CHAPTER 3

Local Theory: Chiral Basics

Алгебраисты обычно определяют группы как множества с операциями, удовлетворяющими длинному ряду трудно-запоминаемых аксиом. Понять такое определение, на мой взгляд, невозможно.

Член-корреспондент АН СССР В. И. Арнольд,
"Математика с человеческим лицом",
Природа, №3, 1988, 117–118.†

3.1. Chiral operations

From now on we assume that X is a smooth curve.

We define chiral operations between $\mathcal{D}$-modules on X in 3.1.1. The action of the tensor category $\mathcal{M}(X)^!$ on $\mathcal{M}(X)^{ch}$ is defined in 3.1.3. The operad $P^{ch}(\omega)$ of chiral operations acting on ω_X identifies canonically with the Lie operad (see 3.1.5). This fact is (the de Rham version of) a theorem of F. Cohen [**C**]; an alternative proof is given in 3.1.6–3.1.9. In 3.1.16 we mention that chiral operations can also be defined in the setting of $(\mathfrak{g}, K)$-modules with $\dim \mathfrak{g}/\mathfrak{k} = 1$ (and, in fact, for $\mathcal{O}$-modules on any c-stack of c-rank 1; see 2.9), and $(\mathfrak{g}, K)$-structures provide pseudo-tensor functors between the $(\mathfrak{g}, K)$-modules and $\mathcal{D}$-modules.

3.1.1. For a finite non-empty set I put $U^{(I)} := \{(x_i) \in X^I : x_{i_1} \neq x_{i_2}$ for every $i_1 \neq i_2\}$. So $U^{(I)}$ is the complement to the diagonal divisor if $|I| \geq 2$. Denote by $j^{(I)} : U^{(I)} \hookrightarrow X^I$ the open embedding. We write $U^{(n)}$ for $U^{(\{1,\dots,n\})}$.

Let L_i, $i \in I$, and M be (right) $\mathcal{D}$-modules on X. Set

(3.1.1.1) $$P_I^{ch}(\{L_i\}, M) := \mathrm{Hom}_{\mathcal{M}(X^I)}(j_*^{(I)} j^{(I)*}(\boxtimes_I L_i), \Delta_*^{(I)} M).$$

The elements of this vector space are called *chiral I-operations.* They have the local nature with respect to the étale topology of X. As usual, we write P_n^{ch} for $P_{\{1,\dots,n\}}^{ch}$; the elements of P_2^{ch} are called *chiral pairings.*

REMARK. Let F_i ,G be quasi-coherent $\mathcal{O}_X$-modules. By 2.1.8 (see, in particular, Example (iii) there) we have $P_I^{ch}(\{F_{i\mathcal{D}}\}, G_{\mathcal{D}}) = \mathrm{Diff}(j_*^{(I)} j^{(I)*}(\boxtimes F_i), \Delta_\cdot G)$; elements of this vector space are called *chiral polydifferential operators.*

† "Algebraists define a group as a set equipped with operations, subject to a long series of axioms which are difficult to remember. Understanding such a definition is, in my opinion, impossible." V. I. Arnold, Corresponding Member of the Academy of Sciences of the USSR, "Mathematics with a Human Face", Priroda, No. 3, 1988, 117–118.

3.1.2. Our P_I^{ch} are left exact k-polylinear functors on $\mathcal{M}(X)$. Let us define the composition of chiral operations. Let $\pi\colon J \twoheadrightarrow I$ be a surjective map of finite non-empty sets, $\{K_j\}$ a J-family of $\mathcal{D}_X$-modules. The composition map

$$P_I^{ch}(\{L_i\}, M) \otimes (\otimes_I P_{J_i}^{ch}(\{K_j\}, L_i)) \longrightarrow P_J^{ch}(\{K_j\}, M) \tag{3.1.2.1}$$

sends $\varphi \otimes (\otimes \psi_i)$ to $\varphi(\psi_i)$ defined as the composition

$$j_*^{(J)} j^{(J)*}(\boxtimes_J K_j) = j_*^{(\pi)} j^{(\pi)*}(\boxtimes_I (j_*^{(J_i)} j^{(J_i)*} \boxtimes_{J_i} K_j)) \xrightarrow{\boxtimes \psi_i} j_*^{(\pi)} j^{(\pi)*}(\boxtimes_I \Delta_*^{(J_i)} L_i)$$
$$= \Delta_*^{(\pi)} j_*^{(I)} j^{(I)*}(\boxtimes_I L_i) \xrightarrow{\Delta_*^{(\pi)}(\varphi)} \Delta_*^{(\pi)} \Delta_*^{(I)} M = \Delta_*^{(J)} M.$$

Here $j^{(\pi)}\colon U^{(\pi)} := \{(x_j) \in X^J : x_{j_1} \neq x_{j_2} \text{ if } \pi(j_1) \neq \pi(j_2)\} \hookrightarrow X^J$; for the rest of the notation see 2.2.3.

The composition is associative, so the P_I^{ch} define on $\mathcal{M}(X)$ an abelian pseudo-tensor structure on $\mathcal{M}(X)$. We call it the *chiral structure* and denote it by $\mathcal{M}(X)^{ch}$. The pseudo-tensor categories $\mathcal{M}(U)^{ch}$, $U \in X_{\text{ét}}$, form a sheaf of pseudo-tensor categories $\mathcal{M}(X_{\text{ét}})^{ch}$ on the étale topology of X.

The de Rham functor h (see 2.1.6) is an augmentation functor on $\mathcal{M}(X_{\text{ét}})^{ch}$ (see 1.2.5).[1] The structure map

$$h_{I,i_0}\colon P_I^{ch}(\{L_i\}, M) \otimes h(L_{i_0}) \to P_{I \smallsetminus \{i_0\}}^{ch}(\{L_i\}, M) \tag{3.1.2.2}$$

sends $\varphi \otimes \bar{\ell}_{i_0}$, where $\varphi \in P_I^{ch}(\{L_i\}, M)$, $\ell_{i_0} \in L_{i_0}$, to the chiral $I \smallsetminus \{i_0\}$-operation $a \mapsto Tr_{i_0}\varphi(a \boxtimes \ell_{i_0})$ (cf. 2.2.7). Here, as in 2.2.7, $Tr_{i_0}\colon pr_{I,i_0}.\Delta_*^{(I)} M \to \Delta_*^{(I \smallsetminus \{i_0\})} M$ is the trace map for the projection $pr_{I,i_0}\colon X^I \to X^{I \smallsetminus \{i_0\}}$. Compatibilities (i) and (ii) in 1.2.5 are immediate.

3.1.3. The action of the tensor category $\mathcal{M}(X)^!$ on $\mathcal{M}(X)$ extends naturally to an action of $\mathcal{M}(X)^!$ on $\mathcal{M}(X)^{ch}$ (see 1.1.6(v)). The corresponding morphisms $P_I^{ch}(\{M_i\}, N) \to P_I^{ch}(\{M_i \otimes A_i\}, N \otimes (\otimes A_i))$ send a chiral operation φ to $\varphi \otimes id_{\boxtimes A_i}$; we use the identification $(\Delta_*^{(I)} N) \otimes (\boxtimes A_i) = \Delta_*^{(I)}(N \otimes (\otimes A_i))$ of (2.1.3.2).

3.1.4. Consider the operad $P^{ch}(\omega_X)$ (see 1.1.6) of chiral operations on ω_X, so $P_I^{ch}(\omega_X) := \operatorname{Hom}(j_*^{(I)} j^{(I)*} \omega_X^{\boxtimes I}, \Delta_*^{(I)} \omega_X)$. Set $\lambda_I := (k[1])^{\otimes I}[-|I|] = (\det(k^I))[|I|]$; this is a line of degree 0 on which $\operatorname{Aut} I$ acts by the sgn character. One has a canonical, hence $\operatorname{Aut} I$-equivariant, isomorphism

$$\varepsilon_I\colon \omega_X^{\boxtimes I} \otimes \lambda_I \xrightarrow{\sim} \omega_{X^I}, \quad (\boxtimes \nu_i) \otimes (e_{i_1} \wedge \cdots \wedge e_{i_n}) \mapsto pr_{i_1}^* \nu_{i_1} \wedge \cdots \wedge pr_{i_n}^* \nu_{i_n}; \tag{3.1.4.1}$$

here $\nu_i \in \omega_X$, $\{i_1, \ldots, i_n\}$ is an ordering of I, and $\{e_i\}$ is the standard base of k^I. Equivalently, $\varepsilon_I[|I|] : (\omega_X[1])^{\boxtimes I} \xrightarrow{\sim} \omega_{X^I}[|I|]$ is the compatibility of Grothendieck's dualizing complexes with products (see 2.2.2).

If $|I| = 2$, then the residue morphism Res: $j_*^{(I)} j^{(I)*} \omega_{X^I} \to \Delta_*^{(I)} \omega_X$ yields a canonical map $r_I\colon \lambda_I \to P_I^{ch}(\omega_X)$.

3.1.5. THEOREM. *There is a unique isomorphism of operads*

$$\kappa : \mathcal{L}ie \xrightarrow{\sim} P^{ch}(\omega_X)$$

which coincides with r_I for $|I| = 2$.

[1] Contrary to the $*$ situation, in the chiral setting h is highly degenerate.

Proof. Let $C(\omega)_{X^I}$ be the Cousin complex for $\omega_{X^I}[|I|] = (\omega_X[1])^{\boxtimes I}$ with respect to the diagonal stratification. It equals $\underset{T\in Q(I)}{\oplus} \Delta_*^{(I/T)} j_*^{(T)} j^{(T)*}(\omega_X[1])^{\boxtimes T}$ as a mere graded module; here $Q(I)$ is the set of quotients $I \twoheadrightarrow T$ (see 1.3.1), and $\Delta^{(I/T)}$ is the diagonal embedding $X^T \hookrightarrow X^I$. The non-zero components of the differential are $\Delta_*^{(I/T)}(\mathrm{Res}_{T'})$ for $X^{T'} \subset X^T$, $|T'| = |T| - 1$, where $\mathrm{Res}_{T'} : j_*^{(T)} j^{(T)*}(\omega_X[1])^{\boxtimes T} \to \Delta_*^{(T/T')} j_*^{(T')} j^{(T')*}(\omega_X[1])^{\boxtimes T'}$ is the residue morphism. The complex C_I is a resolution of $(\omega_X[1])^{\boxtimes I}$.

The existence of κ is easy: the map $r_I : \lambda_I \to P_I^{ch}(\omega_X)$, $|I| = 2$, is $\mathrm{Aut}\, I$-equivariant, so it defines a skew-symmetric operation $[\ ] \in P_2^{ch}(\omega_X)$, and we have only to check that the operation $\alpha := [[1,2],3] + [[2,3]1] + [[3,1],2] \in P_3^{ch}(\omega_X)$ is zero. This is true since α equals the square of the differential in $C(\omega)_{X^3}$. The uniqueness of κ is clear. We will show that κ is bijective in 3.1.8–3.1.9 after a necessary digression of 3.1.6–3.1.7.

3.1.6. Let M be a $\mathcal{D}_{X^I}$-module. We say that a finite increasing filtration $W.$ on M is *special* if for every l the successive quotient $\mathrm{gr}_l^W M$ is a finite sum of copies of $\Delta_*^{(I/T)}\omega_{X^T}$ for $T \in Q(I, -l)$. Such a filtration is unique if it exists. We call M a *special* $\mathcal{D}_{X^I}$-module if it admits a special filtration. Let $\mathcal{M}(X^I)^{sp} \subset \mathcal{M}(X^I)$ be the full subcategory of special $\mathcal{D}$-modules. It is closed under finite direct sums and subquotients, so it is an abelian category. Any morphism in $\mathcal{M}(X^I)^{sp}$ is strictly compatible with $W.$; i.e., gr^W is an exact functor.

Denote by $\mathcal{M}(X^I)^{sp}_{>n}$ the full subcategory of special modules with $W_n = 0$.

3.1.7. Lemma. *$j_*^{(I)} j^{(I)*}\omega_{X^I}$ is a special $\mathcal{D}_{X^I}$-module.*

Proof of Lemma. We want to show that $C(\omega)_{X^I}^{-|I|}$ is special. By induction by $|I|$ we can assume that $C(\omega)_{X^I}^{-|I|+1} \in \mathcal{M}(X^I)^{sp}_{>-|I|}$. Thus such is the image of $d : C(\omega)_{X^I}^{-|I|} \to C(\omega)_{X^I}^{-|I|+1}$. Therefore $C(\omega)_{X^I}^{-|I|}$ is an extension of a special module with vanishing $W_{-|I|}$ by ω_{X^I}, and hence it is special; q.e.d. □

3.1.8. So for any $T \in Q(I)$ the $\mathcal{D}_{X^I}$-module $\Delta_*^{(I/T)} j_*^{(T)} j^{(T)*}\omega_{X^T}$ is special. In fact, this is an injective object of $\mathcal{M}(X^I)^{sp}$. This follows from the exactness of the functor $P \mapsto P/W_{-|T|-1}P$ since $\Delta_*^{(I/T)} j_*^{(T)} j^{(T)*}\omega_{X^T}$ is obviously an injective object of the subcategory $\mathcal{M}(X^I)^{sp}_{>-|T|-1}$. In particular, $\Delta_*^{(I)}\omega_X$ is an injective object of $\mathcal{M}(X^I)^{sp}$.

Suppose that $|I| > 1$, so $\mathrm{Hom}(\omega_{X^I}, \Delta_*^{(I)}\omega_X) = 0$. Since $C(\omega)_{X^I}$ is a resolution of $\omega_{X^I}[I]$ in $\mathcal{M}(X^I)^{sp}$, the complex $C_I := \mathrm{Hom}(C(\omega)_{X^I}, \Delta_*^{(I)}\omega_X)$ is acyclic.

As a mere graded module C_I equals $\underset{T\in Q(I)}{\oplus} P_T^{ch}(\omega_X) \otimes \lambda_T[-|T|]$. The non-zero components of the differential are $P_{T'}^{ch}(\omega_X) \otimes \lambda_{T'}[-|T'|] \to P_T^{ch}(\omega_X) \otimes \lambda_T[-|T|]$ for $T' \in Q(T)$, $|T| = |T'| + 1$. In terms of the operad structure it is the insertion of $\kappa([\ ])_{\alpha_1,\alpha_2} \in P^{ch}_{\{\alpha_1,\alpha_2\}}(\omega_X)$; here $\alpha_i \in T$ are the elements such that T' is the quotient of T modulo the relation $\alpha_1 = \alpha_2$.

3.1.9. Now we can finish the proof of 3.1.5. The surjectivity of the top differential in the complexes C_I means that for $n \geq 2$ every operation in $P_n^{ch}(\omega_X)$ is a sum of $(n-1)$-operations composed with binary operations. Therefore $P^{ch}(\omega_X)$ is generated by $P_2^{ch}(\omega_X)$. Acyclicity of C_I implies that all the relations in our operad come from the Jacobi relation for $[\ ]\in P_2^{ch}(\omega_X)$. □

3.1.10. REMARKS. (i) For $S,T\in Q(I)$ let $\mathcal{L}ie_{S/T}$ be the vector space $\underset{T}{\otimes}\mathcal{L}ie_{S_t}$ if $S\geq T$ and 0 otherwise. Let us define a canonical isomorphism

$$(3.1.10.1)\quad \alpha(I,S)_l\colon\ \mathrm{gr}^W_{-l}(\Delta_*^{(I/S)}j_*^{(S)}j^{(S)*}\omega_X^{\boxtimes S})\xrightarrow{\sim}\bigoplus_{T\in Q(S,l)}\Delta_*^{(I/T)}\omega_X^{\boxtimes T}\otimes\mathcal{L}ie^*_{S/T}.$$

First notice that both parts are supported on $X^S\subset X^I$, so (applying $\Delta^{(I/S)!}$ and replacing S by I) we may assume that $S=I$. The case of the top quotient $l=1$ is just the isomorphism κ_I of 3.1.5. Assume that $l>1$. It suffices to define $\alpha(I,I)_l=\alpha(I)_l$ over the complement to diagonals of dimension $<l$. Its T-component $\alpha(I)_T$ is determined by the restriction to a neighbourhood of X^T. There $j_*^{(I)}j^{(I)*}\omega_X^{\boxtimes I}$ coincides with $\underset{T}{\boxtimes}j_*^{(I_t)}j^{(I_t)*}\omega_X^{\boxtimes I_t}$, and $\alpha(I)_T:=\underset{T}{\boxtimes}\alpha(I_t)_1$.

