# QUANTIZATION AND THE FOURIER-MUKAI TRANSFORM

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Here are the notes of my talk at the University of Chicago. I will post a better version when I have it.

## 1. Deformations

1.1. **Deforming the Fourier-Mukai.** The ground field is $\mathbb{C}$. Let $A$ be a scheme and $R$ an local Artinian ring with maximal ideal $m_R$ such that $R/m_R = \mathbb{C}$. Recall that a deformation quantization of $A$ is a sheaf of flat associative $R$-algebras $\widetilde{O}_A$ on $A$ equipped with an identification $\widetilde{O}_A/m_R\widetilde{O}_A = O_A$. We consider $\widetilde{A} = (A, \widetilde{O}_A)$ as a ringed space.

We will need a slightly more general kind of deformations ('gerby deformations'); they are essentially stacks that are locally isomorphic to $\widetilde{A}/\mathbf{G_m}$, the precise definition will be discussed later. After we introduce the deformations we get the following (relatively simple) theorem (which is probably known to specialists.)

**Theorem 1.** *Let $A \to B$ be a an abelian scheme and let $\widetilde{A} \to B$ be its dual. To any deformation $\widetilde{A}$ of $A$, there corresponds a deformation $\widetilde{A}^\vee$ of $\widetilde{A}^\vee$ such that the Poincaré sheaf $P$ on $A \times A^\vee$ (it is supported by $A \times_B A^\vee$) extends to a flat $R$-module $\widetilde{P}$ on $\widetilde{A} \times (\widetilde{A}^\vee)^{op}$ (that is, $\widetilde{P}/m_R\widetilde{P} = P$).*

**Corollary 2.** *The Fourier-Mukai transform $D(\widetilde{A}) \to D(\widetilde{A}^\vee)$ given by $\widetilde{P}$ is an equivalence of categories.*

*Proof.* (Skip this, probably.) Look at the adjoint functor. (Or just take square of Fourier — its kernel will be a deformation of the diagonal sheaf, so it is invertible.) □

It is also easy to see that $\widetilde{A}^\vee$ is determined by $\widetilde{A}$ up to a canonical isomorphism. Indeed, we claim that $\widetilde{A}^\vee$ is the moduli space of line bundles with vanishing Chern class on fibers on $\widetilde{A}$:

**Definition 3.** Let $\widetilde{X}$ be a deformation quantization of a scheme $X$. By definition, a $\widetilde{X}$-family of line bundles on fibers (vertical line bundles?) on $\widetilde{A}$ is a left quasi-coherent sheaf $l$ on $\widetilde{X}^{op} \times \widetilde{A}$ that is flat over $O_{\widetilde{X}}$ and such that $l/lm_R$ is a line bundle on $X \times_B A$ for some morphism $X \to B$.

We say that $l$ has Chern class 0 if restriction of $l/lm_R$ to the fibers of $X \times_B A \to X$ has Chern class 0.

**Corollary 4.** *For a test scheme $X$ and its deformation quantization $\widetilde{X}$, the morphisms $f : \widetilde{X} \to \widetilde{A}^\vee$ are in one-to-one correspondence with $\widetilde{X}$-families $l$ of line bundles on fibers of $\widetilde{A}$ such that $l$ has Chern class 0. The correspondence is $f \mapsto (f^{op} \times id)^*(\widetilde{P})$.*

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*Proof.* The inverse correspondence goes like this: take adjoint (i.e., tensor rather than hom) Fourier transform of $l$. It is a sheaf $\hat{l}$ on $\widetilde{X}^{op} \times \widetilde{A}^\vee$ that is a flat over $R$, and $\hat{l}/m_R\hat{l}$ is the structure sheaf of the graph of a map $X \to A^\vee$. Therefore, $\hat{l}$ is also the structure sheaf of the graph of a map, at least locally. This map is $f$. □

