

CHARACTERISTIC CLASSES: AN INTRODUCTION

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The theory of characteristic classes plays an important role in the study of vector bundles. Since characteristic classes arise ultimately from certain bundles over the Grassmannian, we first examine the finite and infinite Grassmannians along with their canonical vector bundles. We show the fundamental result that any vector bundle over a paracompact manifold admits a map to such a canonical bundle over the infinite Grassmannian, and this map is unique up to homotopy. We then study one particular set of characteristic class, the Stiefel-Whitney classes, which we compute in limited cases. We use Stiefel-Whitney classes to examine when $\mathbb{R}P^n$ admits an immersion into \mathbb{R}^k and to prove that for n not of the form $2^k - 1$, $\mathbb{R}P^n$ is not parallelizable. As a corollary of the latter, division algebras may only exist in real dimension 2^k . We give a number of other, smaller applications.

1. MOTIVATION AND HISTORY

In the vaguest terms, a characteristic class is a cohomology class x of some space B associated to a bundle over B and such that x is natural with respect to bundle maps. This key naturality property allows us to study the cohomology of a manifold M (and hence the manifold itself) by pulling back the cohomology of other spaces under bundle maps. Although not obvious at first, this turns out to be an immensely powerful tool. We assume basic comfort with vector bundles going forward. The material in chapters 1-3 of Milnor and Stasheff's book *Characteristic Classes* [MS74] or something equivalent should suffice.

The study of characteristic classes arose in the 1930s-1940s as mathematicians including Hassler Whitney, Eduard Stiefel, and Shing-Shen Chern constructed certain natural cohomology classes of certain bundles like tangent bundles, sphere bundles, and complex vector bundles. Using earlier work of Charles Ehresmann on cell decompositions of Grassmannians, Pontrjagin and others were able to unify the theory of characteristic classes with new tools and techniques.

This essay largely follows [MS74], in particular chapters 4 and 5. However, we also include additional remarks and motivational commentary drawn both from other chapters in that book (including at least chapters 7, 8, 11, and 13) as well as from other sources altogether. We also supply some of our own examples, which are either original or constitute our own solutions to exercises in Milnor and Stasheff's book. Some of these are pointed out explicitly when they come as their own lemmas/propositions, but others are worked into remarks/proofs/explanations without comment.

2. GRASSMANNIANS

Although not obvious at first, Grassmannians play a key role in the theory of characteristic classes. In particular, we show how certain natural bundles over infinite Grassmannians constitute natural classifying spaces for all reasonable vector bundles.

2.1. The Finite Grassmannian $G(n, n+k)$. To begin, we define the finite Grassmannian.

Definition 2.1. We let $G(n, n+k)$ be the set of all n -planes through the origin in \mathbb{R}^{n+k} . This object is called the Grassmannian.

We now describe a natural topology on $G(n, n+k)$. Recall that an n -frame in \mathbb{R}^k is simply an (ordered) n -tuple of linearly independent vectors in \mathbb{R}^{n+k} . If we imagine our n -tuple as the columns of an n by $n+k$ matrix, we see that any n -tuple of vectors in \mathbb{R}^{n+k} is a frame if this matrix has full rank. Having full rank is an open condition, so the set $V(n, n+k)$ of n -frames is naturally an open subset of $(\mathbb{R}^{n+k})^n$, and so $V(n, n+k)$ inherits an ambient topology. Now consider the map

$$q : V(n, n+k) \rightarrow G(n, n+k)$$

which sends a frame to its \mathbb{R} -span. This map is clearly surjective since given an n -plane, any basis of that n -plane constitutes a frame. Thus, $G(n, n+k)$ is a quotient of $V(n, n+k)$, and we endow the Grassmannian with the resulting quotient topology.

Notice that instead of working with $V(n, n+k)$, we can also consider $V^0(n, n+k)$ the set of orthonormal frames, which is also open in $(\mathbb{R}^{n+k})^n$. The map

$$q' : V^0(n, n+k) \rightarrow G(n, n+k)$$

is similarly surjective, and so we endow $G(n, n+k)$ with the quotient topology. The Gram-Schmidt process gives a surjection $G : V(n, n+k) \twoheadrightarrow V^0(n, n+k)$ such that $q = q' \circ G$, and so these two topologies on $G(n, n+k)$ are the same.

So $G(n, n+k)$ is a perfectly good topological space, but what we really want is a manifold. After all, our eventual goal is to work with vector bundles over Grassmannians. The proof of the next proposition has several steps, but the key point is to pick a clever set of charts into Euclidean space.

Proposition 2.2. The Grassmannian $G(n, n+k)$ is a compact manifold of dimension nk .

Proof. As a preliminary, we show that $G(n, n+k)$ is Hausdorff. It suffices to show that for any two distinct points $X, Y \in G(n, n+k)$, there is a continuous function $\rho : G(n, n+k) \rightarrow \mathbb{R}$ such that $\rho(X) \neq \rho(Y)$.

Since X and Y are distinct n -planes in \mathbb{R}^{n+k} , pick some $w \in X \setminus Y$. For $Z \in G(n, n+k)$ let $\rho(Z) = (d(w, Z))^2$, where $d(w, Z)$ is the Euclidean distance from w to the plane Z . This is well defined since the Euclidean distance comes from the standard inner product, with respect to which \mathbb{R}^{n+k} is a Hilbert space. Now let z_1, \dots, z_n be an orthonormal basis for Z . Then we have

$$\rho(Z) = (w \cdot w) - \sum_{i=1}^n (w \cdot z_i)^2.$$

Then the composition $\rho \circ q' : V^0(n, n+k) \rightarrow \mathbb{R}$ is continuous, and hence so is ρ . Furthermore, $\rho(X) = 0 \neq \rho(Y)$, and so $G(n, n+k)$ is Hausdorff.

Next, we show compactness. Well, $V^0(n, n+k)$ is a closed and bounded subset of $(\mathbb{R}^{n+k})^n$, and so it is compact. Since $q' : V^0(n, n+k) \twoheadrightarrow G(n, n+k)$ is continuous, $G(n, n+k)$ is compact.

Now we show that $G(n, n+k)$ is actually a manifold. That is, we need to cover $G(n, n+k)$ by open sets which are compatibly homeomorphic to \mathbb{R}^{nk} . Let $X_0 \in G(n, n+k)$ be arbitrary, and we find such a neighborhood of $G(n, n+k)$. Let X_0^\perp be the orthogonal complement to X_0 in \mathbb{R}^{n+k} . Now let $U \subseteq G(n, n+k)$ be the set of n -planes $Y \subset \mathbb{R}^{n+k}$ such that the projection

$$p : X_0 \oplus X_0^\perp \rightarrow X_0$$

is surjective when restricted to Y , i.e., that $p|_Y$ is surjective. We may then regard Y as the graph of a linear function $T(Y) : X_0 \rightarrow X_0^\perp$. We then have a bijection $Y \mapsto T(Y)$ from U to $\text{hom}(X_0, X_0^\perp) \simeq \mathbb{R}^{nk}$. We show that T is a homeomorphism, and hence that $G(n, n+k)$ is a (topological) manifold.

