CLASSICAL MODULAR FORMS AS AUTOMORPHIC FORMS

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The object of this note is to describe how the classical theory of modular forms can be interpreted as a special case of the more general theory of automorphic forms.

We first recall the classical notion of modular form (in the spirit of [4]). and then explain its relation on the one hand to the general notion of automorphic forms on the adelic group $\mathrm{GL}_2(\mathbb{A})$, and on the other hand to the geometrical interpretation of a modular form as a section of a line bundle on a suitable (so-called modular) curve. Both relations have been explained in many other places (for example [2, 3], all of which have very much influenced our presentation). Nevertheless, it might be useful to recall them again here. We have attempted to proceed in as succint and as natural a manner as possible, with a minimal number of *ad hoc* constructions. On the other hand, we have tried to give enough detail that our discussion might provide a useful translation aid for a reader who is familiar with one point of view but not the other. We close the section by recalling the interpretation of classical Eisenstein series (as described for example in [4]) as a special case of the general construction of Eisenstein series in the theory of automorphic forms.

In the following discussion it will be convenient to fix the following basis for the complexified Lie algebra $\mathfrak{gl}_2$ of $\mathrm{GL}_2(\mathbb{R})$:

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad H = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Y_+ = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \quad Y_- = \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}.$$

Note that Z is central in $\mathfrak{gl}_2$, that $[H, Y_\pm] = \pm 2Y_\pm$, and that $[Y_+, Y_-] = 4H$. We also have the formulas $\overline{Z} = Z$, $\overline{H} = -H$, $\overline{Y_\pm} = Y_\mp$. (Here $^-$ denotes complex conjugation on $\mathfrak{gl}_2$.)

We also recall that the centre $\mathfrak{z}(\mathfrak{gl}_2)$ of the universal enveloping algebra of $\mathfrak{gl}_2$ is isomorphic to a polynomial ring in two generators. In terms of the above basis elements, we have

$$\mathfrak{z}(\mathfrak{gl}_2) = \mathbb{C}[Z\,,\, Y_+Y_- + H^2 - 2H].$$

1. Bases

The set of complex numbers $\mathbb{C}$ forms a two-dimensional real vector space. Let $\mathrm{GL}(\mathbb{C})$ denote the group of $\mathbb{R}$-linear automorphisms of $\mathbb{C}$. Since any complex linear automorphism is certainly $\mathbb{R}$-linear, there is a natural inclusion $\mathbb{C}^\times \subset \mathrm{GL}(\mathbb{C})$.

1.1. **Definition.** Let $\mathcal{B}$ denote the set of bases of $\mathbb{C}$ as an $\mathbb{R}$-vector space, i.e. $\mathcal{B} := \mathrm{Iso}_{\mathbb{R}}(\mathbb{R}^2, \mathbb{C}) \xrightarrow{\sim} \{\, (z_1, z_2) \in \mathbb{C}^2 \mid z_1 \text{ and } z_2 \text{ are } \mathbb{R}\text{-linearly independent}\,\}$; here $\mathrm{Iso}_{\mathbb{R}}(\mathbb{R}^2, \mathbb{C})$ denotes the set of $\mathbb{R}$-linear isomorphisms from $\mathbb{R}^2$ to $\mathbb{C}$, and the indicated bijection is given by

$$\mathrm{Iso}_{\mathbb{R}}(\mathbb{R}^2, \mathbb{C}) \ni \imath \mapsto (\imath(1,0), \imath(0,1)).$$

The second description of $\mathcal{B}$ realizes it as an open subset of $\mathbb{C}^2$, and thus endows it with the structure of a complex analytic manifold.

Pre- and post- composition of elements of $\mathcal{B}$ with elements of $\mathrm{GL}_2(\mathbb{R})$ and $\mathrm{GL}(\mathbb{C})$ respectively induce commuting right and left actions of these groups on $\mathcal{B}$. Evidently $\mathcal{B}$ becomes a principal homogeneous space under each of these actions.

Both these actions admits simple descriptions in terms of the realization of $\mathcal{B}$ as an open subset of $\mathbb{C}^2$. On the one hand, if $g \in \mathrm{GL}(\mathbb{C})$, then

$$g \cdot (z_1, z_2) = (g(z_1), g(z_2)).$$

On the other hand, the left action of $\mathrm{GL}_2(\mathbb{C})$ on $\mathbb{C}^2$ restricts to a left action of $\mathrm{GL}_2(\mathbb{R})$, which evidently preserves $\mathcal{B}$, and one immediately confirms that this left action of $\mathrm{GL}_2(\mathbb{R})$ is related to the previously considered right action of this same group via the formula $g \cdot \imath = \imath \circ g^t$. (Here g^t denotes the transpose of g.) In particular, we see that the right action of $\mathrm{GL}_2(\mathbb{R})$ on $\mathcal{B}$ is via complex analytic automorphisms. In the remainder of this note we will work systematically with the right action of $\mathrm{GL}_2(\mathbb{R})$ on $\mathcal{B}$, rather than with the left action obtained by passing to transposes. However, in classical texts (e.g. [4]) it is often the left action that appears. [1]

If we fix a base-point $\imath_0 \in \mathcal{B}$, then acting on $\imath_0$ from the left and the right respectively yields simultaneous identifications $\mathrm{GL}(\mathbb{C}) \xrightarrow{\sim} \mathcal{B}$ and $\mathrm{GL}_2(\mathbb{R}) \xrightarrow{\sim} \mathcal{B}$. The resulting identification $\mathrm{GL}(\mathbb{C}) \xrightarrow{\sim} \mathrm{GL}_2(\mathbb{R})$ is the natural one induced by our choice of basis $\imath_0$. The respective left and right actions of $\mathrm{GL}(\mathbb{C})$ and $\mathrm{GL}_2(\mathbb{R})$ on $\mathcal{B}$ then become identified with the actions of $\mathrm{GL}_2(\mathbb{R})$ on itself by left and right translations.

In addition to the two descriptions of $\mathcal{B}$ given in Definition 1.1, there is a third very useful description, namely the complex analytic isomorphism

$$\mathcal{B} \xrightarrow{\sim} (\mathbb{C} \setminus \mathbb{R}) \times \mathbb{C}^\times \tag{1}$$

defined by $(z_1, z_2) \mapsto (z_1/z_2, z_2)$. We denote a typical element of the target of this isomorphism via (τ, z). The right $\mathrm{GL}_2(\mathbb{R})$-action on $\mathcal{B}$ admits the following description in terms of these coordinates:

$$(\tau, z)\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (\frac{a\tau + c}{b\tau + d}, (b\tau + d)z).$$

The left action of $\mathrm{GL}(\mathbb{C})$ is not so easily described in terms of these coordinates, but the action of its subgroup $\mathbb{C}^\times$ is: if $\alpha \in \mathbb{C}^\times$ then

$$\alpha \cdot (\tau, z) = (\tau, \alpha z).$$

[1] Explicitly, the left $\mathrm{GL}_2(\mathbb{R})$-action on $\mathcal{B}$ is given by the classical formula

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(z_1, z_2) = (az_1 + bz_2, cz_1 + dz_2).$$

Correspondingly, the right $\mathrm{GL}_2(\mathbb{R})$-action is given by the formula

$$(z_1, z_2)\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (az_1 + cz_2, bz_1 + dz_2).$$

2. The holomorphic tangent space of $\mathcal{B}$

Let $\imath : \mathbb{R}^2 \xrightarrow{\sim} \mathbb{C}$ be a point of $\mathcal{B}$. The action of $\mathrm{GL}_2(\mathbb{R})$ on $\imath$ induces an isomorphism $\mathrm{GL}_2(\mathbb{R}) \xrightarrow{\sim} \mathcal{B}$ (since $\mathcal{B}$ is a principal homogeneous space under the right action of $\mathrm{GL}_2(\mathbb{R})$), and hence an isomorphism of (complexified) tangent spaces

$$T_I \mathrm{GL}_2(\mathbb{R}) \xrightarrow{\sim} T_\imath \mathcal{B}. \tag{2}$$

(Here $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the identity of $\mathrm{GL}_2(\mathbb{R})$.) The source of this map is precisely the (complexified) Lie algebra $\mathfrak{gl}_2$ of $\mathrm{GL}_2(\mathbb{R})$.