(ii) Let us describe $\mathcal{M}(X^I)^{sp}$ explicitly. Its irreducible objects are $\Delta_*^{(I/T)}\omega_X^{\boxtimes T}$, $T\in Q(I)$. The object $\mathcal{I}_T:=\Delta_*^{(I/T)}j_*^{(T)}j^{(T)*}\omega_X^{\boxtimes T}$ is an injective envelope of $\Delta_*^{(I/T)}\omega_X^{\boxtimes T}$.[2] Isomorphism (3.1.10.1) yields a natural identification $\mathrm{Hom}(\mathcal{I}_S,\mathcal{I}_T)=\mathcal{L}ie_{S/T}$. The composition of morphisms corresponds to (a tensor product) of compositions of Lie operations.

So let $\mathcal{Q}(I)$ be the k-category whose objects are elements of $Q(I)$ and morphisms $\mathrm{Hom}_{\mathcal{Q}(I)}(S,T)=\mathcal{L}ie_{S/T}$; the composition of morphisms is the tensor product of compositions of Lie operations. We have defined a fully faithful embedding $\mathcal{Q}(I)\to\mathcal{M}(X^I)^{sp}$, $T\mapsto\mathcal{I}_T$. A $\mathcal{Q}(I)$-*module* is a k-linear functor $\mathcal{Q}(I)\to$ (finite-dimensional vector spaces); $\mathcal{Q}(I)$-modules form an abelian category $\mathcal{V}(I)$. There is an obvious functor $\mathcal{M}(X^I)^{sp}\to\mathcal{V}(I)^\circ$ which sends a $\mathcal{D}$-module M to the $\mathcal{Q}(I)$-module $T\mapsto\mathrm{Hom}(M,\mathcal{I}_T)$. This functor is an equivalence of categories.

3.1.11. REMARKS. We assume that $k=\mathbb{C}$.

(i) For any special $\mathcal{D}_{X^I}$-module P one has $H^lDR_{an}(P)=H^lDR_{an}(\mathrm{gr}^W_lP)$; this sheaf is isomorphic to a direct sum of constant sheaves supported on X^S, $|S|=-l$. Here DR_{an} denotes the analytic de Rham complex (used in the Riemann-Hilbert correspondence). If $l=-1$, then $H^lDR_{an}(P)$ is the constant sheaf on the diagonal $X\subset X^I$ with fiber V^* where $V:=\mathrm{Hom}(P,\Delta_*^{(I)}\omega_X)$.

(ii) The filtration $W.$ on $j_*^{(I)}\omega_{U^{(I)}}$ coincides with the weight filtration on the mixed Hodge module $j_*^{(I)}\mathbb{C}(|I|)_{U^{(I)}}[|I|]$.

(iii) By 3.1.4, one has $P_I^{ch}(\omega_X)=\mathrm{Hom}(j_*^{(I)}\omega_{U^{(I)}},\Delta_*^{(I)}\omega_X)\otimes\lambda_I$. So remark (i) above implies that $P_I^{ch}(\omega_X)\otimes\lambda_I=H_{|I|-1}(Y_I,\mathbb{C})$ where Y_I is the configuration space of I-tuples of different points on $\mathbb{C}=\mathbb{R}^2$. Up to homotopy the Y_I's are the spaces of the "small disc" operad. The composition law in the operad $P^{ch}(\omega_X)$ coincides with the one on the homology operad of this topological operad. So 3.1.5 amounts to Cohen's theorem [**C**].

3.1.12. For $M\in\mathcal{M}(X)$ we define the *unit operation* $\varepsilon_M\in P_2^{ch}(\{\omega_X,M\},M)$ as the composition $j_*j^*\omega_X\boxtimes M\to(j_*j^*\omega_X\boxtimes M)/\omega_X\boxtimes M\xrightarrow{\sim}\Delta_*M$ where the last arrow comes from the canonical isomorphism $\omega_X\otimes^!M=M$.

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[2]Indeed, $\mathcal{I}_T$ is injective by 3.1.8; it contains $\Delta_*^{(I/T)}\omega_X^{\boxtimes T}$ and is indecomposable.

3.1.13. LEMMA. (i) *For* $M_i, N \in \mathcal{M}(X)$ *there is a canonical isomorphism*[3]

$$(3.1,13.1)\quad P^{ch}_I(\{M_i\}, N)^I \xrightarrow{\sim} P^{ch}_{\tilde{I}}(\{\omega_X, M_i\}, N), \quad (\varphi_i) \mapsto \sum_{i\in I} \varphi_i(\varepsilon_{M_i}, id_{M_{i'}})_{i'\neq i}.$$

(ii) *Chiral operations satisfy the Leibniz rule with respect to the unit operations: for* $\varphi \in P^{ch}_I(\{M_i\}, N)$ *one has* $\varepsilon_N\varphi = \sum_{i\in I} \varphi(\varepsilon_{M_i}, id_{M_{i'}})_{i'\neq i} \in P^{ch}_{\tilde{I}}(\{\omega_X, M_i\}, N)$.

Proof. (i) Let $\Delta^i : X^I \hookrightarrow X^{\tilde{I}}$ be the ith diagonal section of the projection $X^{\tilde{I}} \to X^I$. The images of the Δ^i's are disjoint over $U^{(I)}$, so one has a short exact sequence

(3.1.13.1)

$$0 \to \omega_X \boxtimes j^{(I)}_* j^{(I)*}(\boxtimes M_i) \to j^{(\tilde{I})}_* j^{(\tilde{I})*}\omega_X \boxtimes (\boxtimes M_i) \to \oplus \Delta^i_* j^{(I)}_* j^{(I)*} \boxtimes M_i \to 0.$$

Here we use the standard identifications $j_*j^*\omega_X \boxtimes M_i/\omega_X \boxtimes M_i = \Delta_*(\omega_X \otimes^! M_i) = \Delta_* M_i$. Since $\mathrm{Hom}(\omega_X \boxtimes j^{(I)}_* j^{(I)*}(\boxtimes M_i), \Delta^{(\tilde{I})}_* N) = 0$, (3.2.4.1) yields a canonical isomorphism $P^{ch}_I(\{M_i\}, N)^I \xrightarrow{\sim} P^{ch}_{\tilde{I}}(\{\omega_X, M_i\}, N)$. One checks in a moment that it coincides with the map in (3.1.13.1).

(ii) Let t be a local coordinate on X, t_0, t_i the corresponding local coordinates on $X^{\tilde{I}}$. For $i \in I$ set $\nu^i := \frac{dt_o}{t_0 - t_i}$. For any $m \in j^{(I)}_* j^{(I)*} \boxtimes M_i$ one has $\varepsilon_N \varphi(\nu^i \cdot m) = \Delta^i_\cdot(\varphi(m)) = \varphi(\varepsilon_{M_i}, id_{M_{i'}})_{i'\neq i}(\nu_i \cdot m) \in \Delta^{(\tilde{I})}_* N$ and $\varphi(\varepsilon_{M_{i''}}, id_{M_{i'}})_{i'\neq i''} = 0$ for $i'' \neq i$. Now use (i). □

3.1.14. VARIANTS. (i) Let L be any Lie* algebra, $M_i, N \in \mathcal{M}(X, L)$. A chiral operation $\varphi \in P^{ch}_I(\{M_i\}, N)$ is *L-compatible* if it satisfies the Leibniz rule with respect to the L-actions. Explicitly, we demand that the composition $\cdot_N \varphi \in \mathrm{Hom}(L \boxtimes j^{(I)}_* j^{(I)*} \boxtimes M_i, \Delta^{(\tilde{I})}_* N)$ equals the sum of I operations $\varphi(\cdot_{M_i}, id_{M_{i'}})_{i'\neq i}$, $i \in I$.[4] This amounts to the fact that φ satisfies the Leibniz rule with respect to the $h(L)$-action on our $\mathcal{D}$-modules. We denote the set of such operations by P^{ch}_{LI}; they define on $\mathcal{M}(X, L)$ a pseudo-tensor structure $\mathcal{M}(X, L)^{ch}$.

(ii) Let R^ℓ be a commutative $\mathcal{D}_X$-algebra and M_i, N are $R^\ell[\mathcal{D}_X]$-modules. Consider the $\mathcal{D}_{X^I}$-algebra $R^{\ell\boxtimes I}$. Then $j^{(I)}_* j^{(I)*} \boxtimes M_i$ and $\Delta^{(I)}_* N$ are $R^{\ell\boxtimes I}$-modules in the obvious way.[5] We say that a chiral operation $\varphi : j^{(I)}_* j^{(I)*} \boxtimes M_i \to \Delta^{(I)}_* N$ is *R^ℓ-polylinear* if it is a morphism of $R^{\ell\boxtimes I}$-modules.

R^ℓ-polylinear chiral operations are closed under composition, so they define a pseudo-tensor structure on $\mathcal{M}(X, R^\ell)$, which we denote by $\mathcal{M}(X, R^\ell)^{ch}$.

According to 3.1.3, the functor $M \mapsto M \otimes R^\ell$ extends naturally to a pseudo-tensor functor $\mathcal{M}(X)^{ch} \to \mathcal{M}(X, R^\ell)^{ch}$. More generally, for any morphism of commutative $\mathcal{D}_X$-algebras $R^\ell \to F^\ell$ the base change functor $N \mapsto N \underset{R^\ell}{\otimes} F^\ell$ extends naturally to a pseudo-tensor functor $\mathcal{M}(X, R^\ell)^{ch} \to \mathcal{M}(X, F^\ell)^{ch}$.

The above constructions make evident sense in the DG super setting.

[3] Here $\tilde{I} := I \sqcup \cdot$ (see 1.2.1).

[4] Here $\cdot_{M_i}$, $\cdot_N$ are the $*$ actions of L.

[5] The $R^{\ell\boxtimes I}$-module structure on $\Delta^{(I)}_* N$ comes from the $R^{\ell\otimes I}$-module structure on N (defined by the product morphism of algebras $R^{\ell\otimes I} \to R^\ell$) and the fact that $\Delta^{(I)}_*$ sends $R^{\ell\otimes I} = \Delta^{(I)*} R^{\ell\boxtimes I}$-modules to $R^{\ell\boxtimes I}$-modules (to define the module structure use (2.1.3.2) for $i = \Delta^{(I)}$, $L = R^{\ell\boxtimes I}$, $M = N$).

Exercises. (i) Suppose L is a Lie* algebra which is a vector $\mathcal{D}_X$-bundle, so L° is a Lie$^!$ coalgebra (see 2.5.7). Let $\mathcal{C}(L)$ be the Chevalley DG $\mathcal{D}_X$-algebra; for an L-module M we have a DG $\mathcal{C}(L)$-module $\mathcal{C}(L,M)$ (see 1.4.10). Show that the functor $\mathcal{C}(L,\cdot):\mathcal{M}(X,L)\to\mathcal{M}(X,\mathcal{C}(L))$ extends naturally to a faithful pseudo-tensor functor $\mathcal{C}(L,\cdot):\mathcal{M}(X,L)^{ch}\to\mathcal{M}(X,\mathcal{C}(L))^{ch}$.

(ii) Extend the above constructions to the setting of modules over a Lie* R-algebroid (see 2.5.16).

3.1.15. The category $\mathcal{M}_{\mathcal{O}}(X)$ of quasi-coherent $\mathcal{O}_X$-modules carries a natural $\mathcal{M}(X)^{ch}$-action (see 1.2.11). The vector space of operations $P^{ch}_{\tilde{I}}(\{M_i,F\},G)$, where F, G are (quasi-coherent) $\mathcal{O}_X$-modules and M_i, $i\in I$, are $\mathcal{D}_X$-modules, is defined as $P^{ch}_{\tilde{I}}(\{M_i,F\},G):=\mathrm{Hom}_{\mathcal{D}_X^{\boxtimes I}\boxtimes\mathcal{O}_X}(j_*^{(\tilde{I})}j^{(\tilde{I})*}((\boxtimes M_i)\boxtimes F),\Delta_*^{(\tilde{I})}G)$. Here $\Delta_*^{(\tilde{I})}G:=(\Delta_\cdot^{(\tilde{I})}G)\underset{\mathcal{O}_{X^{\tilde{I}}}}{\otimes}(\mathcal{D}_X^{\boxtimes I}\boxtimes\mathcal{O}_X)$. The composition of these operations with usual chiral operations between $\mathcal{D}$-modules and morphisms of $\mathcal{O}$-modules (as needed in 1.2.11) is clear.

Consider the induction functor $\mathcal{M}_{\mathcal{O}}(X)\to\mathcal{M}(X)$, $F\mapsto F_{\mathcal{D}}$ (see 2.1.8). One has an obvious natural map $P^{ch}_{\tilde{I}}(\{M_i,F\},G)\hookrightarrow P^{ch}_{\tilde{I}}(\{M_i,F_{\mathcal{D}}\},G_{\mathcal{D}})$. On the other hand, if P,Q are $\mathcal{D}$-modules, then there is an obvious map $P^{ch}_{\tilde{I}}(\{M_i,P\},Q)\hookrightarrow P^{ch}_{\tilde{I}}(\{M_i,P_{\mathcal{O}}\},Q_{\mathcal{O}})$, where $P_{\mathcal{O}}$, $Q_{\mathcal{O}}$ are P,Q considered as $\mathcal{O}$-modules.

Therefore both induction and restriction functors $\mathcal{M}_{\mathcal{O}}(X)\rightleftarrows\mathcal{M}(X)$ are compatible with the $\mathcal{M}(X)^{ch}$-actions. Here the $\mathcal{M}(X)^{ch}$ acts on $\mathcal{M}(X)$ in the standard way; see 1.2.12(i).

3.1.16. The setting of Harish-Chandra modules. Let us mention briefly that chiral operations are also defined for $\mathcal{O}$-modules on any c-stack of c-rank 1 (see 2.9.10). We will not define it in full generality, but we will just consider the case of the category of $(\mathfrak{g},K)$-modules (see 2.9.7; we use notation from loc. cit.) with $\dim\mathfrak{g}/\mathfrak{k}=1$. For $I\in\mathcal{S}$ define the diagonal divisor in G^I/G as the union of preimages of K by all maps $G^I/G\to G$, $(g_i)\mapsto g_{i_1}g_{i_2}^{-1}$. Let $\tilde{F}^{(I)}$ be the localization of $F^{(I)}$ with respect to the equations of the diagonal divisor; this is a G^I-equivariant $F^{(I)}$-module. Now for $M_i,N\in\mathcal{M}(G)$ we set

$$P^{ch}_I(\{M_i\},N):=\mathrm{Hom}((\otimes M_i)\otimes\tilde{F}^{(I)},\Delta_*^{(I)}N)\otimes\lambda_I \tag{3.1.16.1}$$

where the morphisms are taken in the category of G^I-equivariant $F^{(I)}$-modules. The composition of chiral operations is defined similarly to the $\mathcal{D}$-module situation. Theorem 3.1.5 remains valid in this context: one has $P^{ch}(k)=\mathcal{L}ie$. For a $(\mathfrak{g},K)$-structure Y on X (see 2.9.8) the corresponding functor $\mathcal{M}(\mathfrak{g},K)\to\mathcal{M}(X)$, $V\mapsto V^{(Y)}$, extends naturally to a pseudo-tensor functor

$$\mathcal{M}(\mathfrak{g},K)^{ch}\to\mathcal{M}(X)^{ch}. \tag{3.1.16.2}$$

Its definition is similar to the definition of the $*$ pseudo-tensor extension (see 2.9.8), and we skip it.

For example, for $\mathfrak{g}=k$ the category of $\mathfrak{g}$-modules = that of $k[\partial]$-modules, carries a natural chiral pseudo-tensor structure, and we have a natural pseudo-tensor functor from it to $\mathcal{M}(\mathbb{A}^1)$ (see Exercise in 2.9.8). The functor (3.1.16.2) for the Gelfand-Kazhdan structure is a pseudo-tensor equivalence between the chiral pseudo-tensor category of $\operatorname{Aut} k[[t]]$-modules and that of universal $\mathcal{D}$-modules on curves (with chiral operations defined in the obvious way); see 2.9.9.

3.2. Relation to "classical" operations

In this section we explain why "classical" operations from 2.2 (or, more precisely, the corresponding c operations as defined in 1.4.27) can be considered as "symbols", or "classical limits", of chiral operations. The key reason is Cohen's theorem (see 3.1.5).