Let x_1, \dots, x_n be an orthonormal basis of X_0 . For each $Y \in U$, Y has a unique basis such that $p(y_i) = x_i$ for all $1 \leq i \leq n$, and the choice of such n -frame depends continuously on Y itself. This further implies that since $y_i = x_i + T(Y)x_i$, $T(Y)$ is continuous in Y . So T is continuous. This same equation shows that y_i depends continuously then on $T(Y)$. Then Y is continuously dependent on $T(Y)$, or in other words, T^{-1} is continuous. Then T is a homeomorphism and $G(n, n+k)$ is a manifold.

We can also place a smooth structure on $G(n, n+k)$. Recall that a smooth structure on a manifold is equivalent to its set of smooth, real-valued functions. We simply declare that $f : G(n, n+k) \rightarrow \mathbb{R}$ is smooth if $f \circ q : V(n, n+k) \rightarrow \mathbb{R}$ is smooth, where $V(n, n+k)$ is naturally a smooth manifold as an open subset of $(\mathbb{R}^{n+k})^n$. It is an easy check that this yields a valid smooth structure since $q : V(n, n+k) \rightarrow G(n, n+k)$ is a fiber bundle. \square

Remark 2.3. *There are other ways to define smooth manifold structure on $G(n, n+k)$. One common approach is to consider the set of $(n+k) \times n$ matrices of rank n , by identifying such a matrix with the span of its columns. There are of course many matrices which yield the same n -plane in this way, but after some natural identifications we can regard this set of matrices as the Grassmannian. A different proof of smoothness than the one above can also be found in example 1.36 of [Lee13]. Some readers might find it easier to understand the smooth structure with this method, but the author sees the charts given in our proof above as having a more readily available geometric interpretation.*

Remark 2.4. *The Grassmannian $G(n, n+k)$ happens to be not just a smooth manifold but also an algebraic variety (or in more modern language, a scheme). In this way, Grassmannians play important roles in algebraic geometry that are beyond the scope of this essay. See, for example, [Har92].*

Example 2.5. *Projective space \mathbb{RP}^n is just $G(1, n+1)$, and the smooth structure given above matches the induced manifold structure from the natural projection $\pi : S^n \rightarrow \mathbb{RP}^n$.*

Intuitively, it should be clear that the data of an n -plane in \mathbb{R}^{n+k} should be the same as the data of its orthogonal projection, in some sort of smooth way. We formalize this intuition in the following lemma.

Lemma 2.6. *We have a canonical isomorphism $f : G(n, n+k) \rightarrow G(k, n+k)$ given by the orthogonal complement operation.*

Proof. This amounts to showing that Y^\perp depends smoothly on Y , since of course f is a bijection. Let U be the open neighborhoods of $G(n, n+k)$ as above. Let x_1, \dots, x_n form a basis for X_0 and x'_1, \dots, x'_k be a basis for X_0^\perp . Let $g : q^{-1}(U) \rightarrow V(k, n+k)$ by $g(y_1, \dots, y_n)$ is the $n+k$ -frame which results from applying Gram-Schmidt to $(y_1, \dots, y_n, x'_1, \dots, x'_k)$ and then taking the last k entries. By construction, we have $q \circ f = f \circ g|_U$. We see that f is smooth, and so $q \circ f$ is smooth. The result follows. \square

2.2. A natural bundle over $G(n, n+k)$.

Definition 2.7. Let $\gamma^n(n+k)$ be the bundle over $G(n, n+k)$ whose fiber over an n -plane $X \in \mathbb{R}^{n+k}$ consists of all ordered pairs (X, x) such that $x \in \mathbb{R}^{n+k}$ is contained in X . That is, informally, the fiber over X is X itself. This is topologized as a subset of $G(n, n+k) \times \mathbb{R}^{n+k}$. We let $\pi : E \rightarrow G(n, n+k)$ denote the natural projection $(X, x) \mapsto X$. We call $\gamma^n(n+k)$ the tautological bundle or the canonical bundle.

Lemma 2.8. The space $\gamma^n(n+k)$ is indeed a vector bundle. That is, it is locally trivial.

Proof. We trivialize on the same neighborhoods U that we used to give $G(n, n+k)$ a manifold structure. Namely, our goal is to define a homeomorphism $h : U \times X_0 \rightarrow \pi^{-1}(U)$ which is fiber-wise a linear isomorphism. We let $h(Y, x) = (Y, y)$ where $y = (p|_Y)^{-1}(x)$ for $p : \mathbb{R}^{n+k} \rightarrow X_0$ the orthogonal projection. Since $h(Y, x) = (Y, x + T(Y)x)$ and $h^{-1}(Y, y) = (Y, py)$, the result follows. \square

We defined this bundle $\gamma^n(n+k)$, but why? First, the bundle $\gamma^n(n+k)$ helps us understand the tangent bundle of the Grassmannian. We now prove this in the case where $n = 1$, i.e., for projective space.

Lemma 2.9. The tangent bundle $\tau_{\mathbb{R}P^n}$ is isomorphic to $\text{hom}(\gamma_n^1, \gamma^\perp)$, where γ_n^1 is our canonical bundle $\gamma^1(n+1)$ over $\mathbb{R}P^n$, and γ^\perp denotes the orthogonal complement of γ_n^1 in the trivial bundle ε .

Proof. Let L be a line in \mathbb{R}^{n+1} passing through the origin and a point $x \in S^n$, and let L^\perp be its orthogonal complement. We let $f : S^n \rightarrow \mathbb{R}P^n$ be the usual map. Recall that this induces an identification of tangent bundles. That is, $\tau_{\mathbb{R}P^n}$ is isomorphic to $\tau_{S^n}/((x, v) \sim (-x, -v))$. But each such pair $\{(x, v), (-x, -v)\}$ can be identified with a map $\ell : L \rightarrow L^\perp$ such that $\ell(x) = v$. So $T_{(x,v)}\mathbb{R}P^n$ can be identified with $\text{hom}(L, L^\perp)$. Thus, $\tau_{\mathbb{R}P^n}$ is endowed canonically with a global isomorphism to $\text{hom}(\gamma_n^1, \gamma^\perp)$. \square

Now we prove the more general statement. The next two lemmas, Lemma 2.10 and Lemma 2.11, constitute exercise 5b in [MS74].