Since $\mathcal{B}$ is a complex analytic manifold, we may write $T_\imath \mathcal{B} = T_\imath^{\mathrm{hol}} \mathcal{B} \bigoplus \overline{(T_\imath^{\mathrm{hol}} \mathcal{B})}$ as the direct sum of its holomorphic and anti-holomorphic subspaces.

2.1. **Definition.** Let $V_\imath \subset \mathfrak{gl}_2$ (resp. $\overline{V}_\imath \subset \mathfrak{gl}_2$) denote the preimage of $T_\imath^{\mathrm{hol}} \mathcal{B}$ (resp. $\overline{(T_\imath^{\mathrm{hol}} \mathcal{B})}$) under the isomorphism (2). (Note that the isomorphism (2) preserves the underlying real structures on each of the source and target, and so $\overline{V}_\imath$ is simply the complex conjugate of $V_\imath$ in $\mathfrak{gl}_2$.)

Our goal in this section is to describe $V_\imath$ (and hence $\overline{V}_\imath$) explicitly as subspaces of $\mathfrak{gl}_2$. Since $\mathrm{GL}_2(\mathbb{R})$ acts complex analytically on $\mathcal{B}$, one sees that $V_{\imath \circ g} = \mathrm{Ad}_{g^{-1}}(V_\imath)$ for any $g \in \mathrm{GL}_2(\mathbb{R})$. Thus, since $\mathrm{GL}_2(\mathbb{R})$ also acts transitively on $\mathcal{B}$, we see that it suffices to calculate $V_{\imath_0}$ for some fixed choice of base-point $\imath_0 \in \mathcal{B}$.

2.2. **Lemma.** *If $\imath_0 \in \mathcal{B}$ corresponds to the basis $(i,1)$ of $\mathbb{C}$, then the subspace $V_{\imath_0}$ (resp. $\overline{V}_{\imath_0}$) of $\mathfrak{gl}_2$ is equal to the span of $Z + H$ and Y_+ (resp. the span of $Z - H$ and Y_-).*

Proof. We will work in the coordinates (τ, z) provided by the isomorphism (1). Then $\imath_0$ corresponds to the point $(i,1) \in (\mathbb{C} \setminus \mathbb{R}) \times \mathbb{C}^\times$. We compute that

$$(i,1)\begin{pmatrix} a & 0 \\ c & 1 \end{pmatrix} = (c + ai, 1).$$

(for any $(a,b) \in \mathbb{R}^2 \setminus \{(0,0)\}$) and that

$$(i,1)\begin{pmatrix} a & b \\ -b & a \end{pmatrix} = (i, a + bi)$$

Once we recall that on any complex manifold with local coordinates $z_\mu = x_\mu + y_\mu i$, the holomorphic tangent bundle is (locally) spanned by the vectors fields $\partial_{z_\mu} := \frac{1}{2}(\partial_{x_\mu} - i\partial_{y_\mu})$, we conclude that $V_{\imath_0}$ is equal to the span of

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - i\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - i\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}.$$

(We have omitted the factors of $1/2$, since these obviously don't affect the span.) One immediately checks that this coincides with the span of $Z+H$ and Y_+. Passing to complex conjugates then gives the result for $\overline{V}_{\imath_0}$. □

3. Lattices

Let $\mathcal{L}$ denote the set of discrete rank 2 lattices contained in $\mathbb{C}$. The map $\mathcal{B} \to \mathcal{L}$ defined by $\imath \mapsto \imath(\mathbb{Z}^2)$ induces a bijection

$$\mathcal{B}/\mathrm{GL}_2(\mathbb{Z}) \xrightarrow{\sim} \mathcal{L}. \tag{3}$$

Since $\mathrm{GL}_2(\mathbb{Z})$ acts both complex analytically and properly discontinuously on $\mathcal{B}$ (it is a discrete subgroup of $\mathrm{GL}_2(\mathbb{R})$ and $\mathcal{B}$ is a principal homogeneous $\mathrm{GL}_2(\mathbb{R})$-space), we see that $\mathcal{L}$ is naturally a complex analytic manifold.

3.1. **Definition.** For any $N \geq 1$, let $\mathcal{L}(N)$ denote the set of discrete rank 2 lattices contained in $\mathbb{C}$ equipped with a level N structure; i.e.

$$\mathcal{L}(N) := \{ (L, \jmath) \quad | \quad L \in \mathcal{L}, \quad \jmath : (\mathbb{Z}/N\mathbb{Z})^2 \xrightarrow{\sim} L/NL \}.$$

The map $\mathcal{B} \times \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}) \to \mathcal{L}(N)$ defined by $(\imath, g) \mapsto (\imath(\mathbb{Z}^2), \imath_N \circ g^{-1})$ (where $\imath_N$ denotes the isomorphism $(\mathbb{Z}/N\mathbb{Z})^2 \xrightarrow{\sim} \imath(\mathbb{Z}^2)/N\imath(\mathbb{Z}^2)$ induced by $\imath$) induces a bijection

$$\big(\mathcal{B} \times \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})\big)/\mathrm{GL}_2(\mathbb{Z}) \xrightarrow{\sim} \mathcal{L}(N). \tag{4}$$

Thus $\mathcal{L}(N)$ is naturally a complex analytic manifold for any $N \geq 1$. The complex analytic map $\mathcal{L}(N) \to \mathcal{L}$ given by forgetting $\jmath$ realizes $\mathcal{L}(N)$ as a covering space of $\mathcal{L}$, with covering group $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$.

Since the $\mathrm{GL}(\mathbb{C})$-action on $\mathcal{B}$ commutes with the $\mathrm{GL}_2(\mathbb{R})$-action (and so with the $\mathrm{GL}_2(\mathbb{Z})$-action), the bijection (4) yields a left $\mathrm{GL}(\mathbb{C})$-action on $\mathcal{L}(N)$. Explicitly, if $g \in \mathrm{GL}(\mathbb{C})$, then $g \cdot (L, \jmath) = (g(L), g_N \circ \jmath)$ (where g_N denotes the isomorphism $L/NL \xrightarrow{\sim} g(L)/Ng(L)$ induced by g). In particular, we obtain a $\mathbb{C}^\times$-action on $\mathcal{L}(N)$.

The left regular action of $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ on itself induces an action of $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ on $\mathcal{B} \times \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$. This action evidently commutes with the right $\mathrm{GL}_2(\mathbb{Z})$-action on the same space, and so descends via (4) to an action on $\mathcal{L}(N)$. Explicitly, if $g \in \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ and $(L, \jmath) \in \mathcal{L}(N)$, then $g \cdot (L, \jmath) = (L, \jmath \circ g^{-1})$.

4. Modular forms

The following definition is an immediate generalization of the definition of modular forms of weight k and level one given in [4].

4.1. **Definition.** A modular form of weight k and level N is a function $F : \mathcal{L}(N) \to \mathbb{C}$ such that

(1) F is holomorphic;
(2) $F\big(\alpha \cdot (L, \jmath)\big) = \alpha^{-k} F\big((L, \jmath)\big)$ for all $\alpha \in \mathbb{C}^\times$, $(L, \jmath) \in \mathcal{L}(N)$;
(3) F satisfies a growth condition at infinity (which we will make precise in Remark 4.4 below).