3.2.1. The obvious morphism $\boxtimes L_i \to j_*^{(I)} j^{(I)*} \boxtimes L_i$ yields a map

$$\beta_I \colon P_I^{ch}(\{L_i\}, M) \to P_I^*(\{L_i\}, M). \tag{3.2.1.1}$$

It is clear that the β's are compatible with the composition of operations, so we have a pseudo-tensor functor

$$\beta \colon \mathcal{M}(X)^{ch} \to \mathcal{M}(X)^* \tag{3.2.1.2}$$

which extends the identity functor on $\mathcal{M}(X)$. It is compatible with the augmentations, so β is an augmented pseudo-tensor functor.

Remark. The above pseudo-tensor functor commutes the action of $\mathcal{M}(X)^!$ on $\mathcal{M}(X)^{ch}$ and $\mathcal{M}(X)^*$ (see 2.2.9 and 3.1.3).

3.2.2. Consider the tensor category $\mathcal{M}(X)^!$ and the pseudo-tensor category $\mathcal{M}(X)^! \otimes \mathcal{L}ie$ (see 1.1.10). So $\mathcal{M}(X)^! \otimes \mathcal{L}ie$ coincides with $\mathcal{M}(X)$ as a usual category, and the corresponding operations are $P_I^{!\mathcal{L}ie}(\{L_i\}, M) := \operatorname{Hom}(\otimes^! L_i, M) \otimes \mathcal{L}ie_I$ where $\mathcal{L}ie$ is the Lie algebras operad.

There is a canonical faithful pseudo-tensor functor

$$\alpha \colon \mathcal{M}(X)^! \otimes \mathcal{L}ie \to \mathcal{M}(X)^{ch} \tag{3.2.2.1}$$

which extends the identity functor on $\mathcal{M}(X)$. Namely, the map

$$\alpha_I \colon \operatorname{Hom}(\otimes^! L_i, M) \otimes \mathcal{L}ie_I \hookrightarrow P_I^{ch}(\{L_i\}, M) \tag{3.2.2.2}$$

sends $\phi \otimes \nu \in \operatorname{Hom}(\otimes^! L_i, M) \otimes \mathcal{L}ie_I$ to a chiral operation equal to the composition

$$j_*^{(I)} j^{(I)*}(\boxtimes L_i) = (\boxtimes L_i^\ell) \otimes j_*^{(I)} j^{(I)*} \omega_X^{\boxtimes I} \to (\boxtimes L_i^\ell) \otimes \Delta_*^{(I)} \omega_X = \Delta_*^{(I)}(\otimes^! L_i) \to \Delta_*^{(I)} M.$$

Here the first arrow is $\kappa(\nu)$ (see 3.1.5) tensored by the identity morphism of $\boxtimes L_i^\ell$, the second one is $\Delta_*^{(I)}(\phi)$, and the equalities are standard canonical isomorphisms. We leave it to the reader to check injectivity of α_I (notice that the canonical map $j_*^{(I)} j^{(I)*} \omega_X^{\boxtimes I} \to \Delta_*^{(I)} \omega_X \otimes \mathcal{L}ie_I^*$ coming from κ is surjective) and the compatibility of the α_I's with the composition of operations.

3.2.3. Remark. The composition $\beta_I \alpha_I : P_I^{!\mathcal{L}ie} \to P_I^*$ vanishes for $|I| \geq 2$. The sequence $0 \to P_2^{!\mathcal{L}ie} \to P_2^{ch} \to P_2^*$ is exact. Notice that $P_2^! \xrightarrow{\sim} P_2^{!\mathcal{L}ie}$, $\phi \mapsto \phi \otimes [\ ]$.

The above pseudo-tensor functors α, β are parts of the following general picture:

3.2.4. Notice that $j_*^{(I)}j^{(I)*}\boxtimes L_i = (j_*^{(I)}j^{(I)*}\omega_X^{\boxtimes I})\otimes(\boxtimes L_i^\ell)$. So the special filtration $W_\cdot$ on $j_*^{(I)}j^{(I)*}\omega_X^{\boxtimes I}$ (see 3.1.6 and 3.1.7) yields a finite filtration on $j_*^{(I)}j^{(I)*}\boxtimes L_i$ which we denote also by $W_\cdot$. The canonical identifications $gr_{-l}^W j_*^{(I)}j^{(I)*}\omega_X^{\boxtimes I} = \bigoplus_{T\in Q(I,l)} \Delta_*^{(I/T)}\omega_X^{\boxtimes T}\otimes \mathcal{L}ie^*_{I/T}$ of (3.1.10.1) yield the canonical surjections

$$\text{(3.2.4.1)} \qquad \bigoplus_{T\in Q(I,l)} \Delta_*^{(I/T)}(\mathop{\boxtimes}_{t\in T}(\mathop{\otimes}^!_{I_t} L_i)\otimes \mathcal{L}ie^*_{I_t}) \twoheadrightarrow gr_{-l}^W j_*^{(I)}j^{(I)*}\boxtimes L_i$$

which are isomorphisms if the L_i are $\mathcal{O}_X$-flat.

The above filtration yields a canonical filtration on k-modules of the chiral operations:

$$\text{(3.2.4.2)} \quad P_I^{ch}(\{L_i\},M)^n := \mathrm{Hom}(j_*^{(I)}j^{(I)*}\boxtimes L_i/W_{n-1-|I|}j_*^{(I)}j^{(I)*}\boxtimes L_i,\, \Delta_*^{(I)}M).$$

The morphisms (3.2.4.1) yield the canonical embeddings

$$\text{(3.2.4.3)} \qquad gr^n P_I^{ch}(\{L_i\},M) \hookrightarrow \bigoplus_{T\in Q(I,|I|-n)} P_T^*(\{\otimes^!_{I_t} L_i\},M)\otimes \mathcal{L}ie_{I/T}$$

which are isomorphisms if the L_i are projective $\mathcal{D}_X$-modules.

Notice that the maps α, β from 3.2.1 and 3.2.2 occur as "boundary" morphisms of (3.2.4.3). Namely, the morphism (3.2.4.3) for $n = |I|-1$ is always an isomorphism. Its inverse (composed with the embedding $gr^{|I|-1}P_I^{ch} = P_I^{ch\,|I|-1} \hookrightarrow P_I^{ch}$) coincides with α_I of (3.2.2.2). Simlarly, β_I of (3.2.1.1) is the morphism (3.2.4.3) for $n=0$ composed with the projection $P_I^{ch} \twoheadrightarrow P_I^{ch}/P_I^{ch\,1} = gr^0 P_I^{ch}$.

3.2.5. The above filtration is compatible with the composition of chiral operations. Namely, for $J \twoheadrightarrow I$ and a J-family of $\mathcal{D}_X$-modules K_j the composition maps the tensor product $P_I^{ch}(\{L_i\},M)^n\otimes(\mathop{\otimes}_I P_{J_i}^{ch}(\{K_j\},L_i)^{m_i})$ to $P_J^{ch}(\{K_j\},M)^{n+\Sigma m_i}$. Therefore we can define a new pseudo-tensor structure on $\mathcal{M}$ with operations $P^{cl} := gr^\cdot P^{ch}$; this is the *classical limit* of the chiral structure. We can rewrite (3.2.4.3) as a canonical embedding $P_I^{cl}(\{L_i\},M)\hookrightarrow P_I^c(\{L_i\},M)$ (recall that c operations were defined in 1.4.27 using the compound pseudo-tensor structure $\mathcal{M}^{*!}$) which is an isomorphism if the L_i are projective $\mathcal{D}_X$-modules. It follows from the definitions that this embedding is compatible with the composition of operations. So we defined a canonical fully faithful embedding of pseudo-tensor categories

$$\text{(3.2.5.1)} \qquad \mathcal{M}(X)^{cl} \hookrightarrow \mathcal{M}(X)^c$$

which extends the identity functor on $\mathcal{M}$.

3.2.6. The above constructions remain valid in the setting of $(\mathfrak{g},K)$-modules (see 3.1.16) and pseudo-tensor functors defined by $(\mathfrak{g},K)$-structures are compatible with them.

3.3. Chiral algebras and modules

In this section we define the basic objects to play with following the "Lie algebra style" approach (see 0.8). We begin with non-unital chiral algebras which are simply Lie algebras in the pseudo-tensor category $\mathcal{M}(X)^{ch}$. The standard functors relating chiral operations with ! and $*$ operations provide the canonical Lie*

bracket on every chiral algebra and identify commutative $\mathcal{D}_X$-algebras with commutative chiral algebras (see 3.3.1–3.3.2). Unital chiral algebras (or simply chiral algebras) are considered in 3.3.3. We pass then to modules over chiral algebras and operations between them in 3.3.4–3.3.5, define an induction functor from Lie* modules to chiral modules in 3.3.6 (for a more general procedure see 3.7.15), consider various commutativity properties for chiral modules and their sections in 3.3.7–3.3.8, identify modules over commutative $\mathcal{D}_X$-algebras with *central* modules over the corresponding commutative chiral algebras in 3.3.9, consider families of chiral algebras in 3.3.10, explain why coisson algebras are "classic limits" of chiral algebras in 3.3.11, consider the filtered setting in 3.3.12, and homotopy chiral algebras in 3.3.13. Chiral algebras in the setting of $(\mathfrak{g}, K)$-modules are mentioned in 3.3.14.

Chiral operations between modules over a chiral algebra (see 3.3.4) are closely related to the notion of fusion product of A-modules (cf. [**KL**] and [**HL1**]); due to the lack of our understanding, the latter subject will not be discussed in the book.

Below, we deal either with plain $\mathcal{D}_X$-modules or tacitly assume to be in the DG super setting.

3.3.1. For a k-operad $\mathcal{B}$ a $\mathcal{B}^{ch}$ *algebra* on X is a $\mathcal{B}$ algebra in the pseudo-tensor category $\mathcal{M}(X)^{ch}$. The category $\mathcal{B}(\mathcal{M}(X)^{ch})$ of $\mathcal{B}^{ch}$ algebras will also be denoted by $\mathcal{B}^{ch}(X)$. The canonical pseudo-tensor functors α, β from 3.2 yield the functors between the categories of $\mathcal{B}$ algebras

$$\mathcal{B}(\mathcal{M}(X)^! \otimes \mathcal{L}ie) \xrightarrow{\alpha^{\mathcal{B}}} \mathcal{B}^{ch}(X) \xrightarrow{\beta^{\mathcal{B}}} \mathcal{B}^*(X). \tag{3.3.1.1}$$

Our prime objects of interest are Lie^{ch} algebras. According to 1.1.10, one has $\mathcal{L}ie(\mathcal{M}(X)^! \otimes \mathcal{L}ie) = \mathcal{C}om^!(X) := \mathcal{C}om(\mathcal{M}(X)^!)$, so we can rewrite (3.3.1.1) as

$$\mathcal{C}om^!(X) \xrightarrow{\alpha^{\mathcal{L}ie}} \mathcal{L}ie^{ch}(X) \xrightarrow{\beta^{\mathcal{L}ie}} \mathcal{L}ie^*(X). \tag{3.3.1.2}$$

Note that $\beta^{\mathcal{L}ie}$ is a faithful functor, and $\alpha^{\mathcal{L}ie}$ is a fully faithful embedding (since α is a faithful pseudo-tensor functor).

3.3.2. For a Lie^{ch} algebra A we denote by $\mu = \mu_A \in P_2^{ch}(\{A, A\}, A)$ the commutator and by $[\ ] = [\ ]_A \in P_2^*(\{A, A\}, A)$ the $*$ Lie bracket $\beta(\mu)$. We usually call μ the *chiral product*.

Consider the sheaf of Lie algebras $h(A) := h(\beta^{\mathcal{L}ie}(A))$ (see 2.5.3). It acts on the $\mathcal{D}$-module A (the adjoint action) by derivations of μ_A (see the end of 1.2.18).

A Lie^{ch} algebra A is said to be *commutative* if $[\ ]_A = 0$. Denote by $\mathcal{L}ie^{ch}_{com}(X) \subset \mathcal{L}ie^{ch}(X)$ the full subcategory of commutative Lie^{ch} algebras. It follows from 3.2.3 that $\alpha^{\mathcal{L}ie}(\mathcal{C}om^!(X)) = \mathcal{L}ie^{ch}_{com}(X)$; i.e., we have the equivalence of categories

$$\alpha^{\mathcal{L}ie} \colon \mathcal{C}om^!(X) \xrightarrow{\sim} \mathcal{L}ie^{ch}_{com}(X). \tag{3.3.2.1}$$

3.3.3. Let A be a Lie^{ch} algebra. A *unit* in A is a morphism of $\mathcal{D}$-modules $1 = 1_A \colon \omega_X \to A$ (i.e., 1 is a horizontal section of A^ℓ) such that $\mu_A(1, id_A) \in P_2^{ch}(\{\omega_X, A\}, A)$ is the unit operation ε_A (see 3.1.12). A unit in A is unique if it exists.

A Lie^{ch} algebra with unit is called a *unital* $\textit{Lie}^{ch}$ algebra or simply a *chiral algebra*. We will also refer to Lie^{ch} algebras as chiral algebras without unit, or *non-unital* chiral algebras. Let $\mathcal{CA}(X) \subset \mathcal{L}ie^{ch}(X)$ be the subcategory of chiral algebras and morphisms preserving units; the category $\mathcal{L}ie^{ch}(X)$ will also be denoted by $\mathcal{CA}_{nu}(X)$. One has an obvious "adding of unit" functor[6] $\mathcal{CA}_{nu}(X) \to \mathcal{CA}(X)$, $\quad B \mapsto B^+ := B \oplus \omega_X$, left adjoint to the embedding $\mathcal{CA}(X) \hookrightarrow \mathcal{CA}_{nu}(X)$.

In particular, we have the *unit* chiral algebra $\omega = \omega_X$; for a chiral algebra A the structure morphism of chiral algebras $1_A : \omega \to A$ is called the *unit* morphism.

An *ideal* $I \subset A$ means Lie^{ch} ideal; for an ideal I the quotient A/I is a chiral algebra in the evident way.

Chiral algebras form a sheaf of categories $\mathcal{CA}(X_{\text{ét}})$ on the étale topology of X.

For a chiral algebra A we denote the corresponding Lie* algebra $\beta^{\mathcal{L}ie}(A)$ by A^{Lie}, so A^{Lie} is A considered as a Lie* algebra with bracket $[\ ]_A$.

For a Lie* algebra L an L-*action* on A is a Lie* action of L on the $\mathcal{D}$-module A such that μ_A is L-compatible (see 3.1.14(i)).[7] E.g., we have the adjoint A^{Lie}-action on A.

Let $\mathcal{CA}(X)_{com} \subset \mathcal{CA}(X)$ be the full subcategory of commutative chiral algebras. The above equivalence (3.3.2.1) identifies the corresponding subcategories of unital algebras, so we have a canonical equivalence (see 2.3.1 for notation)

$$\mathcal{C}omu_{\mathcal{D}}(X) = \mathcal{C}omu^!(X) \xrightarrow{\sim} \mathcal{CA}(X)_{com}. \tag{3.3.3.1}$$

Every chiral algebra A admits the maximal commutative quotient (which is the quotient of A modulo the ideal generated by the image of $[\ ]_A$).

Exercise. Suppose A is generated as a chiral algebra by a set of sections $\{a_i\}$ that mutually commute (i.e., $\mu_A(a_i \boxtimes a_j) = 0$). Then A is commutative.

3.3.4. For a chiral algebra A an A-*module* (or *chiral* A-*module*) always means a *unital* A-module, i.e., a Lie^{ch} algebra A-module M such that $\mu_M(1_A, id_M) \in P_2^{ch}(\{\omega_X, M\}, M)$ is the unit operation ε_M (see 3.1.12); here $\mu_M \in P_2^{ch}(\{A, M\}, M)$ is the A-action on M. We denote the category of A-modules by $\mathcal{M}(X, A)$.

Remark. Lie^{ch} algebra A-modules are referred to as *non-unital* (chiral) A-modules. They make sense for any non-unital chiral algebra A and coincide with A^+-modules.