Lemma 2.10. The tangent bundle τ of $G(n, n+k)$ is isomorphic to $\text{hom}(\gamma^n(n+k), \gamma^\perp)$, where γ^\perp is the orthogonal complement of $\gamma^n(n+k)$ in the trivial bundle ε .

Proof. We proceed analogously as for projective space. Fiber-wise, we have the natural map $\text{hom}(\gamma^n(n+k), \gamma^\perp) \rightarrow \tau$ on the fiber F_p by $\text{hom}(\gamma^n(n+k), \gamma^\perp)A_p \mapsto \frac{d}{ds}|_{s=0}((1+sA_p)p) \in \tau_p$, the fiber of τ over p . This map is fiber-wise bijective just as in the projective case by examining one of our usual charts U , and clearly the image of a smooth section varies smoothly in p . Then we have a bundle isomorphism. \square

As a point of interest, we also have the following lemma.

Lemma 2.11. *Let $M \subset \mathbb{R}^{n+k}$ be a smooth manifold of dimension n , and let $g\bar{g} : M \rightarrow G(n, n+k)$ be the generalized Gauss map obtained by taking the normal space over each point. Then the derivative $Dg\bar{g}$ induces a section of*

$$\text{hom}(\tau_M, \text{hom}(\tau_m, \nu)) \simeq \text{hom}(\tau_m \otimes \tau_m, \nu)$$

for τ_m the tangent bundle of M and ν the normal bundle.

Proof. Well, \bar{g} is covered by a bundle map $g \oplus g^\perp : \tau \oplus \nu \rightarrow \gamma^n(n+k) \oplus \gamma^\perp$, which acts on tangent and normal spaces in the natural way, and also the differential $Dg\bar{g} : \tau_M \rightarrow \tau_{G(n, n+k)} \simeq \text{hom}(\gamma^n(n+k), \gamma^\perp)$, where the last equality follows by Lemma 2.10. Then this gives a fiber-wise linear map $f : \tau_m \rightarrow \text{hom}_{\tau_M, \nu}$ by $(p, v) \mapsto (g^\perp)_{\bar{g}(p)}^{-1} \circ Dg\bar{g}_p(v) \circ g_p$. Then we get exactly a section of the desired bundle. \square

Remark 2.12. *It turns out that the section defined in the proof of Lemma 2.11 is the “second fundamental form”, an important tool in differential geometry for understanding curvature.*

Now we turn to the most important property of $\gamma^n(n+k)$: it turns out that **every other n -bundle over any compact base B maps into $\gamma^n(n+k)$** . These maps aren't unique, but nonetheless, this constitutes an important step towards understanding the importance of the Grassmannian and its canonical bundle.

Before proving this, we note the following useful observation: for any bundle ξ over a compact base B , a map $\hat{f} : E(\xi) \rightarrow \mathbb{R}^m$ which on each fiber is linear and injective gives rise to a bundle map $f : \xi \rightarrow \gamma^n(m)$. Specifically, we let $f(e) = (\hat{f}(F_e), \hat{f}(e))$, where F_e is the fiber containing e . Local triviality of ξ implies that f is continuous.

Lemma 2.13. *Any n -plane bundle ξ over a compact base B admits a bundle map $\xi \rightarrow \gamma^n(n+k)$, for some large k . (That is, k may depend on ξ .)*

Proof. Let U_i be a cover of B which is also a trivialization of ξ . Since B is a manifold (in particular, it is normal), we can find open sets V_i with $\bar{V}_i \subset U_i$, and similarly W_i such that $\bar{W}_i \subset V_i$. Then we may also construct a function λ_i which is identically 1 on \bar{W}_i and 0 on V_i^c . Since $\pi^{-1}(U_i)$ is trivial, there is a map $h_i : \pi^{-1}(U_i) \rightarrow \mathbb{R}^n$ that maps each fiber linearly. We think of this as ‘projection onto a fiber’. Now weighting h_i by λ_i allows us to extend this projection to all of B . Formally, we let $g_i : E \rightarrow \mathbb{R}^n$ by $g_i(e) = 0$ for $\pi(e) \notin V_i$ and $g_i(e) = h_i(e)\lambda_i(\pi(e))$ for $\pi(e) \in U_i$. Finally, we let $\hat{f}(e) = (g_1(e), \dots, g_r(e)) : E \rightarrow \mathbb{R}^{rn}$. Since \hat{f} is continuous and fiber-wise injective, the result follows. \square

For a space B which is not compact, we say a bundle ξ over B is of *finite type* if it admits a map to $\gamma^n(n+k)$ as in Lemma 2.13. Below in Section 3.5, we use Stiefel-Whitney classes to give an example of a bundle which is not of finite type.

2.3. The Infinite Grassmannians G_n . Our theory so far is encouraging: every n -bundle admits a map into $\gamma^n(m)$ for large enough m . We might hope that $\gamma^n(m)$ could be some sort of universal bundle, but we have two problems. First, we care about bases spaces which are not compact. As mentioned above, we show in Section 3.5 that the bundles $\gamma^n(m)$ do not universally admit maps from bundles over non-compact bases. Second, even in the compact case, we have no control over

m . There are bundles ξ over compact B which require unboundedly high numbers of neighborhoods to be trivialized, so at least with the above construction (and indeed with any construction), maps ξ into $\gamma^n(m)$ might require m to be unboundedly high.

Naively, we could try and take m to infinity and see what happens. We might guess that if we manage to define infinite objects, any ξ might map to $\gamma^n(\infty)$, whatever that might mean. Miraculously, this works! Not only that, but any ξ over a paracompact base (which covers most bases we run into in nature), ξ maps *uniquely* into $\gamma^n(\infty)$ once suitably defined. We now see how this is formalized.

Definition 2.14. Let \mathbb{R}^∞ denote the set of infinite sequences (x_1, x_2, \dots) of real numbers where all but finitely many entries are 0. We can naturally identify \mathbb{R}^k with the subset consisting of sequences for which $x_i = 0$ for all $i > k$. So $\mathbb{R}^1 \subset \mathbb{R}^2 \subset \dots$, and $\cup \mathbb{R}^k = \mathbb{R}^\infty$.

Definition 2.15. Identifying \mathbb{R}^k as a subset of \mathbb{R}^∞ as above, we define the infinite Grassmannian G_n as the n -dimensional linear subspaces of \mathbb{R}^∞ . We give this a topology by taking the direct limit of

$$G(n, n) \subset G(n, n+1) \subset \dots$$

Concretely, this means a set is open if its intersection with each $G(n, k)$ is open.