The descriptions (1) of $\mathcal{B}$ and (4) of $\mathcal{L}(N)$ taken together provide the following alternative description of modular forms.

4.2. **Definition.** A modular form of weight k and level N is a function $f : (\mathbb{C} \setminus \mathbb{R}) \times \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}) \to \mathbb{C}$ such that

(1) f is holomorphic;
(2) $f\big(\frac{a\tau + c}{b\tau + d}, g\gamma\big) = (b\tau + d)^k f\big(\tau, g\big)$ for all $(\tau, g) \in (\mathbb{C} \setminus \mathbb{R}) \times \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z})$;
(3) f satisfies a growth condition at infinity (which we will make precise in Remark 4.4 below).

The passage from F to f is given via $f(\tau, g) = F\big((L, \jmath)\big)$, where $L = \mathbb{Z}\tau + \mathbb{Z}$ and $\jmath : (\mathbb{Z}/N\mathbb{Z})^2 \xrightarrow{\sim} L/NL$ is defined to be the composite $\jmath = \imath_N \circ g^{-1}$, where $\imath_N : (\mathbb{Z}/N\mathbb{Z})^2 \xrightarrow{\sim} L/NL$ is defined via $\imath_N((1,0)) = \tau \bmod NL$, $\imath_N((0,1)) = 1 \bmod NL$.

We let $\mathcal{M}_k(N)$ denote the $\mathbb{C}$-vector space of modular forms of weight k and level N.

4.3. **Remark.** The left action of $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ on $\mathcal{L}(N)$ induces a right action of $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ on $\mathcal{M}_k(N)$, defined via

$$(F \cdot g)\big((L, \jmath)\big) = F\big((L, \jmath \circ g^{-1})\big).$$

We may convert this into a left action in the usual way, by passing to inverses:

$$(g \cdot F)\big((L, \jmath)\big) = F\big((L, \jmath \circ g)\big).$$

4.4. **Remark.** If $F \in \mathcal{M}_k(N)$, corresponding to the function f on $(\mathbb{C} \setminus \mathbb{R}) \times \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ satisfying Definition 4.2, then we see that $f(\tau + N, g) = f(\tau, g)$ for all $(\tau, g) \in (\mathbb{C} \setminus \mathbb{R}) \times \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$. (Apply condition 2 of Definition 4.2 to the matrix $\begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z})$.) In particular, we may expand the function $\tau \mapsto f(\tau, I)$ (where I denotes the identity of $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$, and τ is restricted to the upper half-plane of $\mathbb{C} \setminus \mathbb{R}$) as a Fourier series $f(\tau) = \sum_{n=-\infty}^{\infty} a_n(f) e^{2\pi i n\tau/N}$. The growth condition on f (i.e. condition 3 of Definition 4.2) is as follows:

$$a_n(g \cdot f) = 0 \text{ for all } n < 0 \text{ and all } g \in \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}).$$

4.5. **Remark.** As a consequence of the growth condition (i.e. condition 3) in the definition of modular forms, one finds that $\mathcal{M}_k(N) = 0$ if $k < 0$, and that any element of $M_0(N)$ is constant on each connected component of $\mathcal{L}(N)$. In general $\mathcal{M}_k(N)$ is finite dimensional, and (for fixed N) its dimension grows roughly linearly with k. (See Remark 7.9 below.)

4.6. **Remark.** If $M \mid N$, then pulling back via the natural map $\mathcal{L}(N) \to \mathcal{L}(M)$ induces an embedding $\mathcal{M}_k(M) \hookrightarrow \mathcal{M}_k(N)$. We write $\mathcal{M}_k := \varinjlim_N \mathcal{M}_k(N)$ (where the set of levels $N \geq 1$ is directed by divisibility, and the transition maps are provided by the embeddings just described).

4.7. **Remark.** We may inflate the left $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$-action on $\mathcal{M}_k(N)$ to an action of $\mathrm{GL}_2(\widehat{\mathbb{Z}})$, via the natural surjection $\mathrm{GL}_2(\widehat{\mathbb{Z}}) \to \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$. The $\mathrm{GL}_2(\widehat{\mathbb{Z}})$-action on each $\mathcal{M}_k(N)$ is compatible with the transition maps for varying N introduced in the preceding remark, and so we obtain an induced action of $\mathrm{GL}_2(\widehat{\mathbb{Z}})$ on $\mathcal{M}_k$. (As we observe in Remark 5.8 below, this $\mathrm{GL}_2(\widehat{\mathbb{Z}})$-action in fact extends to an action of $\mathrm{GL}_2(\mathbb{A}_f)$ on $\mathcal{M}_k$.)

5. Classical modular forms as automorphic forms

In this subsection we will explain how to interpret classical modular forms as automorphic forms on the group $\mathrm{GL}_2(\mathbb{A})$.

The general definition of automorphic forms on the adèlic points of a reductive group [1], when applied to the reductive group GL_2 over $\mathbb{Q}$, gives the following:

5.1. **Definition.** The space $\mathcal{A}\big(\mathrm{GL}_2(\mathbb{A})\big)$ of automorphic forms on $\mathrm{GL}_2(\mathbb{A})$ is the $\mathbb{C}$-vector space of functions $\phi : \mathrm{GL}_2(\mathbb{A}) \to \mathbb{C}$ satisfying the following conditions:

(1) For all $\gamma \in \mathrm{GL}_2(\mathbb{Q})$ and $g \in \mathrm{GL}_2(\mathbb{A})$, $\phi(\gamma g) = \phi(g)$ (i.e. ϕ descends to a function on the quotient $\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A})$).
(2) There is a positive integer N such that $\phi(gk) = \phi(g)$ for all $g \in \mathrm{GL}_2(\mathbb{A})$ and $k \in K(N)$.
(3) For each fixed element $g_f \in \mathrm{GL}_2(\mathbb{A}_f)$, the function $g_\infty \to \phi(g_\infty g_f)$ is a smooth function of the variable $g_\infty \in \mathrm{GL}_2(\mathbb{R})$, which *grows moderately* at infinity (i.e. is bounded in absolute value by a polynomial in the absolute value of the entries of g_∞ and g_∞^{-1}); people also say that this function is *slowly increasing.*
(4) There is an ideal I in $\mathfrak{z}(\mathfrak{gl}_2)$ of finite codimension which annihilates ϕ. (Here $\mathfrak{gl}_2$ denotes the Lie algebra of the real Lie group $\mathrm{GL}_2(\mathbb{R})$, and $\mathfrak{z}(\mathfrak{gl}_2)$ denotes the centre of its universal enveloping algebra over $\mathbb{C}$. Note that this centre acts naturally, by differential operators, on the space of functions satisfying the preceding three conditions.)
(5) The function ϕ is $\mathrm{SO}(2)$-finite, i.e. its translates under the right regular action of $\mathrm{SO}(2)$ (which action leaves invariant the space of functions satisfying the preceding four conditions) span a finite dimensional $\mathbb{C}$-vector space. (Note that it an equivalent condition is obtained if $\mathrm{SO}(2)$ is replaced by $\mathrm{O}(2)$. The role of $\mathrm{O}(2)$ here is that it is a maximal compact subgroup of $\mathrm{GL}_2(\mathbb{R})$, while $\mathrm{SO}(2)$ is the connected component of the identity in $\mathrm{O}(2)$.)