By 1.2.18, $\mathcal{M}(X, A)$ carries an abelian augmented pseudo-tensor structure; we denote it by $\mathcal{M}(X, A)^{ch}$. For $M_i, N \in \mathcal{M}(X, A)$ the corresponding vector space of *chiral* A-*operations* is denoted by $P_{AI}^{ch}(\{M_i\}, N) \subset P_I^{ch}(\{M_i\}, N)$. The augmentation functor on $\mathcal{M}(X, A)^{ch}$ is $M \mapsto h^A(M) := h(M)^{h(A)}$.

Exercise. A chiral operation $\varphi \in P_I^{ch}(\{M_i\}, N)$ belongs to $P_{AI}^{ch}(\{M_i\}, N)$ if and only if it satisfies the following condition: For every $m \in \boxtimes M_i$, $a \in A$, and $i \in I$ there exists $n = n(a, m, i) \gg 0$ such that $\mu_N(id_A, \varphi)(fa \boxtimes m) = \varphi(\mu_{M_i}, \{id_{M_{i'}}\}_{i' \neq i})(fa \boxtimes m)$ for every function f on $X \times X^I$ which vanishes of order $\geq n$ at every diagonal $x = x_{i'}$, $i' \neq i$, and may have poles at any other diagonal.

Example. For the unit chiral algebra ω the forgetful functor $\mathcal{M}(X, \omega) \to \mathcal{M}(X)$ is an equivalence of augmented pseudo-tensor categories (see 3.1.13(ii)).

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[6] The Jacobi identity for B^+ follows from 3.1.13(ii).

[7] The unit is automatically fixed by any L-action (as well as any horizontal section of A^ℓ).

REMARKS. (i) If N is a left $\mathcal{D}_X$-module, M a chiral A-module, then $N \otimes M$ is a chiral A-module. The tensor category $\mathcal{M}^\ell(X)$ acts on $\mathcal{M}(X, A)^{ch}$ (see 1.1.6(v)).

(ii) A-modules localize in the obvious manner, so we have a sheaf of augmented pseudo-tensor categories $\mathcal{M}(X_{\acute{e}t}, A)^{ch}$.

(iii) Let $a : P \hookrightarrow A$ be a sub-$\mathcal{D}$-module which generates A as a chiral algebra. Then for M_i, N as above an operation $\varphi \in P_I^{ch}(\{M_i\}, N)$ is a chiral A-operation if and only if the operations $\mu_N(a, \varphi), \sum_{i\in I} \varphi(id_{M_{j\neq i}}, \mu_{M_i}(a, id_{M_i})) \in P_{\tilde{I}}^{ch}(\{P, M_i\}, N)$ coincide.

3.3.5. REMARKS. (i) An A-module structure on a $\mathcal{D}$-module M amounts to a structure of a chiral algebra on $A \oplus M$ such that $A \hookrightarrow A \oplus M$ and $A \oplus M \twoheadrightarrow A$ are morphisms of chiral algebras and the restriction of the chiral product to $M \times M$ vanishes.

(ii) Due to 3.1.15 and 1.2.13, the notion of an A-module structure (= A-action) makes sense not only for $\mathcal{D}_X$-modules, but also for $\mathcal{O}_X$-modules. The corresponding category of unital A-modules is denoted by $\mathcal{M}_{\mathcal{O}}(X, A)$. The induction and restriction yield adjoint faithful functors

(3.3.5.1) $$\mathcal{M}_{\mathcal{O}}(X, A) \rightleftarrows \mathcal{M}(X, A).$$

EXERCISE. An object of $\mathcal{M}(X, A)$ is the same as an $\mathcal{O}$-module M equipped with an A-action and a right $\mathcal{D}$-module structure such that the A-action on M^ℓ is horizontal.

3.3.6. Any A-module is automatically (via the functor β) an A^{Lie}-module. The functor

(3.3.6.1) $$\beta_A \colon \mathcal{M}(X, A) \to \mathcal{M}(X, A^{Lie})$$

extends to a pseudo-tensor functor $\beta_A = \beta_A^{ch} \colon \mathcal{M}(X, A)^{ch} \to \mathcal{M}(X, A^{Lie})^{ch}$. So for any A-module M the Lie algebra $h(A)$ acts canonically on M (considered as a plain $\mathcal{D}$-module), and chiral A-operations satisfy the Leibniz rule with respect to this action.

PROPOSITION. *The functor β_A admits a left adjoint $Ind_A : \mathcal{M}(X, A^{Lie}) \to \mathcal{M}(X, A)$ which extends to a left pseudo-tensor adjoint $Ind_A = Ind_A^{ch}$ of β_A^{ch}.*

Proof. An explicit construction of $\mathrm{Ind}_A N$ for an A^{Lie}-module N takes 2 steps:

(a) First we construct a triple $(\mathcal{I}, \alpha, \mu)$ where $\mathcal{I}$ is an A^{Lie}-module, $\alpha : N \to \mathcal{I}$ a morphism of A^{Lie}-modules, and $\mu \in P_2^{ch}(\{A, \mathcal{I}\}, \mathcal{I})$ a chiral pairing compatible with A^{Lie}-actions such that the corresponding $*$ pairing equals the A^{Lie}-action on $\mathcal{I}$. Our $(\mathcal{I}, \alpha, \mu)$ is a universal triple with these properties.

Notice that $[A^{tors}, N^{tors}]$, where $A^{tors} \subset A$, $N^{tors} \subset N$ are the $\mathcal{O}_X$-torsion submodules, is an A^{Lie}-submodule of N. Denote by N^v the quotient module. The A^{Lie}-action map $A \boxtimes N^v \to \Delta_* N^v$ kills all sections supported at the diagonal, so the push-forward of the exact sequence $A \boxtimes N^v \to j_* j^*(A \boxtimes N^v) \to \Delta_* A \otimes N^v \to 0$ by the above action map is well defined. Let $I(N)$ be its pull-back to the diagonal; it is an extension of $A \otimes N^v$ by N^v. By construction, one has a morphism $\alpha : N \to I(N)$ and a chiral operation $\mu \in P_2^{ch}(\{A, N\}, I(N))$. It is easy to see that $I(N)$ admits a unique structure of the A^{Lie}-module such that i and μ are compatible with A^{Lie}-actions. The $*$ operation that corresponds to μ is α composed with the A^{Lie}-action map.

Iterating this construction, we get a sequence of morphisms $I^n(N) \to I^{n+1}(N)$ of A^{Lie}-modules, $I^0(N) = N$. Our $\mathcal{I}$ is its inductive limit. The above α and μ define $\alpha : N \to \mathcal{I}(N)$ and $\mu \in P_2^{ch}(\{A, \mathcal{I}(N)\}, \mathcal{I}(N))$. This $(\mathcal{I}, \alpha, \mu)$ is the promised universal triple.

(b) The operation μ usually does not satisfy the properties of an A-action. However $\mathcal{I}$ admits a maximal quotient on which μ defines a structure of the unital A-module. This is $\mathrm{Ind}_A N$.

We leave it to the reader to check the adjunction property: for every $N_i \in \mathcal{M}(X, A^{Lie})$, $M \in \mathcal{M}(X, A)$ the maps $P_{AI}^{ch}(\{\mathrm{Ind}_A N_i\}, M) \to P_{A^{Lie}I}^{ch}(\{N_i\}, M)$, $\phi \mapsto \phi(\{i_{N_i}\})$, are isomorphisms. □

REMARK. For any A-module M and its A^{Lie}-submodule N the image I of the chiral operation $(\mu_M)|_N \in P_2^{ch}(\{A, N\}, M)$ is an A-submodule of M; in particular, in the notation of the lemma above, $\mathrm{Ind}_A N$ is a quotient of $I(N)$ already. This can be seen from the following trick. Let $F_1 \subset F_2 \subset j_*^{(3)} j^{(3)*} \mathcal{O}_{X^3}$ be the subalgebras of functions having poles at the diagonals $x_1 = x_2$ and $x_1 = x_2$, $x_2 = x_3$, respectively. Since every function on the diagonal $x_2 = x_3$ with pole at $x_1 = x_2$ is a restriction of some function from F_1, we see that the image of $(\mu_M)|_I \in P_2^{ch}(\{A, I\}, M)$, i.e., the image of $\mu_M(id_A, (\mu_M)|_N) \in P_3^{ch}(\{A, A, N\}, M)$, coincides already with the $\mu_M(id_A, \mu_M)$-image of the F_2-submodule generated by $A \boxtimes A \boxtimes N$ in $j_*^{(3)} j^{(3)^*}(A \boxtimes A \boxtimes N)$. By the Jacobi identity and the condition on N, the latter image equals I.

3.3.7. Let M be an A-module and $m \in \Gamma(X, M)$ a section.

We say that $a \in A$ *kills* m if $\mu_M(fa \boxtimes m) \in \Delta_* M$ vanishes for every $f \in j_* j^* \mathcal{O}_{X \times X}$. Such a form a subsheaf $\mathcal{A}nn(m) \subset A$ called the *annihilator* of m.

The *centralizer* $\mathcal{C}ent(m)$ of m consists of sections $a \in A$ that commute with m, i.e., $\mu_M(a \boxtimes m) = 0$. This is a chiral subalgebra of A. We say that m is an *A-central* section of M if $\mathcal{C}ent(m) = A$.

Both $\mathcal{A}nn(m)$ and $\mathcal{C}ent(m)$ have étale local nature.

EXERCISES. (i) If the $\mathcal{D}$-submodule of M generated by m is lisse, then m is A-central (see 2.2.4(ii)). In particular, every horizontal section of M is A-central.

(ii) The map $\varphi \mapsto \varphi(1)$ identifies $\mathrm{Hom}_{\mathcal{M}(X,A)}(A, M)$ with the vector space of all horizontal sections of M^ℓ.[8]

(iii) The maximal constant $\mathcal{D}_X$-submodule of A (see 2.1.12) is a commutative chiral subalgebra of A which centralizes every A-module.

(iv) If $\mathcal{A}nn(m)$ is a quasi-coherent $\mathcal{O}_X$-module, then it is automatically an ideal in A.[9] The quasi-coherence always holds if M is $\mathcal{O}_X$-flat (since the flatness implies

[8] Let m be a horizontal section of M^ℓ. By (i), $\mu_M(\cdot, m)$ vanishes on $A \boxtimes \omega_X$, so it yields a morphism of $\mathcal{D}$-modules $\varphi_m : A \to M$. The Jacobi identity (restricted to sections of $j_*^{(3)} j^{(3)*} A \boxtimes A \boxtimes M$ which have no pole at the diagonal $x_1 = x_3$) shows that φ_m is a morphism of A-modules. One checks immediately that $m \mapsto \varphi_m$ is the inverse map to that of (ii).

[9] Sketch of a proof (cf. Remark in 3.3.6). Fix some $n \geq 0$. Notice that every function on the diagonal $x_1 = x_2$ in $X \times X \times X$ with possible pole at the diagonal X is the restriction of a function on $X \times X \times X$ with possible pole at $x_2 = x_3$ and zero of order $\geq n$ at $x_1 = x_3$. Now take any $a \in A$ and $b \in \mathcal{A}nn(m)$. We want to show that the μ_A-image P of $j_* \mathcal{O}_U \cdot a \boxtimes b \subset j_* j^* A \boxtimes A$ lies in $\mathcal{A}nn(m)$. The above remark implies that the $\mathcal{D}$-module generated by $\mu_A(j_* \mathcal{O}_U \cdot P \boxtimes m)$ coincides with the $\mathcal{D}$-module generated by the $\mu_M(\mu_A, id_M)$-images of all elements $f \cdot a \boxtimes b \boxtimes m$ where the function f can have arbitrary poles at the diagonals $x_1 = x_2$ and $x_2 = x_3$ and zero of order $\geq n$ at $x_1 = x_3$. Choose n such that $\mu_M(a \boxtimes m) \in \Delta_* M$ is killed by the nth power of the equation of the diagonal, and apply the Jacobi identity.

that a section of A belongs to $\mathcal{A}nn(m)$ iff its restriction to some Zariski open is). If $m \neq 0$ vanishes on a Zariski open subset, then $\mathcal{A}nn(m)$ is *not* $\mathcal{O}_X$-quasi-coherent.

The *annihilator* $\mathcal{A}nn(M) = \mathcal{A}nn(A, M)$ of M is the subsheaf of A whose sections over an affine open U are those a that kill every section m of M over U. Similarly, the *centralizer* $\mathcal{C}ent(M) = \mathcal{C}ent(A, M)$ of M is the subsheaf of A whose sections over U as above are those a that commute with every m. Both these subsheaves are $\mathcal{O}_X$-submodules of A. If we know that $\mathcal{A}nn(M)$ (resp. $\mathcal{C}ent(M)$) is a quasi-coherent $\mathcal{O}_X$-module, then it is an ideal (resp. a chiral subalgebra) in A. This always happens if M is $\mathcal{O}_X$-flat.

EXERCISE. If $\{m_\alpha\}$ are A-generators of M, then $\mathcal{A}nn(M) = \bigcap \mathcal{A}nn(m_\alpha)$.

We say that M is *central* (or the A-action on M is central) if $\mathcal{C}ent(M) = A$, or, equivalently, if every $m \in M$ is A-central.

The centralizer of A (considered as an A-module) is called the *center of* A; we denote it by $Z(A)$. If $Z(A)$ is $\mathcal{O}_X$-quasi-coherent (see above), then this is a commutative subalgebra of A.

We denote by $\mathcal{M}(X, A)^{ch}_{cent} \subset \mathcal{M}(X, A)^{ch}$ the full pseudo-tensor subcategory of central modules. The subcategory $\mathcal{M}(X, A)_{cent} \subset \mathcal{M}(X, A)$ is closed under direct sums and subquotients. Any A-module M has maximal central sub- and quotient modules which we denote, respectively, by M^{cent} and M_{cent}. Notice that M_{cent} is the quotient of M modulo the A-submodule generated by the image of the $*$ action of A^{Lie} on M. So the functor $M \mapsto M_{cent}$ is compatible with the étale localization.

Every chiral algebra A admits the maximal commutative quotient algbera A_{com}. As an A-module it equals the maximal central A-module quotient of A.

3.3.8. LEMMA. *The obvious functor* $\mathcal{M}(X, A_{com})^{ch}_{cent} \to \mathcal{M}(X, A)^{ch}_{cent}$ *is an equivalence of pseudo-tensor categories.*

Proof. Let us prove that the action of A on any central A-module M factors through A_{com}. Let $a, a' \in A$ and $m \in M$ be any sections, and $f(x_1, x_2, x_3)$ any function on $X \times X \times X$ which may have poles at the diagonal $x_1 = x_3$ and is regular elsewhere. We want to show that the operation $\mu_M(\mu_A, id_M) \in P^{ch}_3(\{A, A, M\}, M)$ vanishes on $f \cdot a_1 \boxtimes a_2 \boxtimes m$, which is clear by the Jacobi identity. □

3.3.9. Let R be a commutative chiral algebra, so R^ℓ is a commutative $\mathcal{D}_X$-algebra. The functor α identifies R^ℓ-actions on a $\mathcal{D}_X$-module M with *central* chiral R-actions. One gets a canonical equivalence of categories

$$\mathcal{M}(X, R^\ell) \xrightarrow{\sim} \mathcal{M}(X, R)_{cent}. \tag{3.3.9.1}$$

We leave it to the reader to check that a chiral operation φ between $R^\ell[\mathcal{D}_X]$-modules is R^ℓ-polylinear (see 3.1.14(ii)) if and only if it satisfies the Leibnitz rule with respect to the chiral actions of R on these modules. So we have a canonical equivalence of pseudo-tensor categories

$$\mathcal{M}(X, R^\ell)^{ch} \xrightarrow{\sim} \mathcal{M}(X, R)^{ch}_{cent}. \tag{3.3.9.2}$$

REMARK. The pseudo-tensor category $\mathcal{M}(X, R^\ell)^{ch}$, and hence the category of non-unital chiral R-algebras, makes sense for a *non-unital* commutative $\mathcal{D}_X$-algebra R^ℓ (use 3.1.3).

3.3.10. Everything said in this section remains valid if we replace $\mathcal{M}(X)^{ch}$ by $\mathcal{M}(X, R^\ell)^{ch}$. We leave it to the reader to check that (3.3.9.2) identifies chiral algebras in $\mathcal{M}(X, R^\ell)$ with pairs (A, i) where A is a chiral algebra on X and $i: R \to A$ is a morphism of chiral algebras such that A is a central R-module; we call such a pair (A, i) a *chiral R-algebra*. Denote the corresponding category by $\mathcal{CA}(X, R^\ell)$. An A-module in $\mathcal{M}(X, R^\ell)^{ch}$ is the same as an A-module M such that the R-action on M (via the structure morphism i) is central; such objects are called *R-central A-modules* or *A-modules on* Spec R^ℓ.