Example 2.16. When $n = 1$, we get the infinite projective space $\mathbb{R}P^\infty$. Since S^∞ is contractible and covers $\mathbb{R}P^n$ via the natural 2-to-1 projection, we have that $\mathbb{R}P^\infty$ is a $K(\mathbb{Z}/2\mathbb{Z})$ -space.

Remark 2.17. The higher homotopy groups of G_k are not quite so simple when $k > 1$. However, they are still in their own way remarkable. In particular, we have the fiber bundle

$$O(n) \rightarrow V_k \rightarrow G_k,$$

where V_k is the set of k -frames in \mathbb{R}^∞ , topologized similarly to G_k as the direct limit of the sequence $V(k, m)$ for increasing m . Since $V(k, m)$ is contractible, we get $\pi_i O(n) \simeq \pi_{i+1} G_k$. The homotopy groups of $O(n)$ are still hard, but remarkably, the stable homotopy groups of $O(n)$ are totally known and are 8-periodic. The interested reader can learn more about this surprising result, known as Bott Periodicity, from many sources including [Mil63], IV.23.

Definition 2.18. Let γ^n be the bundle over G_n which is the set of all pairs (X, x) where X is an n -plane in \mathbb{R}^∞ and $x \in \mathbb{R}^\infty$ lies on X , along with the map $\pi : E(\gamma^n) \rightarrow G_n$ by projection onto the first coordinate.

We omit the proof that γ^n is a bundle. It follows nearly identically to the proof in the finite case, although continuity arguments require a slight technical attention to the direct limit topology of G_n .

Recall that an open cover is *locally finite* if every point has a neighborhood which intersects only finitely many elements of the cover. A space is called *paracompact* if any cover has a locally finite refinement. Every paracompact space is normal (proven by Dieudonne). Morita proved the following theorem, whose proof we omit.

Theorem 2.19. The direct limit of $K_1 \subset K_2 \subset \dots$ is paracompact whenever all K_i are compact.

Corollary 2.20. The space G_n is paracompact.

Proof. Since $G(n, n+k)$ is compact, the result follows. □

Lemma 2.21. *Let ξ be a vector bundle over a paracompact space B . Then there is a countable locally finite covering of $\{U_i\}_{i \in \mathbb{N}}$ of B so that ξ is trivial on each U_i .*

Proof. First, pick any locally finite cover $\{V_\alpha\}$ so that $\xi|_{V_\alpha}$ is trivial, and let $\{W_\alpha\}$ such that $\overline{W_\alpha} \subset V_\alpha$. Finally, let $\lambda_\alpha : B \rightarrow \mathbb{R}$ be continuous so that λ_α is identically 1 on $\overline{W_\alpha}$ and 0 outside of V_α . For each finite $S \subset \{\alpha\}$ non-empty, let

$$U(S) = \{b \in B : \min_{\alpha \in S} \lambda_\alpha(b) > \max_{\alpha \notin S} \lambda_\alpha(b)\},$$

and let $U(k)$ be the union of all $U(S)$ which contain exactly k elements. Note that $B = U_1 \cup U_2 \cup \dots$, and each U_i is open. Notice that $U(S) \subset V_\alpha$ for each $\alpha \in S$. Then since $\{V_\alpha\}$ is locally finite, so is $\{U_k\}$, and furthermore ξ is trivial on $U(S)$. Then since U_k is a disjoint union of various $U(S)$, ξ is trivial on $U(k)$, and the result follows. \square

Remark 2.22. *Readers comfortable with fiber bundles should note that the above lemma in fact holds for fiber bundles ξ over paracompact B , with the same proof.*

Theorem 2.23. *Any n -plane bundle ξ over a paracompact base B admits a bundle map $\xi \rightarrow \gamma^n$.*

Proof. The proof follows just as in the finite case of Lemma 2.13. We use a countable locally finite cover U_i of B such that U_i is a trivialization of ξ , and construct h_i, λ_i , and g_i as before. Also similarly to before, we construct $\hat{f} : E \rightarrow \mathbb{R}^\infty$ by $e \mapsto (g_1(e), g_2(e), \dots)$. Locally finiteness ensures that only finitely many $g_i(e)$ are non-zero, and hence that our map really does land in \mathbb{R}^∞ . As before, \hat{f} implies the existence of a bundle map defined by $f(e) = (\hat{f}(F_e), \hat{f}(e))$ where F_e is the fiber containing e . \square

For now on, we assume all bundles are over paracompact base spaces.

Definition 2.24. *We say two bundle maps $f, g : \xi \rightarrow \gamma^n$ for an n -bundle ξ are ‘bundle-homotopic’ (or sometimes just ‘homotopic’) if there is a continuous map $h : \xi \times [0, 1] \rightarrow \gamma^n$ such that $h_0 = f, h_1 = g$, and $h(-, t) : \xi \rightarrow \gamma^n$ is a bundle map for all t .*

As promised, we now present the following theorem.

Theorem 2.25. *Let ξ be an n -plane bundle. Any two bundle maps $f, g : \xi \rightarrow \gamma^n$ are bundle-homotopic.*

Proof. Just as we have seen before, a bundle map $f : \xi \rightarrow \gamma^n$ is equivalent to a map $\hat{f} : E \rightarrow \mathbb{R}^\infty$ where $E = E(\xi)$, by $f(e) = (\hat{f}(F_e), \hat{f}(e))$.

We first examine a special case: suppose that for all $0 \neq e \in E$, $\hat{f}(e)$ is not equal to a negative multiple of $\hat{g}(e)$. Then we claim the linear homotopy $\hat{h}_t = (1-t)\hat{f} + t\hat{g}$ is a valid bundle homotopy. We show h is continuous in e and t . To this end, it suffices to show that the vector space operations of addition and scalar multiplication on \mathbb{R}^∞ are continuous. This follows from the following statement which we recall from point-set topology, which we call (*).

(*) Let $A_1 \subset A_2 \subset \dots, B_1 \subset B_2 \subset \dots$ be two sequences with direct limits A, B , respectively. Then the product topology on $A \times B$ coincides with the topology of the direct limit of $A_1 \times B_1 \subset A_2 \times B_2 \subset \dots$.

The assumption that $\hat{f}(e)$ and $\hat{g}(e)$ do not differ by a negative scalar now ensures that for $e \neq 0$, $h_t(e) \neq 0$. Then \hat{h}_t induces a bundle map $h : \xi \rightarrow \gamma^n$. So it remains to show that h is continuous.