1.2. **Proof of Theorem 1.** We construct $\widetilde{A}^\vee$ locally. Let $U = \mathrm{Spec}(S) \subset A^\vee$ be an open set. Denote by $M$ the Fourier transform of $O_U$; it is a quasi-coherent sheaf on $A$. We claim that there exists a deformation of $M$ to a $\widetilde{A}$-module $\widetilde{M}$ that is $R$-flat, $\widetilde{M}$ is unique up to an isomorphism, and $\widetilde{S} := Hom_{\widetilde{A}}(\widetilde{M}, \widetilde{M})$ is a flat $R$-algebra such that $S = \widetilde{S}/m_R\widetilde{S}$. This is easy to see from deformation theory: obstructions to $\widetilde{M}$ lie in $Ext^2_A(M, M)$; different $\widetilde{M}$'s are controlled by $Ext^1_A(M, M)$, and lifts from $S$ to $\widetilde{S}$ are obstructed by $Ext^1(M, M)$. However, all these vanish because they can be computed on the other side of Fourier transform.

Now we have constructed $\widetilde{A}^\vee$ and $\widetilde{P}$ locally. It remains to glue the pieces together using uniqueness of $\widetilde{M}$ (that is, $\widetilde{P}$). □

1.3. **Gerby deformations.** There's a slight gap in the above proof (which explains where the gerbiness comes from). Namely, gluing is not completely trivial. We end up with the following object:

For a cover $U_i = \mathrm{Spec}(S_i)$ of $A$, we have a

1. A non-commutative deformation of $S_i$ to $\widetilde{S}_i$;
2. On each intersection $U_{ij} = \mathrm{Spec}(S_{ij}) = U_i \cap U_j$, an isomorphism $\phi_{ij}$ between the nc-deformations of $S_{ij}$ induced by $S_i$ and $S_j$; (Here we assume for simplicity that $A$ is separated.)
3. On a triple intersection $U_{ijk} = \mathrm{Spec}(S_{ijk})$, the automorphism of $\widetilde{S}_{ijk}$ induced by $\phi_{ij} \circ \phi_{jk} \circ \phi_{ki}$ must be inner, moreover, there's a chosen $\alpha_{ijk} \in \widetilde{S}_{ijk}$ such that $\alpha_{ijk} = 1 mod m_R$ and the automorphism is conjugation by $\alpha_{ijk}$;
4. $\alpha_{ijk}$'s must satisfy a cocycle condition on intersections of four objects.

This is exactly what we mean by a gerby deformation of $A$ (of course, the interesting part is writing what all these data can be changed by). A precise definition is written in Kontsevich.

The idea behind this definition is that if you have a gerby deformation, you have the notion of descent data for $O_{\widetilde{A}}$-modules. So even though $A$ does not carry a sheaf of algebras, the category of modules over it still makes sense.

*Example* 5. Suppose all $\widetilde{S}_i$ are commutative. Then $\phi_{ij}$ satisfy the cocycle condition, so we get an honest (commutative) deformation $\widetilde{A}'$ of $A$. The class $\alpha_{ijk}$ also satisfy the cocycle condition, so we get $[\alpha] \in H^2(\widetilde{A}', 1 + m_R O_{\widetilde{A}'})$. In characteristic zero, can write $[\alpha] \in H^2(\widetilde{A}', m_R O_{\widetilde{A}'})$.

So commutative gerby deformations can be thought of as a commutative deformation equipped with a gerbe cohomology class.

*Remark* 6. If the deformation is not commutative, there is no reason to expect $\widetilde{A}'$ exists, and if it does, it need not be unique (it can be twisted by $1 + m_R O_{\widetilde{A}'}$). So these gerbes are not gerbes over anything.

*Example* 7. A gerbe is trivial if $\alpha_{ijk} = 1$. Then $\widetilde{A}$ is a usual deformation quantization. Another way to check for triviality is that in the category of sheaves on $\widetilde{A}$ (which makes sense even if it is gerby) the sheaf $O_A$ admits a flat $R$-deformation

$E$. We can then write $O_{\widetilde{A}} = Hom(E, E)$. Notice that there might be more than one trivialization, and corresponding algebras need not be isomorphic.

## 2. First order deformations

Assume $A$ is smooth. (And char is not 2?)

