof here begins like the previous case, only now we can take $n = 1$. That λ is an ω-eigendistribution gives

$$(1-\omega(\pi))\langle \lambda, f \rangle = \langle \lambda, f(x) - f(x\pi^{-1}) \rangle = \frac{c}{1-\omega(\pi)}\int_{K^*-(\pi)} (f(x) - f(x\pi^{-1}))\omega(x)d^*x.$$

We define $z_0(\omega)$ so that

$$\langle z_0(\omega), f \rangle = \int_{K^*-(\pi)} (f(x) - f(x\pi^{-1}))\omega(x)d^*x.$$

Finally, if ω is trivial, meaning λ is K^*-invariant. Then $\langle \lambda, f^0 \rangle = \langle \lambda, f^0(x\pi^{-1}) \rangle$, and so

$$0 = \langle \lambda, f^0(x) - f^0(x^{-1}) \rangle = c\int_{\mathcal{O}_K^*} d^*x$$

for c as above. In particular, $c = 0$, and so λ is a scalar multiple of δ_0. $\square$

For nontrivial ω, the restriction of $z_0(\omega)$ to K^* is also non-zero. We then have some other generator of $S'(\omega)$ which restricts to be exactly $\omega(x)d^*x$.

Definition 4.16. *Let $L(\omega)$ be the constant such that $z(\omega) = L(\omega)z_0(\omega)$. In the proof of Proposition 4.15, we saw that $L(\omega) = (1-\omega(\pi))^{-1}$ for ω unramified, and $L(\omega) = 1$ else.*

We have now defined the L-factor as the ratio of two natural generators of a character space, which is rather more motivated than the definition in Section 4.2. Not only is this definition more concrete, but this ratio of generators also provides unexpected analytic and global information, as we allude to in Section 4.3.4.

4.3.3. *ϵ-factors and the functional equation.* Now we move towards the ϵ-factors. For ψ a nontrivial character of K, K is isomorphic to its own character group $\hat{K}$ via the isomorphism $y \to \psi(y \cdot -)$. Then for $f \in L^1(K) \cap L^2(K)$, the Fourier transform is given by

$$\hat{f}(\xi) = \int_K f(x)\psi(x\xi)dx,$$

and the Fourier transform is an automorphism of $S(K)$. We say the Haar measure dx is *self-dual* with respect to ψ if $\hat{\hat{f}}(x) = f(-x)$. More details can be found in [**RV**]. From here on, fix a character ψ, and assume dx to be this self-dual measure.

Definition 4.17. *For a distribution λ, we define $\hat{\lambda}$ by $\langle \hat{\lambda}, f\rangle = \langle \lambda, \hat{f}\rangle$. For ω a character of K^*, its shifted dual is defined to be $\hat{\omega} = |\cdot|\,\omega^{-1}$.*

Lemma 4.18. *For any λ an ω-eigendistribution, then $\hat{\lambda}$ is an $\hat{\omega}$-eigendistribution.*

Proof. we have

$$\begin{aligned}
\langle \hat{\lambda}, f(xa^{-1})\rangle &= \langle \lambda, \widehat{f(xa^{-1})}\rangle \\
&= \langle \lambda, |a|\,\hat{f}(xa)\rangle \\
&= \langle |a|\,\omega(a^{-1})\langle \lambda, \hat{f}\rangle \\
&= |a|\,\omega^{-1}(a)\langle \hat{\lambda}, f\rangle.
\end{aligned}$$

∎

Using Proposition 4.15 and the fact that z_0 is holomorphic in ω, we immediately obtain the following theorem and proposition from the relevant definitions.

Theorem 4.19. *There exists some non-zero, holomorphic $\epsilon(\omega, \psi)$ such that*

$$\epsilon(\omega,\psi)z_0(\omega) = \widehat{z_0(\hat{\omega})}.$$

If ω is nontrivial, there is also a meromorphic $\gamma(\omega,\psi)$ such that $\gamma(\omega,\psi)z(\omega) = \widehat{z(\hat{\omega})}$. Furthermore, these are related by

$$\gamma(\omega,\psi) = \epsilon(\omega,\psi)\frac{L(\hat{\omega})}{L(\omega)}.$$

So we see that we define ϵ to be a scaling factor that relates our one of our natural generators of the character space to its own sort of 'double dual', again a more concrete and motivated definition than in Section 4.2. The second statement of Theorem 4.19 also gives an explicit relationship between the L and ϵ-factors that is not so obvious from the definitions of Section 4.2. Sometimes Theorem 4.19 is called the Local Functional Equation [**Ta1**].

We do not prove the following proposition, as it follows essentially from definitions.