For this, it suffices to show the induced map on base spaces $\bar{h} : B(\xi) \times [0, 1] \rightarrow G_n$ is continuous. Choose $U \subset B$ such that $\xi|_U$ is trivial, and so we may pick on U s_1, \dots, s_n linearly independent sections. We see that $\bar{h}|_U$ is the composition of the function $\phi : U \times [0, 1] \rightarrow V_n = (\mathbb{R}^\infty)^n$ by $(b, t) \mapsto (\hat{h}_t s_1(b), \hat{h}_t s_2(b), \dots)$, and the function $q : V_n \rightarrow G_n$. Since q is continuous and ϕ is also continuous by (*), then h is continuous, we are done in this case.

For the general case, we simply reduce to the case where $\hat{f}(e)$ and $\hat{g}(e)$ are not negative multiples of each other. Let $d_0 : \gamma^n \rightarrow \gamma^n$ be induced by the map $\mathbb{R}^\infty \rightarrow \mathbb{R}^\infty$ which takes the i th basis vector to the $(2i)$ th. Similarly, let d_1 be induced by taking the i th basis vector to the $(2i - 1)$ th. Now each of the pairs of bundle maps $(f, d_1 f), (d_1 f, d_0 g), (d_0 g, g)$ satisfy the assumption of our special case. Then by composing homotopies, we have that f and g are bundle-homotopic. \square

Remark 2.26. *The reader should note the essential role of infinite-dimensional space in the proof of Theorem 2.25. Intuitively, there is now ‘enough space’ to move one bundle map into any other. This intuition is, of course, made precise by the proof itself.*

Corollary 2.27. *Any bundle ξ over B uniquely determines a homotopy class of maps $\bar{f}_\xi : B \rightarrow G_n$.*

Proof. By Theorem 2.25, ξ admits a unique bundle map to γ^n , which we call f_ξ . Simply take the induced map on base spaces. \square

Finally, we are ready to define a characteristic class.

Definition 2.28. *For any coefficient group Λ , let $c \in H^i(G_n; \Lambda)$. The characteristic class determined by c of a bundle ξ over B is defined to be $\bar{f}_\xi^*(c) \in H^i(B; \Lambda)$.*

The key feature of characteristic classes is their *naturality*. In fact, one way of summarizing our result thus far is that the notion of characteristic class is the most general possible definition of any set of cohomology classes which are natural with respect to bundle maps.

3. STIEFEL-WHITNEY CLASSES

3.1. Definitions and Basic Examples. We now examine one particular set of characteristic classes, called the Stiefel-Whitney classes $w_i(\xi) \in H^*(B, \mathbb{Z}/2\mathbb{Z})$ for a bundle ξ over B . We define them below axiomatically but motivate them now in a different way. We saw above that characteristic classes of a paracompact space B arise precisely as the pullbacks of cohomology classes of G_n along bundle maps. So it seems natural that in order to understand characteristic classes, we might want to understand the cohomology of G_n . Doing so is beyond the scope of this essay, but we present the following theorem.

Theorem 3.1. *We have $H^*(G_n; \mathbb{Z}/2\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}[w_1(\gamma^n), \dots, w_n(\gamma^n)]$, where each $w_i(\gamma^n)$ has degree i . That is, $w_i(\gamma^n) \in H^i(G_n; \mathbb{Z}/2\mathbb{Z})$.*

In fact, one could say that by *definition*, $w_i(\gamma^n)$ are the free generators of $H^*(G_n; \mathbb{Z}/2\mathbb{Z})$ over $\mathbb{Z}/2\mathbb{Z}$, and that $w_i(\xi) := \bar{f}_\xi^*(w_i(\gamma^n))$. Although this is a perfectly reasonable way to proceed it, it requires some more work to get useful properties of the Stiefel-Whitney classes this way. More constructively, we could also define the Stiefel-Whitney classes using the Thom class and Steenrod square, but again, this would distract from our primary goals. Instead, we now define them axiomatically. **From now on, we assume all cohomology groups have coefficients in $\mathbb{Z}/2\mathbb{Z}$.**

Definition 3.2. To any vector bundle ξ an n -plane over B we associate cohomology classes $w_i(\xi) \in H^i(B)$ such that

- (1) $w_0(\xi) = 1 \in H^*(B)$ and $w_i(\xi) = 0$ for $i > n$,
- (2) w_i is natural; that is, for a bundle map $f : \xi \rightarrow \eta$ and the induced map $\bar{f} : B(\xi) \rightarrow B(\eta)$ on base spaces, we have $w_i(\xi) = f^*w_i(\eta)$,
- (3) the following product formula holds (called the Whitney product formula): for ξ, η bundles over the same space B ,

$$w_k(\xi \oplus \eta) = \sum_{i=0}^k w_i(\xi) \smile w_{k-i}(\eta),$$

- (4) and $w_1(\gamma_1^1) \neq 0$ for the tautological line bundle γ_1^1 over $\mathbb{R}P^1$.

The class $w_i(\xi)$ is called the n th Stiefel-Whitney class of ξ .

In particular, we do not prove that these axioms uniquely characterize a set of cohomology classes, or even that there exist any cohomology classes at all satisfying the axioms. As it happens, existence and uniqueness both hold. See [MS74] chapters 6-8 for proofs.

Proposition 3.3. We have the following properties of Stiefel-Whitney classes.

- (1) If $\xi \simeq \eta$, then $w_i(\xi) = w_i(\eta)$ for all i .
- (2) For ε the trivial bundle, $w_i(\varepsilon) = 0$ for all $i > 0$.
- (3) For ε trivial, we have $w_i(\varepsilon \oplus \xi) = w_i(\xi)$.
- (4) If we endow ξ with a Euclidean metric, and if ξ has a non-vanishing section s , then $w_n(\xi) = 0$. If ξ has k non-vanishing sections which are everywhere linearly independent, then $w_i(\xi) = 0$ for all $i > n - k$.

Proof. The first follows immediately by naturality. The second follows since there is a bundle map from ε to the bundle over a point (induced as a pullback of the constant map on base spaces). The third follows from the second by the Whitney product formula. For the fourth, note that the section s defines a sub-bundle of ξ which is the span of $s(b)$ for each point b , and this sub-bundle ε_s is trivial since s does not vanish. Then we write $\xi = \varepsilon_s \oplus \varepsilon_s^\perp$ and use the Whitney product formula to show that $w_n(\xi) = 0$. For the stronger statement with k non-vanishing sections, simply induct. \square

Just as with formal power series in one variable, the collection of infinite sums $a_0 + a_1 + \dots$ for $a_i \in H^i(B)$ forms a ring whose units are the sums with $a_0 = 1$. In fact, the proof is the same as in the case of formal power series in one variable, and so we omit it here. As useful bookkeeping strategy, we introduce the **total Stiefel-Whitney class** $w(\xi)$ of ξ to be the formal sum $\sum_{i=0}^{\infty} w_i(\xi)$, considered as an element of this ring. Since $w_0(\xi) = 1$ by definition, then $w(\xi)$ is invertible. We write \bar{w} for the inverse of w in the power series ring. With this notation, the Whitney product formula simplifies to

$$w(\xi \oplus \eta) = w(\xi)w(\eta).$$

Lemma 3.4. Let τ_M be the tangent bundle of a manifold M , and if the bundle is embedded in \mathbb{R}^n , let ν denote the normal bundle. Then $w(\tau_M) = \bar{w}(\nu)$.

