**Algebra 1 : First homework — due Monday, October 3**

Do the following exercises from Fulton and Harris:

1.2, 1.11, 1.12, 1.13, 3.7

Also do the following exercises:

**1.** Let $A$ be a ring (e.g. a group ring). (In this course, rings always are associative with 1.) Recall that we say that a submodule $M'$ of an $A$-module $M$ is a direct summand of $M$ if there exists another $A$-submodule $N'$ of $M$ such that $M = M' \oplus M''$.

(a) Prove that $M'$ is a direct summand of $M$ if and only if there exists an $A$-linear projection from $M$ to $M'$, i.e. a morphism $\pi : M \to M'$ such that $\pi_{|M'} = \mathrm{id}_{M'}$ and $\pi \circ \pi = \pi$.

(b) Prove that if $M = M' \oplus M''$, then there is an isomorphism $M/M' \cong M''$.

**2.** If $M$ is a module over a ring $A$, and $\pi : M \to M''$ is a surjection of $A$-modules, prove that the kernel of $\pi$ is a direct summand of $M$ if and only if there exists a morphism of $A$-modules $\sigma : M'' \to M$ such that $\pi \circ \sigma = \mathrm{id}_{M''}$. (If these equivalent conditions hold, then we say that the surjection $\pi$ *splits*, or *is split*, and we say that $\sigma$ is a section to, or of, $\pi$.)

**3.** If $M$ is a module over a ring $A$, we say that $M$ is semisimple if every submodule of $M$ is a direct summand of $M$.

(a) Show that $M$ is semisimple if and only if every surjection of $A$-modules $M \to M''$ splits.

(b) Show that if $M$ is semisimple, then any submodule or quotient of $M$ is also semisimple (and hence that any *subquotient — i.e. quotient of a submodule* of $M$ is semisimple).

**4.** Suppose that $A$ is a $k$-algebra, for some field $k$, and that $M$ is an $A$-module that is finite-dimensional as a $k$-vector space. Prove that $M$ is semisimple if and only if $M$ is isomorphic to a direct sum of finitely many simple $A$-modules.

**Remark/bonus question:** It is true in general, i.e. for any module $M$ over any ring $A$, that $M$ is semisimple if only if $M$ is a direct sum of (possibly infinitely many) simple $A$-modules. Can you prove it?