Proposition 4.20. *For any $y \in K$, we have $\epsilon(\omega, \psi(.y)) = \sqrt{|y|}\omega(y)\epsilon(\omega, \psi)$.*

From our earlier comment on the self-duality of K, it then suffices to compute $\epsilon(\omega, \psi)$ for any single choice of ψ in order to compute $\epsilon(\omega, \psi)$. We use the functional equation, and so we just need to pick some test function where we can easily evaluate $z_0(\omega)$ and $\widehat{z_0(\hat{\omega})}$. We define the conductor ν to be the integer such that $\pi^\nu \mathcal{O}_K$ is the kernel of ψ. Then by definition, we have

$$\hat{f}^0(x) = \int_{\mathcal{O}_K} \psi(xy)dy.$$

By Lemma 4.8, this gives $\hat{f}^0(x) = f^0(x\pi^{-\nu}) \int_{\mathcal{O}_K} dy$, and similarly $\hat{\hat{f}}^0(x) = f^0 \cdot |\pi|^\nu \left(\int_{\mathcal{O}_K} dy\right)^2$. Then for the Fourier transform to be a bijection, we must have that dy assigns the measure $|\pi|^{-\nu/2}$ to $\mathcal{O}_K$. Then $\hat{f}^0(x) = |\pi|^{-\nu/2} f^0(x\pi^{-\nu})$.

In order to compute $\epsilon(\omega, \psi)$, first assume that ω is unramified. Then

$$\begin{aligned}\langle z_0, \omega\rangle &= \int_{K^*} (f^0(x) - f^0(x\pi^{-1}))\omega(x)d^*x \\ &= \int_{\mathcal{O}_K^*} d^*x + \int_{K^*\backslash\mathcal{O}_K^*} d^*x \\ &= \int_{\mathcal{O}_K^*} d^*x.\end{aligned}$$

Since $z_0(\hat{\omega})$ is an $\hat{\omega}$-eigendistribution, then

$$\langle \widehat{z_0(\hat{\omega})}, f^0\rangle = \langle z_0(\hat{\omega}), |\pi|^{\nu/2} f^0(x\pi^{-\nu})\rangle = \hat{\omega}(\pi^\nu) |\pi|^{\nu/2} \langle z_0(\hat{\omega}), f^0\rangle.$$

Then by definition $\langle z_0(\omega), f^0\rangle = \langle z_0(\hat{\omega}), f^0\rangle$ from the definition of z_0, and so $\epsilon(\omega, \psi) = \omega(\pi^{-\nu}) |\pi|^{-\nu/2}$.

If ω is ramified, then let c be its conductor. In an analogous calculation using the text function

$$g_\omega(x) = \begin{cases} \psi(x) & x \in \pi^{\nu-c}\mathcal{O}_K \\ 0 & \text{else}\end{cases}$$

with Fourier transform

$$\hat{g}_\omega(\xi) = \begin{cases} |\pi|^{\nu/2-c} & \xi \equiv 1 \pmod{\pi^c \mathcal{O}_K} \\ 0 & \text{else}\end{cases},$$

we see that

$$\epsilon(\omega, \psi) = \int_{\pi^{\nu-c}\mathcal{O}_K^*} \omega^{-1}(x)\psi(x)dx.$$

We can compare this to the definition of the ϵ-factor of the general case in Equation (2). Note that in the abelian case to which Tate's thesis pertains, all representations are one dimensional, and we have no need for an analogous result to Theorem 4.6.

4.3.4. *A further word on Tate's thesis.* Tate's thesis certainly gives a more concrete description, derivation, and motivation for the local ϵ factor of Theorem 2.2 by working with explicit objects and not relying on less constructive existence results like Theorem 4.6, but it also accomplishes much more. Most notably, it establishes a link between the local and global situations. Observe that the local L-factor $(1-\omega(\pi))^{-1}$ looks very much like a multiplicative term of an Euler product. In fact, given a number field K and its completions K_v, we can multiply together each of the local L-factors for primes v (also including infinite primes, which are defined as complex embeddings of K), and the resulting product *is* the Euler product of an associated L-function. Incredibly, multiplying together the corresponding local functional equations gives us the global functional equation of that L-function! In other words, Tate's technology allows us to 'break up' an L-function into local pieces which preserve information. More information can found either in Tate's original thesis [**Ta1**] or in expositions like [**De1**] or [**Po**].

5. Conclusion

We've now spent almost 30 pages in an attempt to understand the objects of the local Langlands correspondence Theorem 2.2. Despite this, we have omitted many details and neglected several other closely related and motivating topics. And more glaringly, we have not proven any part the correspondence at all! So, in short... why? One can certainly find the theory interesting in itself for its own curious interconnectedness and surprising naturality, but the local Langlands program opens one door to a much broader world with far-reaching implications. In Section 4.3.4, we mentioned the latter portion of Tate's thesis (not exposited here), which outlines a local-global connection linking the local factors to global analytic objects like L-functions. On a much larger scale and without Tate's abelian hypothesis, the local correspondence also seems to extend to global objects.

Although we (meaning here the mathematical community) have not yet totally understood or proven the global correspondence, we do nonetheless have an idea of what it looks like and, importantly, what else it would imply. In particular, Lafforgue proved the correspondence for $\mathrm{GL}_n(K)$ for K a global field, but there remain more general formulations [**La**]. On the analogue to this essay's $\mathrm{GL}_n(K)$-side, the global program looks at automorphic forms, and on the other side it pairs them with suitable Galois representations.

When applied to number fields, the global correspondence carries varied implications. Along with encapsulating the modularity of elliptic curves (and hence Fermat's Last Theorem), the global correspondence and its generalized reciprocity seems to relate to problems in algebraic number theory like the Birch and Swinnerton-Dyer Conjecture, algebraic geometric problems like the Hodge conjecture, and analytic number theoretic problems like the Riemann hypothesis. Global Langlands and closely related subfields remain very active areas of research.

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