Example. Let A be a chiral algebra and L a Lie* algebra that acts on A (see 3.3.3). Suppose that L is a vector $\mathcal{D}_X$-bundle, so we have the Chevalley topological DG $\mathcal{D}_X$-algebra $\mathcal{C}(L) = \varprojlim \mathcal{C}(L)/\mathcal{C}^{\geq i}(L)$ (see 1.4.10). Then, by Exercise (i) in 3.1. 14, $\mathcal{C}(L, A)$ is a topological DG chiral $\mathcal{C}(L)$-algebra. It is a $\mathcal{C}(L)$-deformation of $A = \mathcal{C}(L, A)/\mathcal{C}^{\geq 1}(L, A)$. If A is commutative, then $\mathcal{C}(L, A)$ is commutative, and the above construction coincides with the one from 1.4.10.

Recall that $R^\ell[\mathcal{D}_X]$-modules have the local nature with respect to the étale topology of Spec R^ℓ (see 2.3.5). R^ℓ-polylinear chiral operations also have the local nature. So for every algebraic $\mathcal{D}_X$-space $\mathcal{Y}$ we have the pseudo-tensor category $\mathcal{M}(\mathcal{Y})^{ch}$ of $\mathcal{O}_{\mathcal{Y}}[\mathcal{D}_X]$-modules (see 2.3.5), the category $\mathcal{CA}(\mathcal{Y})$ of chiral $\mathcal{O}_{\mathcal{Y}}$-algebras on $\mathcal{Y}$, and for $\mathcal{A} \in \mathcal{CA}(\mathcal{Y})$ the pseudo-tensor category $\mathcal{M}(\mathcal{Y}, \mathcal{A})^{ch}$ of $\mathcal{A}$-modules on $\mathcal{Y}$, as well as the corresponding sheaves of categories on $\mathcal{Y}_{\text{ét}}$.

For example, if $\mathcal{Y}$ is constant, i.e., $\mathcal{Y} = X \times Y$ with a "trivial" connection, then $\mathcal{O}_{\mathcal{Y}}[\mathcal{D}_X]$-modules are the same as Y-families of $\mathcal{D}$-modules on X. A chiral algebra $\mathcal{A}$ on $\mathcal{Y}$ is the same as Y-families of chiral algebras on X. We also call $\mathcal{A}$ a *chiral* $\mathcal{O}_Y$*-algebra on* X.

By 3.1.14(ii), the above objects behave naturally with respect to morphisms of $\mathcal{Y}$'s or R^ℓ's: for any morphism of commutative $\mathcal{D}_X$-algebras $R^\ell \to F^\ell$ we have a base change functor $\mathcal{CA}(X, R^\ell) \to \mathcal{CA}(X, F^\ell)$, $A \mapsto A \underset{R^\ell}{\otimes} F^\ell$, etc.

3.3.11. Let us explain why coisson algebras are "classical limits" of chiral algebras.

Let A_t be a one-parameter flat family of chiral algebras; i.e., A_t is a chiral $k[t]$-algebra which is flat as a $k[t]$-module (see 3.3.10). Assume that $A := A_{t=0} := A_t/tA_t$ is a commutative chiral algebra. This means that the $*$ bracket $[\ ]_t$ on A_t is divisible by h. Thus $\{\ \}_t := t^{-1}[\ ]_t$ is a Lie* bracket on A_t; the corresponding Lie* algebra acts on the chiral algebra A_t (see 3.3.3) by the adjoint action. Reducing this picture modulo t, we see (as in 3.3.3) that A^ℓ is a commutative $\mathcal{D}_X$-algebra, and $\{\ \} := \{\ \}_{t=0}$ a coisson bracket on it.

One calls A_t the *quantization* of the coisson algebra $(A^\ell, \{\ \})$ with respect to the parameter t.

Remark. The above construction is also a direct consequence of 3.2.5 and 1.4.28. Notice the difference between the usual quantization picture and the chiral quantization. The former exploits the fact that the operad $\mathcal{A}ss$ is a deformation of the operad $\mathcal{P}ois$ (see 1.1.8(ii)), while the tensor structure on the category of vector spaces remains fixed. The latter deals with the fixed operad $\mathcal{L}ie$, and the pseudo-tensor structure on $\mathcal{M}(X)$ is being deformed as in 3.2.

As in the usual Poisson setting, one can consider *quantizations* mod t^{n+1}, $n \geq 0$, of a given coisson algebra $(A, \{\ \})$. Namely, these are triples $(A^{(n)}, \{\ \}^{(n)}, \alpha)$ where

$A^{(n)}$ is a chiral $k[t]/t^{n+1}$-algebra flat as a $k[t]/t^{n+1}$-module, $\{\ \}^{(n)}$ a $k[t]/t^{n+1}$-bilinear Lie* bracket on $A^{(n)}$ such that $t\{\ \}$ equals the Lie* bracket for the chiral algebra structure, and $\alpha : A^{(n)}/tA^{(n)} \xrightarrow{\sim} A$ an isomorphism of chiral algebras which sends $\{\ \}^{(n)} \operatorname{mod} t$ to $\{\ \}$.

Quantizations mod t^{n+1} form a groupoid. For $n \geq 1$ the reduction mod t^n of a quantization mod t^{n+1} is a quantization mod t^n. A compatible sequence of such quantizations $A^{(n)}$, $A^{(n)}/t^{n-1}A^{(n)} = A^{(n-1)}$, is a *formal quantization* of $(A, \{\ \})$.

3.3.12. Let A be a not necessarily unital chiral algebra. A *filtration* on A is an increasing sequence of $\mathcal{D}$-submodules $A_0 \subset A_1 \subset \cdots \subset A$ such that $\bigcup A_i = A$ and $\mu(j_*j^*A_i \boxtimes A_j) \subset \Delta_* A_{i+j}$. Then $\operatorname{gr} A$ is a Lie^{ch} algebra in the obvious way.

If A is unital, then, unless it is stated explicitly otherwise, we assume that our filtration is *unital*; i.e., $1_A \in A_0$. Then $\operatorname{gr}.A$ is a chiral algebra. The filtration is *commutative* if $\operatorname{gr}.A$ is a commutative chiral algebra. Then $\operatorname{gr}.A^\ell$ is a coisson algebra: the coisson bracket $\{\ \} \in P_2^*(\{\operatorname{gr}_i A, \operatorname{gr}_j A\}, A_{i+j-1})$ comes from the Lie* bracket on A.

Consider the Rees algebra $A_t := \oplus A_i$. This is a chiral $k[t]$-algebra[10] which is $k[t]$-flat. One has $A_{t=1} = A$, $A_{t=0} = \operatorname{gr} A$. If our filtration is commutative, then we are in the situation of 3.3.11, and the above coisson bracket equals the one from 3.3.11.

3.3.13. We denote by $\mathcal{H}o\mathcal{C}\mathcal{A}(X)$ the homotopy category of chiral algebras, defined as the localization of the category of DG super chiral algebras with respect to quasi-isomorphisms.[11] One can consider the non-unital setting as well. We will not exploit $\mathcal{H}o\mathcal{C}\mathcal{A}(X)$ to any length.

Notice that the category of DG chiral algebras is *not* a closed model category (e.g., since the coproduct of chiral algebras usually does not exist).

The next remarks show the level of our understanding of $\mathcal{H}o\mathcal{C}\mathcal{A}(X)$:

Remarks. (i) For any homotopy Lie operad Lie$\tilde{}$ (i.e., a DG operad equipped with a quasi-isomorphism Lie$\tilde{}$ $\to$ Lie) one has the corresponding category of Lie$\tilde{}$ algebras in $\mathcal{M}(X)^{ch}$. We do *not* know if the corresponding homotopy category is equivalent to the homotopy category of (non-unital) chiral algebras. In particular, we do not know if chiral DG algebras satisfy the "homotopy descent" property. If the answer is negative, then, probably, the right homotopy theory should be based on different objects.

(ii) A morphism of DG chiral algebras $\phi : A \to B$ yields an evident exact DG functor between the DG categories of chiral DG modules $\mathcal{M}(X, B) \to \mathcal{M}(X, A)$. Suppose that ϕ is a quasi-isomorphism. We do *not* know if the corresponding functor of the derived categories $D\mathcal{M}(X, B) \to D\mathcal{M}(X, A)$ is an equivalence of categories.

3.3.14. The above definitions have immediate counterparts in the setting of $(\mathfrak{g}, K)$-modules (see 3.1.16). The functors (3.1.16.2) transform chiral algebras to chiral algebras. In particular, a chiral algebra in the category of $\operatorname{Aut} k[\![t]\!]$-modules

[10] The chiral product on A_t is compatible with the grading and comes from the chiral product on A; multiplication by t is the sum of embeddings $A_i \hookrightarrow A_{i+1}$.

[11] To see that $\mathcal{H}o\mathcal{C}\mathcal{A}(X)$ is well defined (no set-theoretic difficulties occur), one shows that any diagram representing a morphism $A \to B$ in $\mathcal{H}o\mathcal{C}\mathcal{A}(X)$ can be replaced by a subdiagram all of whose terms (but the last one, B) have cardinality less than or equal to that of A.

amounts to a *universal* chiral algebra, i.e., a chiral algebra in the category of universal $\mathcal{D}$-modules on curves; see 2.9.9.

3.4. Factorization

In this section we give another definition of chiral algebras in terms of $\mathcal{O}$-modules on the space $\mathcal{R}(X)$ of finite non-empty subsets of X which satisfy a certain factorization property for the disjoint union of subsets.

Ran's space $\mathcal{R}(X)$ was, in fact, introduced by Borsuk and Ulam [**BU**]; Ziv Ran [**R1**], [**R2**] was, probably, first to consider it in algebraic geometry. Remarkably enough, it is contractible. This result (in fact, a stronger one) is due to Curtis and Nhu (see Lemmas 3.3 and 3.6 from [**CN**]); simple-connectedness of $\mathcal{R}(X)_3$ was established by Bott [**Bo**] who showed that $\mathcal{R}(S^1)_3 = S^3$. We give a simple algebraic proof in 3.4.1(iv); a similar proof was found indepently by Jacob Mostovoy.

We begin in 3.4.1 with a review of the basic topological properties of $\mathcal{R}(X)$. In 3.4.2–3.4.3 we play with $\mathcal{O}$- and $\mathcal{D}$-modules on $\mathcal{R}(X)$. Notice that $\mathcal{R}(X)$ is *not* an algebraic variety but merely an ind-scheme in a broad sense, which brings some (essentially notational) complications. Factorization algebras are defined in 3.4.4; equivalent definitions are considered in 3.4.5 and 3.4.6. In 3.4.7 we show that, in fact, every factorization algebra is a $\mathcal{D}$-module in a canonical way. We define a functor from the category of factorization algebras to that of chiral algebras in 3.4.8. The theorem in 3.4.9 says that it is an equivalence of categories. In the course of the proof (see 3.4.10–3.4.12) some constructions that will be used later in Chapter 4 are introduced. First we realize $\mathcal{M}(X)^{ch}$ as a full pseudo-tensor subcategory in a certain abelian *tensor* category (of a non-local nature) and the same for $\mathcal{M}(X)^*$ (see 3.4.10). This permits us to define the *Chevalley-Cousin* complex of a chiral algebra; the theorem in 3.4.9 amounts to its acyclicity property (see 3.4.12). Having at hand the identification of chiral and factorization algebras, we consider "free" chiral algebras in 3.4.14 (in the setting of vertex algebras this construction was studied in [**Ro**]), define the tensor product of chiral algebras in 3.4.15, play with chiral Hopf algebras in 3.4.16, explain in 3.4.17 what are twists of chiral algebras, describe chiral modules and chiral operations between them in factorization terms (see 3.4.18–3.4.19), and give a "multijet" geometric construction of the factorization algebra that corresponds to a commutative chiral algebra (see 3.4.22).

3.4.1. A topological digression. This subsection plays an important yet purely motivational role.

Let X be a topological space. Its *exponential* $\mathcal{E}xp(X)$ is the set of all its finite subsets; for $S \subset X$ the corresponding point of $\mathcal{E}xp(X)$ is denoted by $[S]$. For any finite index set I we have an obvious map $r_I : X^I \to \mathcal{E}xp(X)$. We equip $\mathcal{E}xp(X)$ with the strongest topology such that all maps r_I are continuous.[12] Thus $\mathcal{E}xp(X)$ is the disjoint union of the base point $[\emptyset]$ and the subspace $\mathcal{R}(X)$ called *Ran's space* associated to X.

For $[S] \in \mathcal{R}(X)$ let $\mathcal{R}(X)_{[S]} \subset \mathcal{R}(X)$ be the subspace whose points are finite subsets of X that contain S. If X is connected, then so are $\mathcal{R}(X)$ and $\mathcal{R}(X)_{[S]}$.

Here are some properties of $\mathcal{E}xp$ and $\mathcal{R}$:

[12]Therefore two points $[S], [S'] \in \mathcal{E}xp(X)$ are close iff S lies in a small neighborhood of S' and vice versa.

(i) The subspaces $\mathcal{E}xp(X)_n$ that consists of $[S]$ with $|S| \leq n$ form an increasing filtration on $\mathcal{E}xp(X)$; the projection $r_n : X^n \to \mathcal{E}xp(X)_n$ is an open map. The same is true for $\mathcal{E}xp$ replaced by $\mathcal{R}$. One has $\mathcal{R}(X)_0 = \emptyset$, $\mathcal{R}(X)_1 = X$, $\mathcal{R}(X)_2 = \mathrm{Sym}^2 X$. For any n the stratum $\mathcal{R}(X)_n^o := \mathcal{R}(X)_n \smallsetminus \mathcal{R}(X)_{n-1}$ is equal to the complement to the diagonals in $\mathrm{Sym}^n X$; this is the space of configurations of n points in X.

(ii) For any surjection $J \twoheadrightarrow I$ we have the corresponding diagonal embedding $\Delta^{(J/I)} : X^I \to X^J$. One has $r_J \Delta^{(J/I)} = r_I$, and $\mathcal{E}xp(X)$ is the inductive limit of the topological spaces X^I with respect to all diagonal embeddings. The same is true for $\mathcal{R}(X)$ (add the condition $I \neq \emptyset$).

(iii) $\mathcal{E}xp(X)$ is a commutative monoid with respect to the operation $[S] \circ [S'] := [S \cup S']$; the unit element is $[\emptyset]$. The map $\mathcal{E}xp(X_1) \times \mathcal{E}xp(X_2) \to \mathcal{E}xp(X_1 \sqcup X_2)$, $([S_1],[S_2]) \mapsto [S_1] \circ [S_2] = [S_1 \sqcup S_2]$, is a homeomorphism.

It is clear that $\mathcal{R}(X)$ is a subsemigroup of $\mathcal{E}xp(X)$, as well as each subspace $\mathcal{R}(X)_{[S]}$, $[S] \in \mathcal{R}(X)$. Notice that $[S]$ is a unit element in $[\mathcal{R}(X)_{[S]}$.

(iv) Proposition. *Suppose that X is linearly connected. Then all the homotopy groups of $\mathcal{R}(X)$ and $\mathcal{R}(X)_{[S]}$ vanish. So if X is a CW complex, then $\mathcal{R}(X)$ and $\mathcal{R}(X)_{[S]}$ are contractible.*

Proof. Since $\mathcal{R}(X)$ and $\mathcal{R}(X)_{[S]}$ are linearly connected, the proposition follows from (iii) above and the next lemma:

Lemma. *Let $\mathcal{R}$ be a linearly connected topological space which admits an associative and commutative product $\circ : \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ such that for any $v \in \mathcal{R}$ one has $v \circ v = v$. Then all the homotopy groups of $\mathcal{R}$ vanish.*

Proof of Lemma. Suppose for a moment that $\circ$ has a neutral element $e \in \mathcal{R}$ (which happens if $\mathcal{R} = \mathcal{R}(X)_{[S]}$). Then $\mathcal{R}$ is an H-space. The space of all continuous maps $\gamma : (S^i, \cdot) \to (\mathcal{R}, e)$ carries a natural product $\circ$ coming from the product on $\mathcal{R}$. Passing to the homotopy classes of the γ's, we get a product on $\pi_i(\mathcal{R}, e)$. It is well known that this product coincides with the usual product on the homotopy group. Since for any γ one has $\gamma \circ \gamma = \gamma$, we see that $\pi_i(\mathcal{R}, e)$ is trivial; q.e.d.