5.2. **Remark.** The space $\mathcal{A}(\mathrm{GL}_2(\mathbb{A}))$ of automorphic forms is invariant under the action by right translation of $\mathrm{GL}_2(\mathbb{A}_f)$. (It is obvious that conditions 1, 3, 4, and 5 are preserved by this action. Since the open subgroups $K(N)$ form a neighbourhood basis of the identity in $\mathrm{GL}_2(\mathbb{A}_f)$, condition 2 is also preserved.) On the other hand, the right translation action of $\mathrm{GL}_2(\mathbb{R})$ does not preserve $\mathcal{A}(\mathrm{GL}_2(\mathbb{A}))$; more precisely, it does not preserve condition 5. However, the actions of each of $\mathrm{O}(2)$ and $\mathfrak{gl}_2$ induced by the right translation action $\mathrm{GL}_2(\mathbb{R})$ do preserve $\mathcal{A}(\mathrm{GL}_2(\mathbb{A}))$; thus $\mathcal{A}(\mathrm{GL}_2(\mathbb{A}))$ is a $\big(\mathfrak{gl}_2, \mathrm{O}(2)\big) \times \mathrm{GL}_2(\mathbb{A}_f)$-module.

5.3. **Remark.** Sometimes condition (5) is omitted from the definition, so that automorphic forms become simply a $\mathrm{GL}_2(\mathbb{A})$-module. But then one needs to invoke results of Harish-Chandra to see the relationship between that modified definition and the definition with (5) included.

For a fixed positive integer N, we will say that an automorphic form $\phi \in \mathcal{A}\big(\mathrm{GL}_2(\mathbb{A})\big)$ has level N if ϕ satisfies condition 2 of Definition 5.1 for the given value of N; equivalently, ϕ descends to a function $\phi : \mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A})/K(N) \to \mathbb{C}$ which satisfies conditions 3, 4, and 5 of Definition 5.1. In order to connect such automorphic forms with more classical objects, it is convenient to describe the double quotient $\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A})/K(N)$ more explicitly.

5.4. **Lemma.** *There are canonical diffeomorphisms*

$$\big(\Gamma(N)\backslash\mathrm{GL}_2(\mathbb{R})_+\big) \times (\mathbb{Z}/N\mathbb{Z})^\times \xrightarrow{\sim} \mathrm{SL}_2(\mathbb{Z})\backslash\big(\mathrm{GL}_2(\mathbb{R})_+ \times \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})\big)$$
$$\xrightarrow{\sim} \mathrm{GL}_2(\mathbb{Z})\backslash\big(\mathrm{GL}_2(\mathbb{R}) \times \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})\big) \xrightarrow{\sim} \mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A})/K(N).$$

Proof. The first of these diffeomorphisms is induced by the embedding

$$\mathrm{GL}_2(\mathbb{R})_+ \times (\mathbb{Z}/N\mathbb{Z})^\times \hookrightarrow \mathrm{GL}_2(\mathbb{R})_+ \times \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$$

which is the product of the identity on the first factors and the map $a \mapsto \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ on the second factors. (To see that we obtain a diffeomorphism, use the fact that the natural map $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ is surjective, with kernel $\Gamma(N)$.) The second is induced by the inclusion

$$\big(\mathrm{GL}_2(\mathbb{R})_+ \times \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})\big) \subset \big(\mathrm{GL}_2(\mathbb{R}) \times \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})\big)$$

(taking into account that $\mathrm{SL}_2(\mathbb{Z}) = \mathrm{GL}_2(Z) \bigcap \mathrm{GL}_2(\mathbb{R})_+$ and $\mathrm{GL}_2(Z)\mathrm{GL}_2(\mathbb{R})_+ = \mathrm{GL}_2(\mathbb{R})$.) The third is induced by the inclusion

$$\mathrm{GL}_2(\mathbb{R}) \times \mathrm{GL}_2(\widehat{\mathbb{Z}})/K(N) \subset \mathrm{GL}_2(\mathbb{R}) \times \mathrm{GL}_2(\mathbb{A}_f)/K(N)$$

(taking into account that $\mathrm{GL}_2(\mathbb{Z}) = \mathrm{GL}_2(\mathbb{Q}) \bigcap \mathrm{GL}_2(\widehat{\mathbb{Z}})$ and $\mathrm{GL}_2(\mathbb{Q})\mathrm{GL}_2(\widehat{\mathbb{Z}}) = \mathrm{GL}_2(\mathbb{A}_f)$.) $\square$

Let us now fix the same base-point $\imath_0 \in \mathcal{B}$ as in Subsection 2; namely, we let $\imath_0$ correspond to the point $(i, 1) \in (\mathbb{C} \setminus \mathbb{R}) \times \mathbb{C}^\times$ under the isomorphism (1). The diffeomorphism $\mathrm{GL}_2(\mathbb{R}) \times \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}) \to \mathcal{B} \times \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ defined via $(g_\infty, g_N) \mapsto (\imath_0 \circ g_\infty^{-1}, g_N^{-1})$ induces a diffeomorphism

$$\mathrm{GL}_2(\mathbb{Z})\backslash\big(\mathrm{GL}_2(\mathbb{R}) \times \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})\big) \xrightarrow{\sim} \big(\mathcal{B} \times \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})\big)/\mathrm{GL}_2(\mathbb{Z}),$$

and hence, after composition with (4) and the third isomorphism of Lemma 5.3, a diffeomorphism

$$\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A})/K(N) \xrightarrow{\sim} \mathcal{L}(N). \tag{5}$$

5.5. **Proposition.** *The pull-back of any classical modular form of weight k and level N via (5) is an automorphic form of level N, and this pull-back induces an isomorphism*

$$\mathcal{M}_k(N) \xrightarrow{\sim} \{\, \phi \in \mathcal{A}(\mathrm{GL}_2(\mathbb{A})) \quad | \quad Z\phi = H\phi = k\phi, \quad Y_-\phi = 0, \quad \phi \text{ has level } N \,\}.$$

Proof. We first note that a holomorphic function F on $\mathcal{L}(N)$ satisfies condition 3 of Definition 4.1 if and only if its pull-back ϕ satisfies condition 3 of Definition 5.1.

We now show that a smooth function F on $\mathcal{L}(N)$ satisfies condition 1 of Definition 4.1, i.e. is holomorphic, if and only if its pull-back ϕ satisfies the conditions

$$(Z - H)\phi = Y_-\phi = 0. \tag{6}$$

Indeed, the smooth function F is holomorphic if and only if it satisfies the Cauchy-Riemann equations on $\mathcal{L}(N)$; i.e. if and only if it is annhilated by sections of the antiholomorphic tangent bundle of $\mathcal{L}(N)$. Now for any $X \in \mathfrak{gl}_2$ and $(g_\infty, g_N) \in \mathrm{GL}_2(\mathbb{R}) \times \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$, we compute that

$$\begin{aligned}(X\phi)(g_\infty, g_N) &= \frac{d}{dt}_{|t=0}\phi(g_\infty e^{Xt}, g_N)\\ &= \frac{d}{dt}_{|t=0}F\big((\imath_0 \circ e^{-Xt} \circ g_\infty^{-1}, g_N^{-1}) \bmod \mathrm{GL}_2(\mathbb{Z})\big)\\ &= \frac{d}{dt}_{|t=0}F\big((\imath_0 \circ g_\infty^{-1} \circ e^{-\mathrm{Ad}_{g_\infty}(X)t}, g_N^{-1}) \bmod \mathrm{GL}_2(\mathbb{Z})\big).\end{aligned}$$

(Here and below we use (4) to write F as a function on $\big(\mathcal{B}\times \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})\big)/\mathrm{GL}_2(\mathbb{Z})$.) In particular, using the notation introduced in Subsection 2, we see that F satisfies the Cauchy-Riemann equations on $\mathcal{L}(N)$ if and only if

$$(X\phi)(g_\infty, g_N) = 0$$

for all $X \in \mathrm{Ad}_{g_\infty^{-1}} \overline{V}_{\imath_0 \circ g_\infty^{-1}} = \overline{V}_{\imath_0}$ and $(g_\infty, g_N) \in \mathrm{GL}_2(\mathbb{R}) \times \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$, i.e. F satisfies the Cauchy-Riemman equations on $\mathcal{L}(N)$ if and only if ϕ is annhilated by $\overline{V}_{\imath_0}$. Lemma 2.2 shows that this condition on ϕ is equivalent to (6).