Proof. Since $\varepsilon := \tau_M \oplus \nu$ is trivial, then we have $w(\tau_M)w(\nu) = w(\varepsilon) = 1$. The result follows. \square

We may now compute a number of increasingly interesting examples.

Example 3.5. Notice that the normal bundle ν of S^n under the standard embedding into \mathbb{R}^{n+1} is trivial. Then $w(\tau_{S^n}) = 1$. Even though we used a specific embedding of the sphere, our result is totally intrinsic.

Example 3.6. Recall that $H^*(\mathbb{R}P^n) \simeq \mathbb{Z}/2\mathbb{Z}[a]/(a^{n+1})$ where a has degree 1. Consider the inclusion $i : \mathbb{R}P^1 \hookrightarrow \mathbb{R}P^n$. The resulting pullback of γ_1^n is then isomorphic to γ_1^1 , since γ_1^1 is the restriction of γ_1^n to fibers over $\mathbb{R}P^1 \subset \mathbb{R}P^n$. Since by assumption $w_1(\gamma_1^1) \neq 0$, we have by naturality that

$$i^*w_1(\gamma_1^n) = w_1(\gamma_1^1) \neq 0,$$

and so $w_1(\gamma_1^n) = a$. Then since γ_1^n is a line-bundle, $w(\gamma_1^n) = 1 + a$.

Example 3.7. By construction, the bundle γ_n^1 sits naturally in the trivial $(n+1)$ -bundle ε^{n+1} over $\mathbb{R}P^n$. Let γ^\perp denote the orthogonal complement bundle. Then we have

$$w(\gamma^\perp) = \bar{w}(\gamma_n^1) = (1 + a)^{-1} = 1 + a + \cdots + a_n.$$

3.2. Calculating $w(\mathbb{R}P^n)$. For ease of notation, we let $w(M) = w(\tau_M)$ for a manifold M . The tangent bundle τ_{S^n} had simple Stiefel-Whitney class, but what about $\tau_{\mathbb{R}P^n}$? We might hope to leverage the natural covering map $S^n \rightarrow \mathbb{R}P^n$. In fact, we have already done so back in Lemma 2.9, which we use to prove the following theorem.

Theorem 3.8. We have $w(\mathbb{R}P^n) = (1 + a)^{n+1}$.

Proof. Notice that the bundle $\text{hom}(\gamma_n^1, \gamma_n^1)$ has a non-vanishing section, given at each point by the identity map. Since $\text{hom}(\gamma_n^1, \gamma_n^1)$ is a line bundle, it is then globally trivial. So by Lemma 2.9, we have

$$\begin{aligned} \tau_{\mathbb{R}P^n} \oplus \varepsilon^1 &\simeq \text{hom}(\gamma_n^1, \gamma^\perp) \oplus \text{hom}(\gamma_n^1, \gamma_n^1) \\ &\simeq \text{hom}(\gamma_n^1, \gamma^\perp \oplus \gamma_n^1) \simeq \text{hom}(\gamma_n^1, \varepsilon^{n+1}) \\ &\simeq \text{hom}(\gamma_n^1, \varepsilon^1 \oplus \cdots \oplus \varepsilon^1) \\ &\simeq \text{hom}(\gamma_n^1, \varepsilon^1) \oplus \cdots \oplus \text{hom}(\gamma_n^1, \varepsilon^1) \\ &\simeq \gamma_n^1 \oplus \cdots \oplus \gamma_n^1. \end{aligned}$$

Then by the Whitney product formula, we have

$$w(\tau) = w(\tau \oplus \varepsilon^1) = w(\gamma_n^1)^{n+1} = (1 + a)^{n+1}.$$

□

Example 3.9. Calculating $w(\mathbb{R}P^n)$ is now easy using binomial coefficients. For example,

$$\begin{aligned} w(\mathbb{R}P^5) &= 1 + a^2 + a^4, \\ w(\mathbb{R}P^{12}) &= 1 + a + a^4 + a^5 + a^8 + a^9 + a^{12}. \end{aligned}$$

3.3. Paralellizable Projective Space and Division Algebras.

Corollary 3.10. If n is not of the form $2^k - 1$, then $\mathbb{R}P^n$ is not paralellizable.

Proof. If $\mathbb{R}P^n$ is parallelizable, then $\tau_{\mathbb{R}P^n}$ is trivial (by definition). But then

$$w(\mathbb{R}P^n) = (1 + a)^n = 1.$$

This is only satisfied when n is of the form $2^k - 1$, by the binomial formula. \square

In fact, only $\mathbb{R}P^1$, $\mathbb{R}P^3$, and $\mathbb{R}P^7$ are parallelizable (as can be seen using the \mathbb{C} , the quaternions \mathbb{H} , and the octonions \mathbb{O} , respectively). To prove this, however, is more difficult. In fact, parallelizability of $\mathbb{R}P^n$ is intimately linked to division algebras, as we see now.

Theorem 3.11. *If there exists a bilinear product operation $p : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ without zero-divisors (not necessarily associative), then $\mathbb{R}P^{n-1}$ is parallelizable. Then by Corollary 3.10, n must be a power of 2.*

Proof. Choose a basis b_1, \dots, b_n for \mathbb{R}^n . Since p has no 0-divisors, the kernel of $y \mapsto p(y, b_i)$ is trivial, and so this map defines an isomorphism of \mathbb{R}^n . Define $v_i : \mathbb{R}^n \rightarrow \mathbb{R}^n$ by

$$v_i(p(y, b_i)) = p(y, b_1),$$

and observe that each $v_1(x), \dots, v_n(x)$ are all linearly independent by construction for $x \neq 0$. Additionally, $v_1(x) = x$.