If we do not assume the existence of a neutral element, the above argument should be modified as follows. Below we write the usual group law on the homotopy groups π_i, $i \geq 1$, multiplicatively.

Let $v \in \mathcal{R}$ be any base point. Consider the maps

$$\mathcal{R} \xrightarrow{\delta} \mathcal{R} \times \mathcal{R} \xrightarrow{\circ} \mathcal{R} \tag{3.4.1.1}$$

where δ is the diagonal embedding; one has $\circ\delta = id_{\mathcal{R}}$. For $\gamma \in \pi_i(\mathcal{R}(X), v)$ one has $\delta(\gamma) = i_1(\gamma) \cdot i_2(\gamma) \in \pi_i(\mathcal{R} \times \mathcal{R}, (v,v))$ where $i_1, i_2 : \mathcal{R} \hookrightarrow \mathcal{R} \times \mathcal{R}$ are the embeddings $r \mapsto (r,v)$, (v,r). Therefore $\gamma = \circ\delta(\gamma) = (v \circ \gamma)^2$. We see that $\gamma = 1$ if $v \circ \gamma = 1$. Notice that $v \circ v \circ \gamma = v \circ \gamma$, so, replacing γ in the previous formula by $v \circ \gamma$, we get $v \circ \gamma = (v \circ \gamma)^2$. Thus $v \circ \gamma = 1$, so $\gamma = 1$, and we are done. □

(v) A *factorization algebra* is a sheaf B of vector spaces on $\mathcal{E}xp(X)$ together with natural identifications $B_{[S_1]} \otimes B_{[S_2]} \xrightarrow{\sim} B_{[S_1] \circ [S_2]}$ defined for *disjoint* S_1, S_2, which are associative and commutative in the obvious manner. Denote by B_X the restriction of B to $X \subset \mathcal{E}xp(X)$. The restriction of B to $\mathcal{E}xp(X)_i^o$ equals to the restriction of $\mathrm{Sym}^i(B_X)$ to $\mathrm{Sym}^i X \smallsetminus$(diagonals). A morphism between factorization algebras is completely determined by its restriction to the B_X's, so one can consider B as a sheaf B_X on X equipped with a certain *factorization structure*. It has the X-local nature, so one can speak of factorization algebras *on* X.

3.4.2. $\mathcal{O}$-modules and left $\mathcal{D}$-modules on $\mathcal{R}(X)$. We want to play with factorization algebras in the algebro-geometric setting of quasi-coherent sheaves. From now on X is our curve. Notice that the $\mathcal{R}(X)_i$ are *not* algebraic varieties for $i \geq 3$:

EXERCISE. Let f be a germ of a holomorphic function on $\mathcal{R}(X)_3$ at a point $x \in X = \mathcal{R}(X)_1 \subset \mathcal{R}(X)_3$; i.e., f is a germ of a holomorphic function on X^3 at (x,x,x) which is symmetric and satisfies the relation $f(y,z,z) = f(y,y,z)$. Then the restriction of f to X is constant.

Nevertheless, we will consider $\mathcal{O}$- and $\mathcal{D}$-modules on $\mathcal{R}(X)$; they will be introduced directly using 3.4.1(ii).

An $\mathcal{O}$-*module on* $\mathcal{R}(X)$, or an $\mathcal{O}_{\mathcal{R}(X)}$-*module*, is a rule F that assigns to every finite non-empty I a quasi-coherent $\mathcal{O}_{X^I}$-module F_{X^I} and to every $\pi : J \twoheadrightarrow I$ an identification

$$\nu^{(\pi)} = \nu_F^{(\pi)} = \nu^{(J/I)} : \Delta^{(\pi)*} F_{X^J} \xrightarrow{\sim} F_{X^I} \tag{3.4.2.1}$$

compatible with the composition of the π's. We demand that the F_{X^I} have no non-zero local sections supported at the diagonal divisor.

$\mathcal{O}_{\mathcal{R}(X)}$-modules form a k-category $\mathcal{M}_{\mathcal{O}}(\mathcal{R}(X))$. This category is exact in the sense of [**Q3**].[13] It is also a tensor k-category in the obvious way. The unit object is $\mathcal{O}_{\mathcal{R}(X)}$ (:= the $\mathcal{O}$-module with components $\mathcal{O}_{X^I}$).

Replacing $\mathcal{O}$-modules in the above definition by left $\mathcal{D}$-modules, we get the notion of the left $\mathcal{D}$-module on $\mathcal{R}(X)$. These objects form an exact tensor k-category $\mathcal{M}^\ell(\mathcal{R}(X))$ with unit object $\mathcal{O}_{\mathcal{R}(X)}$.

3.4.3. Let F be an $\mathcal{O}_{\mathcal{R}(X)}$-module. Fix some $J \in \mathcal{S}$; let $\ell : V \hookrightarrow X^J$ be the complement to the diagonal strata of codimension ≥ 2.

LEMMA. (i) *The $\mathcal{O}_{X^J}$-module F_{X^J} is flat along every diagonal $X^I \hookrightarrow X^J$; i.e., $\mathcal{T}or_i(F_{X^J}, \mathcal{O}_{X^I}) = 0$ for $i > 0$.*

(ii) *The morphism $F_{X^J} \to \ell_* \ell^* F_{X^J}$ is an isomorphism.*[14]

Proof. (i) Our statement amounts to the claim that for every $\pi : J \twoheadrightarrow I$ the projection $L\Delta^{(\pi)*} F_{X^J} \to \Delta^{(\pi)*} F_{X^J}$ is an isomorphism. Let $\tilde{\nu}^{(\pi)} : L\Delta^{(\pi)*} F_{X^J} \to F_{X^I}$ be its composition with the structure isomorphism $\nu^{(\pi)}$. Our $\tilde{\nu}^{(\pi)}$'s are compatible with the composition of the π's, so (since every diagonal embedding is a composition of diagonal embeddings of codimension 1) it suffices to treat the codimension 1 case. Here our statement amounts to the property that F_{X^J} has no sections supported on X^I, and we are done.

(ii) We know that our map is injective. For every diagonal $X^I \subset X^J$ of codimension ≥ 2 one has $\mathcal{E}xt^1(\mathcal{O}_{X^I}, F_{X^J}) = 0$ by (i) (indeed, $\mathcal{E}xt^j(\mathcal{O}_{X^I}, F_{X^J})$ is locally isomorphic to $\mathcal{T}or_{|J|-|I|-j}(\mathcal{O}_{X^I}, F_{X^J})$[15]). This implies the surjectivity of our map. Namely, take the smallest m such that $C := \operatorname{Coker}(F_{X^J} \to \ell_* \ell^* F_{X^J})$ vanishes outside of diagonals of dimension $\leq m$. Then for a diagonal $X^I \subset X^J$ of dimension m our C is non-zero on $X^I \smallsetminus$ (diagonals); hence $\mathcal{H}om(\mathcal{O}_{X^I}, C) \neq 0$. The latter sheaf equals $\mathcal{E}xt^1(\mathcal{O}_{X^I}, F_{X^J})$, and we are done. □

[13] A short sequence is exact if and only if all the corresponding sequences of $\mathcal{O}_{X^n}$-modules are exact. Probably, $\mathcal{M}_{\mathcal{O}}(\mathcal{R}(X))$ is not an abelian category.

[14] Here ℓ_* is the naive sheaf-theoretic direct image (no higher derived functors are considered).

[15] Compute them using a Koszul resolution of $\mathcal{O}_{X^I}$.

EXERCISE. F is flat (as an object of the exact tensor category $\mathcal{M}_{\mathcal{O}}(\mathcal{R}(X))$) if and only if every F_{X^I} is a flat $\mathcal{O}_{X^I}$-module.

We say that a complex of $\mathcal{O}_{\mathcal{R}(X)}$-modules is *homotopically flat* if each of the complexes of $\mathcal{O}_{X^I}$-modules is homotopically $\mathcal{O}_{X^I}$-flat (see 2.1.1).

3.4.4. Factorization algebras. For $\pi : J \twoheadrightarrow I$ let $j^{[J/I]} : U^{[J/I]} \hookrightarrow X^J$ be the complement to all the diagonals that are transversal to $\Delta^{(J/I)} : X^I \hookrightarrow X^J$. Therefore one has $U^{[J/I]} = \{(x_j) \in X^J : x_{j_1} \neq x_{j_2} \text{ if } \pi(j_1) \neq \pi(j_2)\}$.

Let B be an $\mathcal{O}$-module on $\mathcal{R}(X)$. A *factorization structure* on B is a rule that assigns to every $J \twoheadrightarrow I$ an isomorphism of $\mathcal{O}_{U^{[J/I]}}$-modules

$$c_{[J/I]} : j^{[J/I]*}(\underset{I}{\boxtimes} B_{X^{J_i}}) \xrightarrow{\sim} j^{[J/I]*} B_{X^J}. \tag{3.4.4.1}$$

We demand that the c's are mutually compatible: for $K \twoheadrightarrow J$ the isomorphism $c_{[K/J]}$ coincides with the composition $c_{[K/I]}(\boxtimes c_{[K_i/J_i]})$. And c should be compatible with ν: for every $J \twoheadrightarrow J' \twoheadrightarrow I$ one has $\nu^{(J/J')}\Delta^{(J/J')*}(c_{[J/I]}) = c_{[J'/I]}(\boxtimes \nu^{(J_i/J'_i)})$.

Notice that $c_{[J/J]}$ identifies $j^{(J)*}B_{X^J}$ with $j^{(J)*}B_X^{\boxtimes J}$, so we have a canonical embedding

$$B_{X^J} \hookrightarrow j_*^{(J)} j^{(J)*} B_X^{\boxtimes J}. \tag{3.4.4.2}$$

$\mathcal{O}$-modules on $\mathcal{R}(X)$ equipped with the factorization structure form a tensor category in the obvious manner. This category is *not* additive.

As follows from (3.4.4.2), the functor $B \mapsto B_X$ is faithful. Therefore we can consider an $\mathcal{O}$-module B on $\mathcal{R}(X)$ equipped with a factorization structure as an $\mathcal{O}$-module B_X on X equipped with a certain structure which we also call a *factorization structure*. Notice that the factorization structure has the X-local nature, so $\mathcal{O}$-modules with the factorization structure form a sheaf of tensor categories on the étale topology of X.

In the above discussion one may replace $\mathcal{O}$-modules everywhere by left $\mathcal{D}$-modules. It also renders to the super or DG super setting in the obvious way.

REMARKS. (i) If an $\mathcal{O}_{\mathcal{R}(X)}$-module B admits a factorization structure, then B is flat if (and only if) B_X is a flat $\mathcal{O}_X$-module.[16] Similarly, if we play with complexes of $\mathcal{O}$-modules, then B is homotopically flat if (and only if) B_X is homotopically $\mathcal{O}_X$-flat.

(ii) Let $\varphi : B \to B'$ be a morphism of DG $\mathcal{O}_{\mathcal{R}(X)}$-modules equipped with a factorization structure such that $\varphi_X : B_X \to B'_X$ is a quasi-isomorphism. Then every $\varphi_{X^I} : B_{X^I} \to B'_{X^I}$ is a quasi-isomorphism.

DEFINITION. Let B be an $\mathcal{O}$-module equipped with a factorization structure. We say that the factorization structure is *unital* or B is a *factorization algebra* if there exists a global section $1 = 1_B$ of B_X such that for every $f \in B_X$ one has $1 \boxtimes f \in B_{X^2} \subset j_* j^* B_X^{\boxtimes 2}$ and $\Delta^*(1 \boxtimes f) = f$.

Notice that such a section 1 is uniquely defined; it is called the *unit* of B.

[16]Assume that B_X is flat. We want to show that every B_{X^n} is flat. It amounts to the vanishing of $Tor_{>0}(B_{X^n}, K)$ where $\operatorname{Spec} K$ is the generic point of any irreducible subscheme of X^n. Using induction by n and 3.4.3(i) we can assume that $\operatorname{Spec} K \subset U^{(n)}$. Now $B_{U^{(n)}}$ is flat by factorization, and we are done.

Factorization algebras form a tensor category (we demand that the morphisms preserve units) denoted by $\mathcal{FA}(X)$. They are objects of the X-local nature, so we have a sheaf of categories $\mathcal{FA}(X_{\text{ét}})$ on the étale topology of X.

The unit object of our tensor category is the "trivial" factorization algebra $\mathcal{O}$. For any $B \in \mathcal{FA}(X)$ there is a unique morphism $\mathcal{O} \to B$. Therefore for a collection B_α of factorization algebras and $B := \otimes B_\alpha$ one has canonical morphisms $\nu_\alpha : B_\alpha \to B$ (defined as tensor product of id_{B_α} and the canonical morphisms $\mathcal{O} \to B_{\alpha'}$, $\alpha' \neq \alpha$).

Remark. In the category $\mathcal{FA}(X)$ infinite tensor products are well defined.[17] Namely, if B_α is any collection of factorization algebras then $\otimes B_\alpha$ is the inductive limit of $\underset{\alpha \in S}{\otimes} B_\alpha$ where S runs the set of finite subsets of indices α. Here for $S \subset S'$ the corresponding arrow $\underset{\alpha\in S}{\otimes} B_\alpha \to \underset{\alpha\in S'}{\otimes} B_\alpha$ is $\otimes b_\alpha \mapsto (\otimes b_\alpha) \otimes (\underset{\alpha' \in S' \smallsetminus S}{\otimes} 1'_\alpha)$.

3.4.5. Factorization algebras can be viewed from a slightly different perspective. Consider the category $\hat{\mathcal{S}}$ of all finite sets and arbitrary maps. For any morphism $\pi : J \to I$ in $\hat{\mathcal{S}}$ we have the corresponding map $\Delta^{(\pi)} : X^I \to X^J$, $(x_i) \mapsto (x_{\pi(j)})$. We also have the open subset $j^{[\pi]} : U^{[\pi]} := \{(x_j) \in X^J : x_{j_1} \neq x_{j_2} \text{ if } \pi(j_1) \neq \pi(j_2)\} \hookrightarrow X^J$. We write $\Delta^{(J/I)} := \Delta^{(\pi)}$, etc.

Suppose we have a rule B which assigns to every $I \in \hat{\mathcal{S}}$ a quasi-coherent $\mathcal{O}_{X^I}$-module B_{X^I}, and to every $\pi : J \to I$ a morphism of $\mathcal{O}_{X^I}$-modules

$$\nu^{(\pi)} : \Delta^{(\pi)*} B_{X^J} \to B_{X^I} \tag{3.4.5.1}$$

and an isomorphism of $\mathcal{O}_{U^{[\pi]}}$-modules

$$c_{[\pi]} : j^{[\pi]*}(\underset{I}{\boxtimes} B_{X^{J_i}}) \xrightarrow{\sim} j^{[\pi]*} B_{X^J}. \tag{3.4.5.2}$$

We demand that our datum satisfies the following properties:
(a) $\nu^{(\pi)}$ are compatible with the composition of π's;
(b) for π surjective $\nu^{(\pi)}$ is an isomorphism;
(c) for $K \to J \to I$ one has $c_{[K/J]} = c_{[K/I]}(\boxtimes c_{[K_i/J_i]})|_{U^{[K/J]}}$;
(d) for $J \to J' \to I$ one has $\nu^{(J/J')}\Delta^{(J/J')*}(c_{[J/I]}) = c_{[J'/I]}(\boxtimes \nu^{(J_i/J'_i)})$;
(e) the B_{X^I} have no non-zero local sections supported at the diagonal divisor;
(f) the vector space $B_{X^\emptyset}$ is non-zero.

Notice that c yields an isomorphism of the vector spaces $B_{X^\emptyset} \otimes B_{X^\emptyset} \xrightarrow{\sim} B_{X^\emptyset}$. This map is a commutative and associative product on $B_{X^\emptyset}$. This implies that $\dim B_{X^\emptyset} = 1$,[18] so one has a canonical identification of algebras $B_{X^\emptyset} = k$.

Lemma. *The above objects are the same as factorization algebras.*

Proof. Any B as above is automatically an $\mathcal{O}$-module on $\mathcal{R}(X)$ equipped with a factorization structure. The section 1_B of B_X is the image of 1 by the morphism $\nu : B_{X^\emptyset} \otimes \mathcal{O}_X \to B_X$.