We next show that F satisfies condition 2 of Definition 4.1 if and only if ϕ satisfies the conditions

$$Z\phi = H\phi = k\phi. \tag{7}$$

Note first that (since ϕ is smooth, $\mathbb{C}^\times$ is connected, and Z and H together span the Lie algbebra of $\mathbb{C}^\times$) these conditions are equivalent to the condition

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \phi = (a+bi)^k \phi, \text{ for all } a+bi \in \mathbb{C}^\times. \tag{8}$$

Now the choice of base-point $\imath_0$ induces an isomorphism $\mathrm{GL}(\mathbb{C}) \xrightarrow{\sim} \mathrm{GL}_2(\mathbb{R})$, and hence an embedding

$$\mathbb{C}^\times \hookrightarrow \mathrm{GL}_2(\mathbb{R}), \tag{9}$$

given explicitly via

$$a+bi \mapsto \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

Writing $g = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ and $\alpha = a+bi$, we then have

$$\imath_0 \circ g = \alpha\, \imath_0. \tag{10}$$

In particular, if $(g_\infty, g_N) \in \mathrm{GL}_2(\mathbb{R})\times\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ and $\alpha = a+bi \in \mathbb{C}^\times$, then, again writing $g = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, we find (using (10) to rewrite $\imath_0 \circ g^{-1}$) that

$$\begin{aligned}(g\phi)(g_\infty, g_N) &= \phi(g_\infty g, g_N) \\ &= F\big((\imath_0 \circ g^{-1} \circ g_\infty^{-1}, g_N^{-1}) \bmod \mathrm{GL}_2(\mathbb{Z})\big) \\ &= F\big(\alpha^{-1}(\imath_0 \circ g_\infty^{-1}, g_N^{-1}) \bmod \mathrm{GL}_2(\mathbb{Z})\big)\end{aligned}$$

Thus

$$(g\phi)(g_\infty, g_N) = \alpha^k \phi(g_\infty, g_N)$$

if and only if

$$F\big(\alpha^{-1}(\imath_0 \circ g_\infty^{-1}, g_N^{-1}) \bmod \mathrm{GL}_2(\mathbb{Z})\big) = \alpha^{-k} F\big((\imath_0 \circ g_\infty^{-1}, g_N^{-1}) \bmod \mathrm{GL}_2(\mathbb{Z})\big),$$

and so indeed F satisfies condition 2 of Definition 4.1 if and only if ϕ satisfies (8), or equivalently (7).

Altogether, we have shown that pulling back via (5) identifies $\mathcal{M}_k(N)$ with the space of smooth functions on $\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A})/K(N)$ satisfying condition 3 of Definition 5.1, as well as conditions (6) and (7). It remains to show that any ϕ satisfying (6) and (7) also satisfies conditions 4 and 5 of Definition 5.1 (and thus is in fact an automorphic form). On the one hand, conditions (6) and (7) taken together imply that ϕ is annihilated by the codimension one ideal

$$I = (Z-k,\, Y_+Y_- + H^2 - 2H - k^2 + 2k) \subset \mathfrak{z}(\mathfrak{gl}_2).$$

Thus ϕ satisfies condition 4 of Definition 5.1. On the other hand, condition (8) (which is equivalent to (7)) implies that the span of ϕ is invariant under the right translation action of SO(2), and thus that ϕ satisfies condition 5 of Definition 5.1. This completes the proof of the proposition. □

5.6. **Remark.** There is a relation between our choice of base-point $\imath_0$ and our choice of maximal compact subgroup O(2) of $\mathrm{GL}_2(\mathbb{R})$; namely, it is important that the image of the embedding (9) contain the connected component of O(2). If we were to replace $\imath_0$ by the base-point $\imath_0 \circ g \in \mathcal{B}$ (for some $g \in \mathrm{GL}_2(\mathbb{R})$) in the construction of the diffeomorphism (5), then in order for the analogue of Proposition 5.4 to hold, we would have to replace O(2) by its conjugate $g^{-1}\mathrm{O}(2)g$ in condition 5 of Definition 5.1.

5.7. **Remark.** The preceding proposition identifies the space $\mathcal{M}_k(N)$ (for each k and N) with a certain space of automorphic forms of level N. These identifications are evidently compatible with change of N, and so we obtain an embeding $\mathcal{M}_k \hookrightarrow \mathcal{A}(\mathrm{GL}_2(\mathbb{A}))$. This subspace admits a more conceptual representation theoretic interpretation, as we now explain.

As was noted in Remark 4.5, $\mathcal{M}_k$ is non-zero only if $k \geq 0$, and so we may as well assume that this inequality holds. Suppose first that $k = 0$. As was also noted in Remark 4.5, a modular form of weight 0 and level N is constant on the connected components of $\mathcal{L}(N)$. Using this one easily sees that modular forms of weight zero correspond precisely to automorphic forms of the form $\mathrm{GL}_2(\mathbb{A}) \ni g \mapsto \chi(\det(g))$, where $\chi : \mathbb{Q}^\times\backslash\mathbb{A}^\times$ is a finite order idèle class character.

Suppose now that $k \geq 1$. One may form the cyclic $\big(\mathfrak{gl}_2, \mathrm{O}(2)\big)$-module D_k, with generator v say, satisfying the relations

$$Zv = Hv = kv,\ Y_- v = 0.$$

It is easy to give an explicit description of D_k: integrating the first two equations satisfied by v determines the action of the image of (9) on v. Namely, we find that for any $\alpha = a + bi \in \mathbb{C}^\times$, if we write $g = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, then $gv = \alpha^k v$. (Compare the equivalence of (7) and (8) above.) If we write $n = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ (so n is a representative of the non-trivial coset of SO(2) in O(2)), then $Ad_n(Y_\pm) = Y_\mp$, and so writing $\overline{v} := nv$, we find that $Y_+\overline{v} = 0$. Now using the commutation relations in $\mathfrak{gl}_2$, it is easy to see that D_k has as a basis over $\mathbb{C}$ the vectors $Y_+^i v$ $(i \geq 0)$ and $Y_-^i v$ $(i \leq 0)$, and that D_k is irreducible as a $\big(\mathfrak{gl}_2, \mathrm{O}(2)\big)$-module. (As a $\mathfrak{gl}_2$-module it is reducible, being the direct sum of the irreducible submodules generated by v and $\overline{v}$ respectively.)

The representation D_k is called a discrete series representation of $(\mathfrak{gl}_2, \mathrm{O}(2))$ if $k \geq 2$, and a limit of discrete series if $k = 1$. (This is a particular instance of terminology used in the representation theory of real Lie groups.)

Now evidently, if M is any $\big(\mathfrak{gl}_2, \mathrm{O}(2)\big)$-module, then $\psi \mapsto \psi(v)$ induces an isomorphism

$$\mathrm{Hom}_{(\mathfrak{gl}_2, \mathrm{O}(2))}(D_k, M) \xrightarrow{\sim} \{\, \phi \in M \quad | \quad Z\phi = Hm = km, \quad Y_- v = 0 \,\}.$$

Combining this isomorphism with those of the proposition, we obtain an isomorphism

$$\mathcal{M}_k \xrightarrow{\sim} \mathrm{Hom}_{(\mathfrak{gl}_2, \mathrm{O}(2))}\big(D_k, \mathcal{A}(\mathrm{GL}_2(\mathbb{A}))\big). \tag{11}$$

Thus the classical modular forms of weight k correspond to the various realizations of the irreducible $(\mathfrak{gl}_2, \mathrm{O}(2))$-module D_k inside the space of automorphic forms.