2.1. **First order deformations splitting.** Things simplify when $R = \mathbb{C}[\hbar]/(\hbar^2)$. In this case, any gerby deformation is over something, and this something can be written as a 'sum' of something purely commutative and something purely quantum. More precisely, any deformation is a 'sum' of the following three kinds:

(1) Purely gerby: $O_A$ does not change, but there is a non-trivial class $[\alpha] \in H^2(A, O_A)$;
(2) Purely commutative: 'usual' deformations, classified by $H^1(A, T_A)$;
(3) Purely quantum: $O_A$ does not change as a sheaf, but the product on it becomes $a \star b = ab + \hbar\tau(a, b)$. Classified by $\tau \in H^0(A, \Lambda^2 T_A)$.

This splitting can be written as

$$\begin{array}{ccccccc} HH^2(A) & = & H^2(A, O_A) & \oplus & H^1(A, T_A) & \oplus & H^0(A, \Lambda^2 T_A) \\ \| & & \| & & \| & & \| \\ \text{First order deformations} & = & \text{Purely gerby} & \oplus & \text{Purely commutative} & \oplus & \text{Purely quantum} \end{array} \tag{2.1}$$

2.2. **Duality on first order deformations.** Recall now that $\pi : A \to B$ is an abelian scheme. $\pi$ gives a filtration on $T_A$:

$$0 \to T_{A/B} \to T_A \to \pi^* T_B \to 0;$$

this yields a filtration on exterior powers $\Lambda^k T_A$, and therefore on the terms of (2.1). For instance, $H^1(A, T_A)$ fits into an exact sequence

$$H^1(A, T_{A/B}) \to H^1(A, T_A) \to H^1(A, \pi^* T_B).$$

We can then write

$$H^1(A, \pi^* T_B) = H^1(B, \pi_* \pi^* T_B) \oplus H^0(B, R^1\pi_* \pi^* T_B) = H^1(B, T_B) \oplus H^0(B, T_B \otimes R^1\pi_* T_{A/B}).$$

Applying this argument to all terms of (2.1), we get the following diagram:

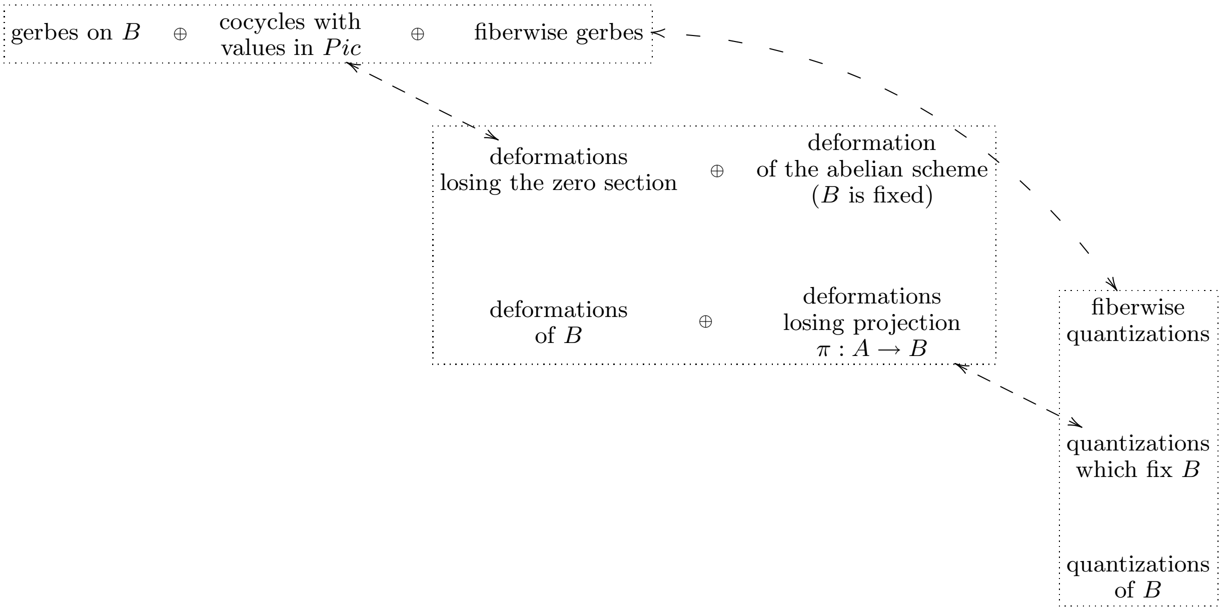
$$\boxed{H^2(O_B) \quad \oplus \quad H^1(R^1\pi_* O_A) \quad \oplus \quad H^0(R^2\pi_* O_A)}$$