Conversely, suppose we have a factorization algebra B. We set $B_{X^\emptyset} = k$ and recover the $\nu^{(\pi)}$ as the unique system of morphisms compatible with the composition of the π's such that for π surjective $\nu^{(\pi)}$ is the structure isomorphism from 3.4.2, and

[17] Just as in the usual category of *unital* associative algebras.

[18] Proof. Since the product map for $B_{X^\emptyset}$ is injective and is commutative, we see that for any $a, b \in B_{X^\emptyset}$ one has $a \otimes b = b \otimes a \in B_{X^\emptyset}^{\otimes 2}$, and we are done by (e).

for π injective our $\nu^{(\pi)} : \Delta^{(\pi)*}B_{X^J} = B_{X^J}\boxtimes\mathcal{O}_{X^{I\smallsetminus J}} \to B_{X^I}$ is $b\boxtimes f \mapsto c(b\boxtimes f1^{\boxtimes I\smallsetminus J})$. Here c is the factorization isomorphism (3.4.4.1); to see that the latter section belongs to B_{X^I} (see (3.4.4.2)), use (ii) of the lemma in 3.4.3 and the definition of the unit in B (see 3.4.4).

The two constructions are clearly mutually inverse. □

3.4.6. Here is still another description of the factorization algebras.

Let Z be a test affine scheme. Recall that an effective Cartier divisor in $X\times Z/Z$ proper over Z is the same as a subscheme $S\subset X\times Z$ which is finite and flat over Z (see [**KM**] 1.2.3). Denote by $\mathcal{C}(X)_Z$ the set of equivalence classes of such an S where S' is said to be equivalent to S if $S'_{red} = S_{red}$. This is an ordered set (where $S' \le S$ iff $S'_{red} \subset S_{red}$), so we can consider it as a category. Our $\mathcal{C}(X)_Z$ form a category $\mathcal{C}(X)$ fibered over the category of affine schemes.

Consider a pair (B, c) where:

(i) B is a morphism from the fibered category $\mathcal{C}(X)$ to that of quasi-coherent $\mathcal{O}$-modules. So B assigns to every $S\in\mathcal{C}(X)_Z$ an $\mathcal{O}_Z$-module $B_S = B_{S,Z}$, for $S'\le S$ we have a morphism $B_{S'}\to B_S$, and everything is compatible with the base change.

In particular, for each $n\ge 0$ the universal effective divisor of degree n (see [SGA 4] Exp. XVII 6.3.9) yields an $\mathcal{O}_{\mathrm{Sym}^n X}$-module $B_{\mathrm{Sym}^n X}$.

(ii) c is a rule that assigns to every pair of *mutually disjoint* divisors $S_1, S_2 \in \mathcal{C}(X)_Z$ an identification $c_{S_1,S_2} : B_{S_1}\otimes B_{S_2} \xrightarrow{\sim} B_{S_1+S_2}$. We demand that these identifications are commutative and associative in the obvious manner and that they are compatible with the natural morphisms from (i).

Assume that the following conditions hold:

(a) The $\mathcal{O}$-modules $B_{\mathrm{Sym}^n X}$ on $\mathrm{Sym}^n X$ have no non-zero local sections supported at the discriminant divisor.

(b) One has $B_{\mathrm{Sym}^0 X} \neq 0$.

We will show that such a (B, c) is the same as a factorization algebra. For the moment we call such an object a factorization algebra in the sense of 3.4.6. They form a category which we denote by $\mathcal{FA}(X)'$. This is a tensor category in the obvious manner (one has $(\otimes B_i)_S = \otimes(B_{iS})$.

Remarks. (i) The associativity and commutativity of c show that for any finite family of *mutually disjoint* divisors S_α there is a canonical identification

$$c_{\{S_\alpha\}} : \otimes B_{S_\alpha} \xrightarrow{\sim} B_{\Sigma S_\alpha}. \tag{3.4.6.1}$$

(ii) As in 3.4.5, one gets a canonical identification $B_{\mathrm{Sym}^0 X} = k$. For any $S\in\mathcal{C}(X)_Z$ the embedding $\emptyset\subset S$ yields a canonical morphism $\mathcal{O}_Z = B_{\emptyset,Z}\to B_{S,Z}$, i.e., a canonical section of $B_{S,Z}$. The structure morphisms for B preserve these sections; we refer to all of them as the *unit section* of B.

(iii) If S', S are Cartier divisors such that $S'_{red}\subset S_{red}$, then $S'\subset nS$ for some $n\ge 1$. Thus $\mathcal{C}(X)_Z$ is the localization of the category of the Cartier divisors and inclusions with respect to the family of all morphisms $S\subset nS$, $n\ge 1$.

Therefore in the definition of B we can replace $\mathcal{C}(X)_Z$ by the ordered set of Cartier divisors in $X\times Z/Z$ proper over Z, adding the condition that the canonical morphisms $B_S\to B_{nS}$, $n\ge 1$, are all isomorphisms.

Proposition. *There is a canonical equivalence of tensor categories*

$$\mathcal{FA}(X)' \xrightarrow{\sim} \mathcal{FA}(X). \tag{3.4.6.2}$$

Proof. Let (B, c) be a factorization algebra in the sense of 3.4.6. For any finite I the union of the subschemes $x = x_i$ of $X \times X^I$, $i \in I$, is a Cartier divisor of degree $|I|$ in $X \times X^I/X^I$. So we have the corresponding $\mathcal{O}_{X^I}$-module B_{X^I}; equivalently, B_{X^I} is the pull-back of $B_{\mathrm{Sym}^{|I|}X}$ by the projection $p_{|I|} : X^I \to \mathrm{Sym}^{|I|}X$. These $\mathcal{O}$-modules together with the evident $\nu^{(\pi)}$ and $c_{[\pi]}$ form a datum from 3.4.5 which satisfies properties (a)–(f) from loc. cit. So we have a factorization algebra; we denote it again by B.

The construction is obviously functorial and compatible with tensor products. We have defined the promised tensor functor $\mathcal{FA}(X)' \to \mathcal{FA}(X)$.

Let us construct the inverse functor. Let B be a factorization algebra in the sense of 3.4.5. We want to define the corresponding factorization algebra (B, c) in the sense of 3.4.6.

First let us construct the $\mathcal{O}$-modules $B_{\mathrm{Sym}^n X}$. Consider the projection $p_n : X^n \to \mathrm{Sym}^n X$. The symmetric group Σ_n acts on X^n and on B_{X^n}. Set $B_{\mathrm{Sym}^n X} := (p_{n*}B_{X^n})^{\Sigma_n}$. By 3.4.2 it satisfies condition (a) in 3.4.6.

LEMMA. *The canonical morphism* $\psi : p_n^* B_{\mathrm{Sym}^n X} \to B_{X^n}$ *is an isomorphism.*

Proof of the lemma. (a) Let $\ell : V \hookrightarrow X^n$ be the complement to the diagonals of codimension ≥ 2 and $\bar{\ell} : \bar{V} \hookrightarrow \mathrm{Sym}^n X$ be its p_n-image. By (ii) in the lemma in 3.4.3 we know that $B_{X^n} = \ell_* \ell^* B_{X^n}$; thus $B_{\mathrm{Sym}^n X} = \bar{\ell}_* \bar{\ell}^* B_{\mathrm{Sym}^n X}$. Since p_n is flat, we conclude that it suffices to check that ψ is an isomorphism over V. By factorization it suffices to consider the case $n = 2$.

(b) By étale descent our ψ is an isomorphism over the complement to the diagonal divisor. Therefore ψ is injective (since p_n is flat).

(c) It remains to prove that $\psi : p_2^* B_{\mathrm{Sym}^2 X} \to B_{X^2}$ is surjective. Consider its cokernel C. This is a Σ_2-equivariant quotient of B_{X^2} such that $(p_{2*}C)^{\Sigma_2} = 0$. We know that C is supported (set-theoretically) on the diagonal divisor X, so one has $C^{\Sigma_2} = 0$.

Let f be a (local) equation of the diagonal $X \subset X \times X$ which is Σ_2-anti-invariant. Since Σ_2 acts on C by the non-trivial character *sgn* multiplication by f kills C, i.e., C is supported on the diagonal scheme-theoretically. Thus C is a quotient of the pull-back of B_{X^2} to X, i.e., of B_X. But Σ_2 acts on B_X trivially. So $C = 0$, and we are done. □

REMARK. Let $G \subset \Sigma_n$ be a subgroup, so we have the projections $X^n \xrightarrow{\pi_1} G \setminus X^n \xrightarrow{\pi_2} \mathrm{Sym}^n X$. The lemma implies that $\pi_2^* B_{\mathrm{Sym}^n X} = (\pi_{1*} B_{X^n})^G$.

Now let us construct our B following Remark (iii) above. Let us define $B_{S,Z}$ for any Cartier divisor S in $X \times Z/Z$ proper over Z. Our problem is Z-local, so we can assume that the degree n of S over Z is constant. Such an S amounts to a morphism $Z \to \mathrm{Sym}^n X$, and $B_{S,Z}$ is the pull-back of $B_{\mathrm{Sym}^n X}$.

Let us define for $S' \subset S$ a canonical morphism $B_{S'} \to B_S$. We have $S = S' + T$ for an effective Cartier divisor T. It suffices to construct our morphism in the universal situation when $Z = \mathrm{Sym}^n X \times \mathrm{Sym}^m X$ and S', T are the universal divisors of order n and m, respectively. So let $q^{(n)} : \mathrm{Sym}^n X \times \mathrm{Sym}^m X \to \mathrm{Sym}^n X$ be the projection and let $p : \mathrm{Sym}^n X \times \mathrm{Sym}^m X \to \mathrm{Sym}^{n+m} X$ be the addition map. We want to define a canonical morphism $q^{(n)*} B_{\mathrm{Sym}^n X} \to p^* B_{\mathrm{Sym}^{n+m} X}$. By the above lemma and remark this amounts to a $\Sigma_n \times \Sigma_m$ -equivariant morphism $B_{X^n} \boxtimes \mathcal{O}_{X^m} \to B_{X^{n+m}}$. This is the structure morphism $\nu^{(\pi)}$ from (3.4.5.1) for $\pi : \{1, \ldots, n\} \hookrightarrow \{1, \ldots, n+m\}$.

The condition from Remark (iii) is satisfied,[19] so we have defined our B. The factorization isomorphisms c come from the isomorphisms (3.4.5.2) (consider the universal situation and use the above remark). So we have defined the factorization algebra in the sense of 3.4.6; i.e., we have a functor $\mathcal{FA}(X) \to \mathcal{FA}(X)'$. It follows from the construction (use the lemma) that it is inverse to the previously defined functor in the opposite direction. We are done. □

3.4.7. The canonical connection. Let B be a factorization algebra.

Proposition. *The $\mathcal{O}$-module B on $\mathcal{R}(X)$ admits a left $\mathcal{D}$-module structure compatible with the factorization structure and such that the section 1_B of B_X is horizontal. Such a $\mathcal{D}$-module structure is unique. We call it the canonical $\mathcal{D}$-module structure (or the canonical connection) on B.*

Proof. Existence. It is evident from the picture of 3.4.6. Indeed, suppose we have two morphisms of schemes $f, g : Z' \rightrightarrows Z$ which coincide on Z'_{red}. Then the pull-back maps $f^*, g^* : \mathcal{C}(X)_Z \to \mathcal{C}(X)_{Z'}$ coincide, so for any $S \in \mathcal{C}(X)_Z$ one has a canonical identification

$$f^* B_S \xrightarrow{\sim} g^* B_S \tag{3.4.7.1}$$

of $\mathcal{O}_{Z'}$-modules. The identifications are transitive, so B_S carries a canonical action of the universal formal groupoid on Z (whose space is the formal completion of $X \times X$ at the diagonal). The structure morphisms of B are compatible with this action. If Z is smooth, then, according to Grothendieck [**Gr2**], such an action amounts to an integrable connection. The compatibility with the base change shows that 1_B is a horizontal section. Looking at B_{X^I}'s, we get our canonical connection.

Here is an equivalent construction that starts from the picture of 3.4.5. We will define the connection on B_X; the connections on B_{X^I}'s are defined similarly.

For $I \in \mathcal{S}$ let $X^{<I>}$ be the formal completion of X^I at the diagonal $X \hookrightarrow X^I$; for π as above let $\Delta^{<\pi>} : X^{<I>} \to X^{<J>}$ be the formal completion of $\Delta^{(\pi)}$. Let $B_{X^{<I>}}$ be the pull-back of B_{X^I} to $X^{<I>}$ (= the formal completion of B_{X^I} at $X \subset X^I$) and $\nu^{<\pi>} : \Delta^{<\pi>*} B_{X^{<J>}} \to B_{X^{<I>}}$ be the completion of $\nu^{(\pi)}$.

Key remark: each $\nu^{<\pi>}$ is an isomorphism. This follows from statement (i) of the lemma in 3.4.3 and the fact that the pull-back of $\nu^{(\pi)}$ to the diagonal $X \hookrightarrow X^I$ is an isomorphism (equal to id_{B_X}).

The isomorphisms $\nu^{<\pi>}$ are compatible with the composition of the π's. A datum of such isomorphisms amounts to an action on B_X of the universal formal groupoid on X (the action itself is the identification of the pull-backs of B_X by the two projections $X^{<2>} \rightrightarrows X$). We have defined the canonical connection on B_X.

Uniqueness. Our condition just says that all morphisms $\nu^{(\pi)}$ from (3.4.5.1) are compatible with the connections. Therefore the above action of the universal formal groupoid on B_X is compatible with the connection, which determines the connection uniquely. □

Remark. Here is a down-to-earth definition of the canonical $\mathcal{D}$-module structure (which follows directly from the proof of 3.4.7). Let t be a local coordinate on

[19] Use the fact that $p_{n*}\mathcal{O}_{X^n}$ is isomorphic to $\mathcal{O}_{\mathrm{Sym}^n X}[\Sigma_n]$ as an $\mathcal{O}_{\mathrm{Sym}^n X}[\Sigma_n]$-module locally on $\mathrm{Sym}^n X$.

X. Then for $b \in B$ one has

$$1 \boxtimes b - b \boxtimes 1 = (\partial_t b) \boxtimes 1 (t_2 - t_1) \mathrm{mod} (t_2 - t_1)^2 B_{X \times X}. \tag{3.4.7.2}$$

In fact, the restriction of $1 \boxtimes b$ to the formal neighbourhood of the diagonal $X \subset X \times X$ equals $\sum \frac{(t_2 - t_1)^i}{i!} (\partial_t^i b) \boxtimes 1$. This defines our $\mathcal{D}$-module structure on B_X. And B_{X^I} is a $\mathcal{D}$-submodule of $j_*^{(I)} j^{(I)*} B_X^{\boxtimes I}$.

3.4.8. From factorization algebras to chiral algebras. Let $B^r = \omega_X \otimes B_X$ be the right $\mathcal{D}_X$-module that corresponds to the left $\mathcal{D}_X$-module B_X (see 2.1.1). Let us show that it carries a canonical structure of a chiral algebra.

For each surjection $J \twoheadrightarrow I$ we have a natural isomorphism of left $\mathcal{D}$-modules $\Delta^{(J/I)*} B_{X^J} \xrightarrow{\sim} B_{X^I}$. By (2.1.3.2), we can rewrite it as an isomorphism of right $\mathcal{D}_{X^J}$-modules

$$\Delta_*^{(J/I)} \omega_X^{\boxtimes I} \otimes B_{X^J} \xrightarrow{\sim} \Delta_*^{(J/I)} (\omega_X^{\boxtimes I} \otimes B_{X^I}). \tag{3.4.8.1}$$

In particular, we get a canonical map

$$P^{ch}(\omega_X)_J \longrightarrow P^{ch}(B_X^r)_J \tag{3.4.8.2}$$

which sends an operation $\varphi : j_*^{(J)} j^{(J)*} \omega_X^{\boxtimes J} \longrightarrow \Delta_*^{(J)} \omega_X$ to the composition

$$j_*^{(J)} j^{(J)*} B^{r \boxtimes J} = j_*^{(J)} j^{(J)*} \omega_X^{\boxtimes J} \otimes B_{X^J} \longrightarrow \Delta_*^{(J)} \omega_X \otimes B_{X^J} = \Delta_*^{(J)} B^r. \tag{3.4.8.3}$$

Here the first equality is the factorization isomorphism tensored by $id_{\omega_X^{\boxtimes J}}$, the arrow is φ tensored by $id_{B_{X^J}}$, and the last equality is (3.4.8.1) (for $I = \cdot$).