5.8. **Remark.** There are other irreducible $(\mathfrak{gl}_2, \mathrm{O}(2))$-modules besides the representations D_k (and their twists by powers of the determinant character), namely the so-called principal series representations. (Actually, the limit of discrete series representation D_1 and its twists are also principal series representations; by "principal series" we really mean "principal series other than D_1 and its twists".) One can correspondingly consider the realizations of these representations inside the space $\mathcal{A}(\mathrm{GL}_2(\mathbb{A}))$. An analogue of Proposition 5.4 identifies the space of such realizations with a certain space of non-holomorphic modular forms, known as Maass forms.

5.9. **Remark.** Since $\mathcal{A}(\mathrm{GL}_2(\mathbb{A}))$ is a $\big(\mathfrak{gl}_2, \mathrm{O}(2)\big) \times \mathrm{GL}_2(\mathbb{A}_f)$-module, there is an induced action of $\mathrm{GL}_2(\mathbb{A}_f)$ on the target of (11). We may transport this action via the isomorphism (11) to obtain an action of $\mathrm{GL}_2(\mathbb{A}_f)$ on $\mathcal{M}_k$. We leave it to the reader to check that the restriction of this action to $\mathrm{GL}_2(\widehat{\mathbb{Z}})$ coincides with the $\mathrm{GL}_2(\widehat{\mathbb{Z}})$-action on $\mathcal{M}_k$ described in Remark 4.7.

6. Modular curves

As noted in Subsection 3, there is a left action of $\mathbb{C}^\times$ on $\mathcal{L}(N)$, which is evidently holomorphic. The isomorphisms (1) and (4) together yield an isomorphism

$$\mathbb{C}^\times \backslash \mathcal{L}(N) \xrightarrow{\sim} \big((\mathbb{C} \setminus \mathbb{R}) \times \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})\big)/\mathrm{GL}_2(\mathbb{Z}). \tag{12}$$

Thus $\mathbb{C}^\times \backslash \mathcal{L}(N)$ is naturally a Riemann surface, which in fact is (the space of $\mathbb{C}$-valued points of) an affine algebraic curve over $\mathbb{C}$. [2]

6.1. **Definition.** Write $Y(N) := \mathbb{C}^\times \backslash \mathcal{L}(N)$, and let $X(N)$ denote the smooth projective algebraic curve over $\mathbb{C}$ obtained by completing $Y(N)$. We refer to $Y(N)$ (resp. $X(N)$) as the open (resp. complete) modular curve of level N.

6.2. **Remark.** If $M \mid N$, then the projection $\mathcal{L}(N) \to \mathcal{L}(M)$ induces a finite map $Y(N) \to Y(M)$, which extends to a finite map $X(N) \to X(M)$.

6.3. **Remark.** The $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$-action on $\mathcal{L}(N)$ induces a $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$-action on the modular curve $Y(N)$. Evidently the element $-I := \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ acts trivially on $Y(N)$, and so this action factors to give an action of $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/\langle -I \rangle$ on $Y(N)$, which is then faithful. This action extends to an action of $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/\langle -I \rangle$ on its completion $X(N)$. If $M \mid N$ and $H \subset \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/\langle -I \rangle$ denotes the kernel of the projection $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/\langle -I \rangle \to \mathrm{GL}_2(\mathbb{Z}/M\mathbb{Z})/\langle -I \rangle$, then $Y(N)/H \xrightarrow{\sim} Y(M)$ and $X(N)/H \xrightarrow{\sim} X(M)$. If $M \geq 3$ then H acts freely on $Y(N)$ (compare the proof of Lemma 7 below), and so $Y(N)$ is a covering space of $Y(M)$. However, the map $X(N) \to X(M)$ is always ramified at the points in the complement of $Y(N)$ (except of course in the trivial case when $M = N$).

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[2] One way to see this is to note that the natural projection $\mathcal{L}(N) \to \mathcal{L}$ realizes $\mathbb{C}^\times \backslash \mathcal{L}(N)$ as a finite (ramified) cover of $\mathbb{C}^\times \backslash \mathcal{L}$, which (12) identifies with $(\mathbb{C} \setminus \mathbb{R})/\mathrm{GL}_2(\mathbb{Z})$. This latter quotient is well known to be isomorphic to $\mathbb{C}$ (for example, via the j-function – see [4]), and hence may be regarded as a copy of the affine line over $\mathbb{C}$.

6.4. **Remark.** The points of $X(N) \setminus Y(N)$ are refered to as the cusps of $X(N)$. (This is classical terminology, which is a bit misleading geometrically, since $X(N)$ is by construction a smooth curve, and so these points are not cusps of $X(N)$ in the algebro-geometric sense.) As was noted in footnote 2, there is an isomorphism $Y(1) \xrightarrow{\sim} \mathbb{C}$, and so $X(1) \xrightarrow{\sim} \mathbb{P}^1(\mathbb{C})$ has just one cusp. Since the projection $X(N) \to X(1)$ restricts to a finite map $Y(N) \to Y(1)$, we see that the cusps of $X(N)$ are precisely the preimage under the projection of the cusps of $X(1)$. From the isomorphism $X(N)/\big(\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/\langle -I\rangle\big) \xrightarrow{\sim} X(1)$ we conclude that $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/\langle -I\rangle$ acts transitively on the set of cusps.

It is simple enough to exhibit the cusps explicitly. For any $T \in \mathbb{R}_{>0}$, write

$$R_T := \{\, \tau \in \mathbb{C} \quad | \quad \mathrm{Im}(\tau) > T \,\} \subset \mathbb{C} \setminus \mathbb{R}.$$

Let I denote the identity of $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$. The matrix $\gamma := \begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z})$ leaves $R_T \times \{\, I \,\} \subset (\mathbb{C} \setminus \mathbb{R}) \times \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ invariant, and if T is sufficiently positive, then the induced map

$$(R_T \times \{\, I \,\})/\langle \gamma \rangle \to \big((\mathbb{C} \setminus \mathbb{R}) \times \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})\big)/\mathrm{GL}_2(\mathbb{Z}) \xrightarrow{\sim} Y(N) \subset X(N) \tag{13}$$

identifies the source (which is isomorphic to a punctured open disk, via the map $\tau \mapsto e^{2\pi i \tau/N}$) with the deleted neighbourhood of a cusp in $X(N)$. (The map extends to the unpunctured disk by mapping the origin of the disk to the cusp itself.) Since $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/\langle -I\rangle$ acts transitively on the set of cusps, all the cusps of $X(N)$ (or rather their deleted neighbourhoods, which carry essentially the same information) can be obtained by translating the map (13) by elements of $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/\langle -I\rangle$. In particular, we find that the set of cusps is in bijection with $\langle \gamma \rangle \backslash \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})/\langle -I\rangle$.

7. Classical modular forms as sections of line bundles

In this subsection, we recall the algebro-geometric interpretation of modular forms as sections of certain line bundles on modular curves. We begin by constructing the appropriate line bundles.