$$\boxed{\begin{array}{ccc} H^1(\pi_* T_{A/B}) & \oplus & H^0(R^1\pi_* T_{A/B})^{quot} \\ \\ \Big(H^1(T_B) & \oplus & H^0(T_B \otimes R^1\pi_* O_A)\Big)^{sub} \end{array}}$$

$$\boxed{\begin{array}{c} H^0(\pi_* \Lambda^2 T_{A/B}) \\ \\ H^0(T_B \otimes \pi_* T_{A/B})^{sub} \\ \\ H^0(\Lambda^2 T_B)^{sub} \end{array}}$$

The rectangular blocks on the diagram correspond to terms of (2.1). Each term has a filtration; the corresponding subquotients are written vertically in the block. Finally, the superscripts ('sub' and 'quot') indicate that the subquotients is actually a subspace or a quotient space of the vector space in the diagram.

The spaces in the diagram admit the following interpretation:




As we pass from $A$ to $A^\vee$, we see non-trivial isomorphisms of the form $R^i\pi_*\Lambda^j T_{A/B} = R^j\pi_*\Lambda^i T_{A^\vee/B}$. This is indicated by arrows in the above diagrams. They lead to the following interesting Fourier transforms:

1. $H^0(R^1\pi_*T_{A/B}) = H^0(R^1\pi_*T_{A^\vee/B})$: usual Fourier-Mukai; (if $B$ is not fixed, it also includes the identity $H^1(T_B) = H^1(T_B)$);
2. $H^1(R^0\pi_*T_{A/B}) = H^1(R^1\pi_*O_{A^\vee})$: Fourier-Mukai for torsors (follows from 1-motive picture, written in Donagi-Pantev);
3. $H^0(\Lambda^2 T_{A/B}) = H^0(R^2\pi_*O_{A^\vee})$: Ben-Bassat/Block/Pantev;
4. $H^0(R^1\pi_*T_{A/B}) = H^0(T_{A/B} \otimes T_A)$: The situation I was talking about last time (from my thesis).

## 3. Duality and properties

Now let us get back to deformations of order higher than one. What kinds of properties do they have?

**Proposition 8.** *$\widetilde{A}^\vee$ is not gerby if and only if $\widetilde{A}$ admits a 'section': that is, the structure sheaf of the zero section in $A$ deforms.*

*Proof.* Obvious: $A^\vee$ is not gerby if $O_{A^\vee}$ deforms. □

**Proposition 9.** *$\widetilde{A}^\vee$ is commutative if and only if locally on $B$, there exists a projection $\widetilde{A} \to \widetilde{B}$ for a commutative $\widetilde{B}$ (in particular, $\widetilde{A}$ is not gerby locally on $\widetilde{B}$).*

*Proof.* Suffices to show that $\widetilde{A}^\vee$ is commutative if and only if locally on $B$ there is a deformation of the zero section to a module $\widetilde{Z}$ such that $Hom_{\widetilde{A}^\vee}(\widetilde{Z}, \widetilde{Z})$ is a deformation of $B$. The 'only if' direction is clear, the 'if' follows when one considers the Poisson bracket. □

**Proposition 10.** *A projection $\widetilde{A} \to \widetilde{B}$ means $\widetilde{A}^\vee$ has a 'two-sided section' (if $\widetilde{A}$ or $\widetilde{A}^\vee$ are gerby, need to be more careful).*

In particular, suppose $\widetilde{A}^\vee$ is commutative and not gerby. Then locally on $B$, we are exactly in the situation of the previous talk: $\widetilde{A}$ has a section and a projection. This is exactly the case when $O_{\widetilde{A}^\vee}$ has a tensor product, which translates into existence of convolution product (and so also a quantum groupoid structure) for $\widetilde{A}$! To make things reasonable, assume $\widetilde{A}$ is not gerby, so $\widetilde{A}^\vee$ admits a section. It is automatically a two-sided section, so we get a projection $\widetilde{A} \to \widetilde{B}$ defined everywhere, not just locally. That's what we did last time.