As a result, the functions v_2, \dots, v_n define $n - 1$ linearly independent sections of $\tau_{\mathbb{R}P^n} \simeq \text{hom}(\gamma_{n-1}^\perp, \gamma^\perp)$ in the following way. Recall that at a point, the tangent space at $x \in \mathbb{R}P^{n-1}$ is defined by a map $L \rightarrow L^\perp$ for L the line in \mathbb{R}^n representing x . So we let $\bar{v}_i(x) = \pi_L \circ v_i$, where π_L is the orthogonal projection $\mathbb{R}^n \rightarrow L^\perp$. Since $\bar{v}_2, \dots, \bar{v}_n$ are linearly independent at each point, $\tau_{\mathbb{R}P^{n-1}}$ is trivial. \square

3.4. Immersions and Plane-Fields. We study immersions of M^n into \mathbb{R}^{n+k} . Let ν be the resulting normal bundle. (Even if the immersion is not an embedding, recall that we may still define ν locally to be the orthogonal complement to the tangent space. Or without a metric, we let it be the quotient of the ambient tangent space by $T_p M$.) Since $\tau_M \oplus \nu$ is the trivial bundle, we have $w_i(\nu) = \bar{w}_i(M)$. Since ν is a k -plane bundle, this implies that $w_i(\nu) = 0$ for $i > k$. We can use this to rule out the possibility of certain immersions.

Example 3.12. *Consider $\mathbb{R}P^6$. Then*

$$\bar{w}(\mathbb{R}P^5) = (1 + a^2 + a^4)^{-1} = (1 + a^2).$$

Thus if we want to immerse $\mathbb{R}P^6$ into \mathbb{R}^{6+k} , we must have $k \geq 2$.

Example 3.13. *Consider $\mathbb{R}P^{2^n}$. We have $w(\mathbb{R}P^{2^n}) = 1 + a + a^n$, and so $\bar{w}(\mathbb{R}P^{2^n}) = 1 + a + a^2 + \dots + a^{n-1}$. Thus if $\mathbb{R}P^{2^n}$ may be immersed into \mathbb{R}^{2^n+k} , then $k \geq 2^n - 1$. Remarkably, this simple result gives us the following corollary.*

Corollary 3.14. *Whitney's strong immersion theorem, which states that any M^n admits an immersion into \mathbb{R}^{2n-1} , is sharp. (See [Kha11] for a proof of Whitney's theorem.) That is, $2n - 1$ is the lowest possible universal linear bound.*

Returning again to general manifolds, we show the following result. (The next two statements of Proposition 3.15 and Example 3.16 constitute exercise 4d of [MS74].)

Proposition 3.15. *If M^n admits an immersion into \mathbb{R}^{n+1} , then $w_i(M) = w_1(M)^i$, where exponentiation is taken to mean repeated cup-product.*

Proof. As usual, we have $w(M)w(\nu) = 1$ for ν the normal bundle. Well, since ν is a line-bundle, then $w(\nu) = 1 + w_1(\nu)$. If $w_1(\nu) = 0$, then $w(M) = 1$, and the result follows trivially. If $w_1(\nu) \neq 0$, then $(1 + w_1(\nu))^{-1} = (1 + w_1(\nu) + \cdots + w_1(\nu)^n) = w(M)$, and again the result follows. \square

Example 3.16. *Let $M = \mathbb{R}P^n$ in Proposition 3.15. Then since $H^*(\mathbb{R}P^n) = \mathbb{Z}/2\mathbb{Z}[a]/(a^{n+1})$, it follows that $w(\mathbb{R}P^n)$ is either 1 or $1 + a + \cdots + a^n$, depending on $w_1(\mathbb{R}P^n)$. By the binomial formula, this means that n is of the form $2^r - 1$ or $2^r - 2$, respectively.*

The same techniques we have been using to study immersions have other applications, as we see in the following two examples (which together are exercise 4c in [MS74]).

Example 3.17. *We say M has a field of tangent k -planes if τ_M has a sub-bundle of dimension k . Suppose that $\mathbb{R}P^n$ admits a field of tangent lines (i.e., 1-planes), ξ . Then we have $w(\xi)w(\xi^\perp) = w(\mathbb{R}P^n) = (1 + a)^{n+1}$. Since ξ is a line bundle, then either $w(\xi) = 1 + a$ or $w(\xi) = 1$. In the first case, we have $w(\xi^\perp) = (1 + a)^n$. But ξ^\perp is an $(n - 1)$ -plane bundle, so $w_n(\xi^\perp)$ must be 0, and we have a contradiction. If $w(\xi) = 1$, then $w(\xi^\perp) = (1 + a)^{n+1}$. The condition $w_n(\xi^\perp) = 0$ now tells us that n must be odd, by the binomial formula.*

If n is odd, then S^n admits a nonvanishing vector field. To see this, embed $S^n \hookrightarrow \mathbb{R}^{n+1}$ in the standard way. Choose any set of pairwise orthogonal coordinate 2-planes in \mathbb{R}^{n+1} , and each one then defines a transformation of \mathbb{R}^{n+1} induced by rotating around the unit circle in this plane. The vector field is then described by simultaneous rotations in each plane. Put another way, the relevant flow is induced by a block-diagonal matrix whose diagonal blocks are 2x2 rotation matrices acting on \mathbb{R}^{n+1} , restricted to S^n .

In summary, $\mathbb{R}P^n$ admits a field of 1-planes if and only if n is odd.

Example 3.18. *We may similarly study which $\mathbb{R}P^n$ admit fields of higher-dimensional planes. For example, neither $\mathbb{R}P^4$ nor $\mathbb{R}P^6$ admits a field of 2-planes. Since $w(\mathbb{R}P^4) = (1 + a)^5 = 1 + a + a^4$, a field of 2-planes ξ imply that $w(\xi)w(\xi^\perp) = 1 + a + a^4$, where $w(\xi)$ and $w(\xi^\perp)$ are each polynomials in a of degree at most 2, since they are both 2-plane bundles. Simple combinatorics tells us this is impossible. There are only four possibilities for each of these, namely $1, 1 + a, 1 + a^2, 1 + a + a^2$, and by simply multiplying all pairs we see that $w(\xi)w(\xi^\perp) = 1 + a + a^4$ is never satisfied.*

The argument for $\mathbb{R}P^6$ is similar. We must have $1 + a + a^2 + a^3 + a^4 = p(a) + q(a)$ for polynomials $p, q \in \mathbb{Z}/2\mathbb{Z}[a]/(a^7)$ with p at most degree 2 and q at most degree 4. A similar (but more tedious) calculation tells us this is impossible.