7.1. **Lemma.** *If $N \geq 3$, then $\mathbb{C}^\times$ acts on $\mathcal{L}(N)$ without fixed points.*

Proof. If $\alpha(L, \jmath) = (L, \jmath)$ for some $\alpha \in \mathbb{C}^\times$ and $(L, \jmath) \in \mathcal{L}(N)$, then firstly $\alpha L = L$, and secondly the automorphism of L induced by multiplication by α becomes trivial when reduced modulo N. The first condition implies that either $\alpha = \pm 1$, or else that L is a square lattice and α is a 4th root of unity, or else that L is a triangular lattice and α is a 6th root of unity. Since $N \geq 3$, it is immediately seen that the only way to meet the second condition is if $\alpha = 1$. □

The preceding lemma shows that if $N \geq 3$, then the natural projection

$$\mathcal{L}(N) \to Y(N)$$

realizes $\mathcal{L}(N)$ as the total space of a holomorphic principal $\mathbb{C}^\times$-bundle over $Y(N)$.

7.2. **Definition.** We denote by $\underline{\omega}$ the holomorphic line bundle over $Y(N)$ associated to the $\mathbb{C}^\times$-bundle $\mathcal{L}(N)$ (for any $N \geq 3$).

7.3. **Remark.** There is an ambiguity in our notation, since we are using the one symbol $\underline{\omega}$ to denote not a single line bundle, but a collection of line bundles, one on each of the curves $Y(N)$ for $N \geq 3$. However, this should not cause confusion.

7.4. **Remark.** The $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$-action on $\mathcal{L}(N)$ induces a $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$-action on the total space of $\underline{\omega}$, which is obviously compatible with the $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$-action on $Y(N)$. Thus $\underline{\omega}$ is naturally a $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$-equivariant line bundle over $Y(N)$.

7.5. **Remark.** If $M \geq 3$ and $M \mid N$, then the diagram

$$\begin{array}{ccc} \mathcal{L}(N) & \longrightarrow & \mathcal{L}(M) \\ \downarrow & & \downarrow \\ Y(N) & \longrightarrow & Y(M) \end{array}$$

(in which the arrows are the various natural projections) is evidently Cartesian. Thus the line bundle $\underline{\omega}$ on $Y(N)$ is canonically isomorphic to the pull-back of the line bundle $\underline{\omega}$ on $Y(M)$ under the projection $Y(N) \to Y(M)$.

7.6. **Remark.** The isomorphisms (1) and (4) provide the following explicit description of the total space of $\underline{\omega}$:

$$\begin{array}{ccc} \underline{\omega} & \xrightarrow{\sim} & \big((\mathbb{C} \setminus \mathbb{R}) \times \mathbb{C} \times \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})\big)/\mathrm{GL}_2(\mathbb{Z}) \\ \downarrow & & \downarrow \\ Y(N) & \xrightarrow{\sim} & \big((\mathbb{C} \setminus \mathbb{R}) \times \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})\big)/\mathrm{GL}_2(\mathbb{Z}), \end{array}$$

where the $\mathrm{GL}_2(\mathbb{Z})$ action on $\big((\mathbb{C} \setminus \mathbb{R}) \times \mathbb{C} \times \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})\big)$ is given via

$$(\tau, z, g)\gamma = (\frac{a\tau + c}{b\tau + d}, (b\tau + d)z, g\gamma),$$

for any $(\tau, z, g) \in \big((\mathbb{C} \setminus \mathbb{R}) \times \mathbb{C} \times \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})\big)$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z})$.

More generally, for any $k \in \mathbb{Z}$, we see obtain an analogous description of the total space of the tensor power $\underline{\omega}^{\otimes k}$:

$$\begin{array}{ccc} \underline{\omega} & \xrightarrow{\sim} & \big((\mathbb{C} \setminus \mathbb{R}) \times \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}) \times \mathbb{C}\big)/\mathrm{GL}_2(\mathbb{Z}) \\ \downarrow & & \downarrow \\ Y(N) & \xrightarrow{\sim} & \big((\mathbb{C} \setminus \mathbb{R}) \times \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})\big)/\mathrm{GL}_2(\mathbb{Z}), \end{array}$$

where now the $\mathrm{GL}_2(\mathbb{Z})$ action on $(\mathbb{C} \setminus \mathbb{R}) \times \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}) \times \mathbb{C}$ is given via

$$(\tau, g, z)\gamma = (\frac{a\tau + c}{b\tau + d}, g\gamma, (b\tau + d)^k z),$$

for any $(\tau, z, g) \in (\mathbb{C} \setminus \mathbb{R}) \times \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}) \times \mathbb{C}$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z})$.

7.7. **Construction.** Given the holomorphic line bundle $\underline{\omega}$ on $Y(N)$, there are many possible ways to extend it to a holomorphic line bundle on $X(N)$. (In fact any holomorphic line bundle on an affine curve over $\mathbb{C}$ is trivializable, and so we may extend $\underline{\omega}$ to any line bundle on $X(N)$ that we choose.) However, we are going to specify a particular extension, which we will continue to denote by $\underline{\omega}$. By stipulation, this extension will also be $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$-equivariant.

Fix T sufficiently large so that (13) is an embedding. Write $D^{\times} := R_T/\langle\gamma\rangle$; then (13) identifies $D^{\times}$ with the deleted neighbourhood of a cusp in $X(N)$. Let D

denote the undeleleted neighbourhood obtained by adding the cusp to $D^\times$. Since $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ acts transitively on the cusps, to specify a $\mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$-equivariant extension of $\underline{\omega}$, it suffices to specify the conditions under which a section of $\underline{\omega}$ over $D^\times$ extends to a section over D.

Remark 7.6 provides the following description of $\underline{\omega}_{|\,D^\times}$:

$$\begin{array}{ccc} \underline{\omega}_{|\,D^\times} & \xrightarrow{\sim} & \big(R_T \times \{\, I \,\} \times \mathbb{C}\big)/\langle\gamma\rangle \\ \downarrow & & \downarrow \\ D^\times & \xrightarrow{\sim} & \big(R_T \times \{\, I \,\}\big)/\langle\gamma\rangle. \end{array}$$

Thus any section of $\underline{\omega}$ over $D^\times$ is of the form

$$R_T \times \{\, I \,\} \ni (\tau, I) \mapsto (\tau, I, f(\tau)) \in R_T \times \{\, I \,\} \times \mathbb{C},$$

where $f(\tau)$ is a holomorphic function on R_T that is invariant under $\tau \mapsto \tau + N$. We may expand such a function in a Fourier series $f(\tau) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n\tau/N}$. We declare that the section of $\underline{\omega}$ over $D^\times$ associated to f extends over D precisely when $a_n = 0$ for $n < 0$.

7.8. **Proposition.** *There is a canonical isomorphism between the space of sections $H^0(Y(N), \underline{\omega}^{\otimes k})$ and the space of functions $f : (\mathbb{C} \setminus \mathbb{R}) \times \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}) \to \mathbb{C}$ satisfying conditions 1 and 2 of Definition 4.2. Under this identification, the subspace $H^0(X(N), \underline{\omega}^{\otimes k})$ of $H^0(Y(N), \underline{\omega}^{\otimes k})$ is identified with the space of functions f that furthermore satisfy condition 3 of Definition 4.2.*

Proof. The description of $\underline{\omega}^{\otimes k}$ provided by Remark 7.6 shows that any such function f satisfying conditions 1 and 2 corresponds to the section of $\underline{\omega}$ defined by

$$\begin{aligned} &\big((\mathbb{C} \setminus \mathbb{R}) \times \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})\big)/\mathrm{GL}_2(\mathbb{Z}) \ni (\tau,\, g) \bmod \mathrm{GL}_2(\mathbb{Z}) \\ &\qquad \mapsto (\tau,\, g,\, f(\tau, g)) \bmod \mathrm{GL}_2(\mathbb{Z}) \in \big((\mathbb{C} \setminus \mathbb{R}) \times \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z}) \times \mathbb{C}\big)/\mathrm{GL}_2(\mathbb{Z}), \end{aligned}$$

and conversely, that any such section of $\underline{\omega}$ over $Y(N)$ arises in this manner. (Condition 1 shows that this section is holomorphic, while condition 2 ensures that it is well-defined.) A comparison of the description of condition 3 given in Remark 4.4 and the extension of $\underline{\omega}$ to $X(N)$ described in Construction 7.7 shows that f does indeed satisfy condition 3 precisely when the associated section of $\underline{\omega}^{\otimes k}$ extends over $X(N)$. □

7.9. **Remark.** Proposition 7.8 identifies $\mathcal{M}_k(N)$ with $H^0(X(N), \underline{\omega}^{\otimes k})$. Since $\mathcal{M}_k(N)$ is trivial for negative values of k, we see that the line bundle $\underline{\omega}$ on $X(N)$ must have positive degree. The Riemann-Roch formula then shows that the dimension of $\mathcal{M}_k(N)$ grows essentially linearly with k when k is positive.