Remark 3.19. *Since $\mathbb{R}P^n$ for n even does not admit a field of lines, neither does S^n . For if such a line field existed on S^n , the projection $S^n \rightarrow \mathbb{R}P^n$ would induce a line field on $\mathbb{R}P^n$. For readers familiar with complex manifolds and complex vector bundles, this has the following interesting consequence.*

Since $\mathbb{C}P^1$ is diffeomorphic as a real manifold to S^2 , then $\mathbb{C}P^1$ does not admit any (real) line field. This implies that $T\mathbb{C}P^1$ and $\overline{T\mathbb{C}P^1}$ are not isomorphic as complex line bundles, where $T\mathbb{C}P^1$ is the complex tangent space and $\overline{T\mathbb{C}P^1}$ denotes the complex conjugate bundle of $T\mathbb{C}P^1$. This is because

locally at $T_p\mathbb{C}\mathbb{P}^1 \simeq \mathbb{C}$ for $p \in \mathbb{C}\mathbb{P}^1$, any such isomorphism would induce an orientation-reversing real vector space isomorphism, which is simple reflection across some line. But since $\mathbb{C}\mathbb{P}^1$ has no real line field, then globally, constructing an isomorphism is impossible.

Along with asking for single sub-bundles of TM , we can also ask about collections of linearly independent sections. (The following proposition is exercise 4b in [MS74].)

Proposition 3.20. *Let $n = 2^r m - 1$ with m odd. Then there do not exist 2^r vector fields on $\mathbb{R}\mathbb{P}^n$.*

Proof. By Theorem 3.8, $w(\mathbb{R}\mathbb{P}^n) = (1+a)^{2^r m}$. Since our coefficient ring is $\mathbb{Z}/2\mathbb{Z}$, then $(1+a)^2 = 1+a^2$, and so $w(\mathbb{R}\mathbb{P}^n) = (1+a^{2^r})^m$. Then

$$w_{n-2^r+1}(\mathbb{R}\mathbb{P}^n) = w_{2^r(m-1)}(\mathbb{R}\mathbb{P}^n) \neq 0$$

since $(1+a^{2^r})^m$ has coefficient $m \neq 0$ in degree 2^r . Then the result follows by part (4) of Proposition 3.3. \square

3.5. Bundles of Finite Type. In this final section, we show that the bundle γ^1 over $\mathbb{R}\mathbb{P}^\infty$ does not admit a bundle map into $\gamma^n(n+k)$ for any k . This demonstrates some interesting mathematics, and it also somehow gives us more philosophical grounding for the importance of moving from finite to infinite Grassmannians. (This section also constitutes a solution to exercise 5e of [MS74].)

Definition 3.21. *We say an n -plane bundle ξ over B has finite type if it admits a bundle map $\xi \rightarrow \gamma^n(n+k)$, the canonical n -plane bundle over $G(n, n+k)$.*

Lemma 3.22. *The bundle ξ is of finite type if and only iff there is some (finite-dimensional) η also over B such that $\xi \oplus \eta$ is trivial.*

Proof. Recall that a map $f : \xi \rightarrow \gamma^n(n+k)$ is equivalent to a map \hat{f} to \mathbb{R}^{n+k} which is fiber-wise linear and injective. We may view \hat{f} as a map $\xi \rightarrow \varepsilon^{n+k}$, the trivial bundle over B . Then take η to be ξ^\perp in ε^{n+k} .

Conversely, given η , we take $\xi \rightarrow \xi \oplus \eta \simeq B \times \mathbb{R}^{n+k} \rightarrow \mathbb{R}^{n+k}$ where the first map is inclusion and the last is projection. Then we take the composition to be \hat{f} . \square

Proposition 3.23. *The bundle γ^1 over $\mathbb{R}\mathbb{P}^\infty$ does not have finite type.*

Proof. Suppose $\gamma^1 \oplus \eta$ is trivial. Then $w(\eta) = (1+a)^{-1} = 1+a+a^2+\dots$. But η is a k -plane bundle for some k , and so $w_i(\eta) = 0$ for all $i > k$. This is a contradiction. \square

So not all bundles have finite type. But how pathological is the example γ^1 ? The next lemma and proposition give us some more insight into when a bundle has finite type.

Lemma 3.24. *If B is paracompact (and hence normal), then ξ has finite type if and only if B is covered by finitely many U_i on which ξ is trivial.*

Proof. If such U_i exist, then we already constructed such a map $f : \xi \rightarrow \gamma^n(n+k)$ in Lemma 2.13. Conversely, if there exists such f , we may take trivializations of $\gamma^n(n+k)$ on $G(n, n+k)$ and pull them back along f . Since $G(n, n+k)$ is compact, we can then cover B with finitely many of these pull-backs. \square

Lemma 3.25. *Suppose B is paracompact with finite covering dimension. Then every ξ over B has finite type.*

Proof. We recall the notations in the proof of Lemma 2.21. Finite covering dimension implies that for large enough k , $U(S) = \emptyset$ for $|S| > k$. Then there are finitely many sets U_i on which cover B and which trivialize ξ . So letting $\pi_i : U_i \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ be the projection, we have the map $\xi \rightarrow \mathbb{R}^{nk}$,

$$(p, v) \mapsto \bigoplus_{i=1}^k \lambda_i(p) \pi_i(p, v).$$

Then ξ has finite type. □

4. FURTHER TOPICS

We have seen a general framework for characteristic classes coming from the infinite Grassmannians G_n , and we have developed a range of applications for the Stiefel-Whitney classes.

Another surprising connection with Stiefel-Whitney classes that we omit here is their relation to (unoriented) cobordism. In particular, the Stiefel-Whitney classes give rise to a collection of elements of $\mathbb{Z}/2\mathbb{Z}$ called the Stiefel-Whitney numbers. A compact manifold M is the boundary of some other manifold if and only if all these numbers are 0 (see [MS74] for the forward direction and [Sto58] for the harder converse).

We might also look at other coefficient rings than $\mathbb{Z}/2\mathbb{Z}$. With coefficients in \mathbb{Z} , the top Stiefel-Whitney class has an analogue called the Euler class. Analogously to how the Stiefel-Whitney classes gave sharpness of Whitney's immersion theorem, we can use the Euler class to give sharpness to Whitney's embedding theorem. A proof of that theorem can be found in [Ada93], and some key ideas (i.e., the 'Whitney trick') are explained also in [Mil65]. The Euler class arises also as an obstruction class, an object which measures the failure to extend maps onto higher skeleta of CW -complexes. The obstruction perspective is available in [Ste51] chapters 25, 35, and 38.

For complex manifolds, the natural analogues of Stiefel-Whitney classes are the Chern classes, which arise from the natural generators of the ring $H^*(G_n(\mathbb{C}^\infty), \mathbb{Z})$ where $G_n(\mathbb{C}^\infty)$ is the set of complex n -planes in \mathbb{C}^∞ (defined analogously to \mathbb{R}^∞), viewed as a complex manifold. Chern classes and the closely related Pontryagin classes contain information about obstructions, linear independence of sections, and even oriented cobordism, among other topics. For an introduction to these topics we suggest [MS74].

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