7.10. **Remark.** If we think of $\mathbb{C}^\times$ as a real Lie group, rather than a complex one, then it has many more characters than just the characters $z \mapsto z^k$. Indeed, for any $s \in \mathbb{C}$ and $k \in \mathbb{Z}$, we have the complex valued character $z \mapsto |\, z\,|^s\, z^k$. Note that this character is smooth but non-holomorphic if $s \neq 0$. Starting with the $\mathbb{C}^\times$-bundle $\mathcal{L}(N)$ over $Y(N)$, this character gives rise to an associated (smooth, but non-holomorphic, if $s \neq 0$) complex line bundle over $Y(N)$, which we will denote by $\underline{\omega}^{s,k}$ (so $\underline{\omega}^{0,k} = \underline{\omega}^{\otimes k}$, the holomorphic line bundle considered above).

Conditions 4 and 5 of Definition 5.1 imply that any automorphic form ϕ of level N generates some finite dimensional representation under the action of $\mathbb{C}^\times$

by right translation. Let us suppose that ϕ in fact transforms under the character $z \mapsto |z|^s\, z^k$. (Any finite dimensional representation of $\mathbb{C}^\times$ will be an extension of such characters.) Then an analogue of Propositions 5.4 and 7.8 will associate to ϕ a (typically non-holomorphic) section of $\underline{\omega}^{s,k}$. Thus one can in part regard the passage from classical modular forms to more general modular forms as arising from enlarging the collection of characters of $\mathbb{C}^\times$ under consideration, by viewing it merely as a smooth, rather than holomorphic, structure group.

8. Eisenstein series

In the general theory of automorphic forms, Eisenstein series are a method for realizing elements of parabolically induced representations as automorphic forms. In the context of GL_2, let B denote the Borel subgroup of lower triangular matrices, and T the maximal torus consisting of diagonal matrices. If $\chi : T(\mathbb{A}) \to \mathbb{C}^\times$ is a continuous character, then (thinking of $T(\mathbb{A})$ as a quotient of $B(\mathbb{A})$ in the obvious way) we may also regard χ as a character of $B(\mathbb{A})$. We may then form the induced representation $\mathrm{Ind}_{B(\mathbb{A})}^{\mathrm{GL}_2(\mathbb{A})}\chi$, which is defined to be the space of functions $f : \mathrm{GL}_2(\mathbb{A}) \to \mathbb{C}$ such that $f(bg) = \chi(b)f(g)$ for all $b \in B(\mathbb{A})$, $g \in \mathrm{GL}_2(\mathbb{A})$. To be precise, we have to impose appropriate regularity conditions on the function f. In the context of studying automorphic forms, it is natural to ask that f be locally constant in the finite adèlic variables, be smooth and slowly growing at infinity in the real variables, and be SO(2)-finite, i.e. f should satisfy conditions 2, 3, and 5 of Definition 5.1. Note that f automatically satisfies condition 4 of that definition; indeed, it is an eigenvector for $\mathfrak{z}(\mathfrak{gl}_2)$, with eigenvalues depending on the derivative of the ∞-component of χ. With this definition of $\mathrm{Ind}_{B(\mathbb{A})}^{\mathrm{GL}_2(\mathbb{A})}\chi$, it becomes a $\big(\mathfrak{gl}_2, \mathrm{O}(2)\big) \times \mathrm{GL}_2(\mathbb{A}_f)$-module via the right regular action on functions.

If χ is a character of $T(\mathbb{Q})\backslash T(\mathbb{A})$ (i.e. an automorphic form on $T(\mathbb{A})$), then Eisenstein series provide an embedding of $\big(\mathfrak{gl}_2, \mathrm{O}(2)\big) \times \mathrm{GL}_2(\mathbb{A}_f)$-modules of $\mathrm{Ind}_{B(\mathbb{A})}^{\mathrm{GL}_2(\mathbb{A})}\chi$ into the space of automorphic forms on $\mathrm{GL}_2(\mathbb{A})$, in the case when χ is character of $T(\mathbb{Q})\backslash T(\mathbb{A})$ (i.e. an automorphic form on $T(\mathbb{A})$). In order to obtain such an embedding, for each $f \in \mathrm{Ind}_{B(\mathbb{A})}^{\mathrm{GL}_2(\mathbb{A})}\chi$ we must construct a corresponding automorphic form. Now f itself already satisfies conditions 2 through 5 of Definition 5.1, and is left-invariant under $B(\mathbb{Q})$ (since χ is trivial on $T(\mathbb{Q})$). To obtain a function satisfying condition 1 of that Definition, it is thus natural to average f over $B(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{Q})$. To this end, we define the Eisenstein series $E_f : \mathrm{GL}_2(\mathbb{A}) \to \mathbb{C}$ via the formula

$$E_f(g) = \sum_{\gamma \in B(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{Q})} f(\gamma g),$$

provided that this sum converges. Assuming that the convergence is reasonable, the Eisenstein series $E_f(g)$ will then be an automorphic form on $\mathrm{GL}_2(\mathbb{A})$, and $f \mapsto E_f(g)$ will provide the required embedding.

The following lemma will let us give a more concrete description of $\mathrm{Ind}_{B(\mathbb{A})}^{\mathrm{GL}_2(\mathbb{A})}\chi$.

8.1. **Lemma.** *There is a canonical diffeomorphism*

$$B(\mathbb{Z})\backslash\big(\mathrm{GL}_2(\mathbb{R}) \times \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})\big) \xrightarrow{\sim} B(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A})/K(N).$$

Proof. The required diffeomorphism is induced by the inclusion

$$\mathrm{GL}_2(\mathbb{R}) \times \mathrm{GL}_2(\widehat{\mathbb{Z}})/K(N) \subset \mathrm{GL}_2(\mathbb{R}) \times \mathrm{GL}_2(\mathbb{A}_f)/K(N),$$

taking into account the equalities $B(\mathbb{Z}) = B(\mathbb{Q}) \bigcap \mathrm{GL}_2(\widehat{\mathbb{Z}})$, $\mathrm{GL}_2(\mathbb{Q}) = B(\mathbb{Q})\mathrm{GL}_2(\mathbb{Z})$, and $\mathrm{GL}_2(\mathbb{Q})\mathrm{GL}_2(\widehat{\mathbb{Z}}) = \mathrm{GL}_2(\mathbb{A}_f)$. (The last two equalities taken together imply that $B(\mathbb{Q})\mathrm{GL}_2(\widehat{\mathbb{Z}}) = \mathrm{GL}_2(\mathbb{A}_f)$.) □

Which Eisenstein series will give rise to classical modular forms of weight k and level N under the isomorphism of Proposition 5.4?

References

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