We leave it to the reader to check that (3.4.8.2) is compatible with the composition of chiral operations. By 3.1.5 we can rewrite (3.4.8.2) as a canonical morphism of operads $\mathcal{L}ie \to P^{ch}(B^r)$ which is the same as a Lie^{ch} algebra structure on B_X^r. The section 1 of B_X is a unit in the Lie^{ch} algebra B^r. So B^r is a chiral algebra.

3.4.9. The next theorem is the principal result of this section:

THEOREM. *The functor $\mathcal{FA}(X) \to \mathcal{CA}(X)$, $B \mapsto B^r$, is an equivalence of categories.*

REMARKS. (i) The factorization algebra corresponding to a *commutative* chiral algebra (i.e., to an affine $\mathcal{D}_X$-scheme) can be constructed geometrically (see 3.4.21–3.4.22). But to prove that the construction from 3.4.21 indeed gives a factorization algebra, we compare it in 3.4.22 with the construction from 3.4.11–3.4.12.

(ii) The theorem and its proof remain valid in the super and DG super setting. It also generalizes immediately to the case of families of chiral algebras (see 3.3.10). Namely, for a scheme (or algebraic space) Z one defines the notion of Z-family of factorization algebras and the above functor in the evident way. It establishes an equivalence between the category of $\mathcal{O}_Z$-flat families of factorization algebras and the category of $\mathcal{O}_Z$-flat families of chiral algebras.

Proof. It is found in 3.4.10–3.4.12. We will construct the inverse functor. So for every chiral algebra A we want to define on the left $\mathcal{D}_X$-module A^ℓ a canonical structure of a factorization algebra. The key point is the acyclicity property of the Chevalley-Cousin complex $C(A)$ of A. We define $C(A)$ in 3.4.11 after the necessary preliminaries of 3.4.10; the acyclicity lemma is proven in 3.4.12.

3.4.10. The category $\mathcal{M}(X^{\mathcal{S}})$ and its two tensor structures. In this subsection we embed $\mathcal{M}(X)$ in a larger abelian category $\mathcal{M}(X^{\mathcal{S}})$ so that both $*$ and chiral pseudo-tensor structures on $\mathcal{M}(X)$ are induced from some *tensor* structures on $\mathcal{M}(X^{\mathcal{S}})$. The reader should compare this construction with the universal one from Remark in 1.1.6(i).

A *right* $\mathcal{D}$*-module* M *on* $X^{\mathcal{S}}$ is a rule that assigns to $I \in \mathcal{S}$ a right $\mathcal{D}$-module M_{X^I} on X^I and to $\pi : J \twoheadrightarrow I$ a morphism of $\mathcal{D}$-modules $\theta^{(\pi)} = \theta^{(\pi)}_M : \Delta^{(\pi)}_* M_{X^I} \to M_{X^J}$. We demand that the $\theta^{(\pi)}$ are compatible with the composition of the π's (i.e., $\theta^{(\pi_1\pi_2)} = \theta^{(\pi_2)}\Delta^{(\pi_2)}_*(\theta^{(\pi_1)})$) and $\theta^{(id_I)} = id_{M_{X^I}}$.

These objects form an abelian k-category $\mathcal{M}(X^{\mathcal{S}})$. There is an exact fully faithful embedding $\Delta^{(\mathcal{S})}_* : \mathcal{M}(X) \hookrightarrow \mathcal{M}(X^{\mathcal{S}})$ defined by $\Delta^{(\mathcal{S})}_* M_{X^I} := \Delta^{(I)}_* M$, $\theta^{(\pi)} = id_{\Delta^{(J)}_* M}$. This embedding is left adjoint to the projection functor $\mathcal{M}(X^{\mathcal{S}}) \to \mathcal{M}(X)$, $M \mapsto M_X$.

Remark. For $n \geq 1$ let $\mathcal{S}_n \subset \mathcal{S}$ be the subcategory of sets of order $\leq n$. The above category has an obvious "truncated" version $\mathcal{M}(X^{\mathcal{S}_n})$. The "restriction" functor $\mathcal{M}(X^{\mathcal{S}}) \xrightarrow{\pi_n} \mathcal{M}(X^{\mathcal{S}_n})$ admits a left adjoint $a_n : \mathcal{M}(X^{\mathcal{S}_n}) \to \mathcal{M}(X^{\mathcal{S}})$, $a_n(N)_{X^I} = \varinjlim \Delta^{(I/S)}_* N_{X^S}$ where the inductive limit is taken over the ordered set $Q(I, \leq n)$. Since $\pi_n a_n$ is the identity functor, we see that π_n identifies $\mathcal{M}(X^{\mathcal{S}_n})$ with the quotient category of $\mathcal{M}(X^{\mathcal{S}})$ modulo the full subcategory $\mathcal{M}(X^{\geq n+1})$ of such M's that $\pi_n M = 0$; i.e., $M_{X^I} = 0$ if $|I| \leq n$. In particular, $\mathrm{gr}_n \mathcal{M}(X^{\mathcal{S}}) := \mathcal{M}(X^{\geq n})/\mathcal{M}(X^{\geq n+1})$ equals the category of Σ_n-equivariant $\mathcal{D}$-modules on X^n.

The category $\mathcal{M}(X^{\mathcal{S}})$ carries two tensor products $\otimes^*$ and $\otimes^{ch}$ defined as follows. Let M_i, $i \in I$, be a finite non-empty family of objects of $\mathcal{M}(X^{\mathcal{S}})$. One has

$$(\otimes^*_I M_i)_{X^J} := \bigoplus_{J \twoheadrightarrow I} \boxtimes_I (M_i)_{X^{J_i}}, \tag{3.4.10.1}$$

where the arrows $\theta^{(\pi)}$ are the obvious ones. Similarly, set

$$(\otimes^{ch}_I M_i)_{X^J} := \bigoplus_{J \twoheadrightarrow I} j^{[J/I]}_* j^{[J/I]*} \boxtimes (M_i)_{X^{J_i}}, \tag{3.4.10.2}$$

where the arrows $\theta^{(\pi)}$ are obvious ones. Here $j^{[J/I]}$ was defined in 3.4.4.

Our tensor products are associative and commutative in the obvious way, so they define on $\mathcal{M}(X^{\mathcal{S}})$ (non-unital) tensor category structures which we denote by $\mathcal{M}(X^{\mathcal{S}})^*$, $\mathcal{M}(X^{\mathcal{S}})^{ch}$. There is an obvious canonical morphism

$$\otimes^* M_i \to \otimes^{ch} M_i \tag{3.4.10.3}$$

compatible with the constraints, so the identity functor for $\mathcal{M}(X^{\mathcal{S}})$ extends to a pseudo-tensor functor

$$\beta^{\mathcal{S}} : \mathcal{M}(X^{\mathcal{S}})^{ch} \to \mathcal{M}(X^{\mathcal{S}})^*. \tag{3.4.10.4}$$

Examples. For $N_i \in \mathcal{M}(X)$ one has $(\otimes^*_I \Delta^{(\mathcal{S})}_* N_i)_{X^J} = \bigoplus_{J \twoheadrightarrow I} \Delta^{(J/I)}_* \boxtimes_I N_i$. In particular, $(\otimes^*_I \Delta^{(\mathcal{S})}_* N_i)_{X^I} = \bigoplus_{\nu \in \mathrm{Aut}\, I} \nu^*(\boxtimes N_i)$. Similarly, for $N \in \mathcal{M}(X)$ one has $(\mathrm{Sym}^m_* \Delta^{(\mathcal{S})}_* N)_{X^I} = \bigoplus_{T \in Q(I,m)} \Delta^{(I/T)}_* N^{\boxtimes T}$. Here Sym^m_*, $m \geq 1$, is the mth symmetric power with respect to $\otimes^*$, and $Q(I,m) \subset Q(I)$ is formed by quotient sets of order m. These formulas are modified in the obvious way in the $\otimes^{ch}$ case.

The embedding $\mathcal{M}(X) \hookrightarrow \mathcal{M}(X^{\mathcal{S}})$ extends canonically to fully faithful pseudo-tensor embeddings

$$\Delta_*^{(\mathcal{S})} : \mathcal{M}(X)^* \hookrightarrow \mathcal{M}(X^{\mathcal{S}})^*, \quad \mathcal{M}(X)^{ch} \hookrightarrow \mathcal{M}(X^{\mathcal{S}})^{ch}. \tag{3.4.10.5}$$

Namely, for $M_i, N \in \mathcal{M}(X)$ our $\mathrm{Hom}(\otimes_I^* \Delta_*^{(\mathcal{S})} M_i, \Delta_*^{(\mathcal{S})} N) \xrightarrow{\sim} P_I^*(\{M_i\}, N)$ assigns to $\varphi : \otimes_I^* \Delta_*^{(\mathcal{S})} M_i \to N$ the restriction of φ_{X^I} to $\underset{I}{\boxtimes} M_I \subset (\otimes_I^* \Delta_*^{(\mathcal{S})} M_i)_{X^I}$. One defines $\mathrm{Hom}(\otimes_I^* \Delta_*^{(\mathcal{S})} M_i, N) \xrightarrow{\sim} P_I^*(\{M_i\}, N)$ in a similar way. Notice that the restriction of $\beta^{\mathcal{S}}$ to $\mathcal{M}(X)^*$ is the pseudo-tensor functor β from (3.2.1.2).

The category $\mathcal{M}(X^{\mathcal{S}})$ admits a natural action of the tensor category $\mathcal{M}^\ell(\mathcal{R}(X))$ (see 3.4.2): the action functor

$$\otimes : \mathcal{M}^\ell(\mathcal{R}(X)) \times \mathcal{M}(X^{\mathcal{S}}) \to \mathcal{M}(X^{\mathcal{S}}) \tag{3.4.10.6}$$

sends $F \in \mathcal{M}^\ell(\mathcal{R}(X))$, $M \in \mathcal{M}(X^{\mathcal{S}})$ to $M \otimes F$ with $(M \otimes F)_{X^I} := M_{X^I} \otimes F_{X^I}$ and $\theta^{(\pi)}_{M \otimes F} : \Delta_*^{(\pi)} M_{X^I} \otimes F_{X^I} \to M_{X^J} \otimes F_{X^J}$ equal to $\theta^{(\pi)}_M \otimes \nu_F^{(\pi)-1}$ via (2.1.3.2). The associativity constraint for $\otimes$ is the evident one.

3.4.11. The Chevalley-Cousin complex. Let A be a (not necessarily unital) chiral algebra on X. According to (3.4.10.5) the right $\mathcal{D}$-module $\Delta_*^{(\mathcal{S})} A$ on $X^{\mathcal{S}}$ is a Lie algebra in the tensor category $\mathcal{M}(X^{\mathcal{S}})^{ch}$. We define the *Chevalley-Cousin complex* $C(A)$ of A as the reduced Chevalley complex of this Lie algebra.[20]

Let us describe $C(A)$ explicitly. Forget about the differential for a moment. As a plain $\mathbb{Z}$-graded $\mathcal{D}$-module, $C(A)$ is the free commutative (non-unital) algebra in $\mathcal{M}(X^{\mathcal{S}})^{ch}$ generated by $\Delta_*^{(\mathcal{S})} A[1]$. Therefore one has

$$C(A)^{\cdot}_{X^I} = \underset{T \in Q(I)}{\oplus} \Delta_*^{(I/T)} j_*^{(T)} j^{(T)*} (A[1])^{\boxtimes T}. \tag{3.4.11.1}$$

In other words, $C(A)^{\cdot}_{X^I}$ is a $Q(I)$-graded module, and for $T \in Q(I)$ its T-component is $\Delta_*^{(I/T)} j_*^{(T)} j^{(T)*} A^{\boxtimes T} \otimes \lambda_T$ sitting in degree $-|T|$.

The differential looks as follows. Its component $d_{T,T'} : \Delta_*^{(I/T)} j_*^{(T)} j^{(T)*} (A[1])^{\boxtimes T} \to \Delta_*^{(I/T')} j_*^{(T')} j^{(T')*} (A[1])^{\boxtimes T'}$ can be non-zero only for $T' \in Q(T, |T|-1)$. Then $T = T'' \sqcup \{\alpha', \alpha''\}$, $T' = T'' \sqcup \{\alpha\}$ and $d_{T,T'}$ is the exterior tensor product of the chiral product map $\mu_A[1] : j_* j^* (A_{\alpha'}[1] \boxtimes A_{\alpha''}[1]) \to \Delta_* A_\alpha[1]$ and the identity map for $A^{\boxtimes T''}$ (localized at the diagonals transversal to $X^{T'}$).

The reader can skip the next remark at the moment.

Remark. As with any Chevalley complex, $C(A)$ carries a canonical structure of a BV algebra with respect to $\otimes^{ch}$ (see 4.1.7). The product $\cdot : C(A)^{\cdot} \otimes^{ch} C(A)^{\cdot} \to C(A)^{\cdot}$ and the bracket $\{\ \} : C(A)[-1] \otimes^{ch} C(A)[-1] \to C(A)[-1]$ can be described as follows. For $J = J_1 \sqcup J_2$ the morphism $\cdot_{J_1 J_2}$ is the obvious embedding $C(A)^{\cdot}_{X^{J_1}} \boxtimes C(A)^{\cdot}_{X^{J_2}} \to C(A)^{\cdot}_{X^J}$ localized at appropriate diagonals. For $T_1 \in Q(J_1)$, $T_2 \in Q(J_2)$, $T \in Q(J)$ the corresponding component of $\{\ \}_{J_1 J_2} : \tilde{j}_* \tilde{j}^* C(A)_{X^{J_1}}[-1] \boxtimes C(A)_{X^{J_2}}[-1] \to C(A)_{X^J}[-1]$ (here $\tilde{j}$ is the embedding of the complement to the diagonals $x_{j_1} = x_{j_2}$, $j_1 \in J_1$, $j_2 \in J_2$) can be non-zero only if T is the quotient of $T_1 \sqcup T_2$ modulo a relation $t_1 = t_2$ where $t_1 \in T_1$, $t_2 \in T_2$. Then this component is

[20] We must consider the *reduced* Chevalley complex since $\mathcal{M}(X^{\mathcal{S}})^{ch}$ is a *non-unital* tensor category. The basic facts about the Chevalley complex are recalled in 4.1.6.

the chiral product $j_*j^*A \boxtimes A \to \Delta_* A$ tensored by the identity map for $(A[1])^{\boxtimes T'}$ where $T' := T \smallsetminus \{t\} = T_1 \smallsetminus \{t_1\} \sqcup T_2 \smallsetminus \{t_2\}$ (localized at the other diagonals).

Notice that if $v : V \hookrightarrow X^I$ is an open embedding such that $X^I \smallsetminus V$ is a union of diagonal strata, then the canonical map $C(A)_{X^I} \to v_* v^* C(A)_{X^I}$ admits an obvious canonical section

$$s : v_* v^* C(A)^{\cdot}_{X^I} \to C(A)^{\cdot}_{X^I} \tag{3.4.11.2}$$

which preserves the $Q(I)$-grading but does not commute with the differentials.

The complex $C(A)$ enjoys the following two properties:

(i) For every $\pi : J \twoheadrightarrow I$ the structure morphism yields an isomorphism of complexes

$$C(A)_{X^I} \xrightarrow{\sim} \Delta^{(\pi)!} C(A)_{X^J}, \tag{3.4.11.3}$$

and this plain $\Delta^{(\pi)!}$ pull-back equals the derived functor pull-back. In other words, there is a canonical short exact sequence of complexes

$$0 \to \Delta^{(\pi)}_* C(A)_{X^I} \to C(A)_{X^J} \to v_* v^* C(A)_{X^J} \to 0. \tag{3.4.11.4}$$

Here $v : X^J \smallsetminus \Delta^{(\pi)}(X^I) \hookrightarrow X^J$ is the open embedding complementary to $\Delta^{(\pi)}$; the naive v_* coincides here with the derived one.

(ii) (Factorization) In the setting of 3.4.4 for $J \twoheadrightarrow I$ the product morphism for the commutative algebra structure on $C(A)^{\cdot}$ yields an isomorphism of complexes

$$c_{[J/I]} : j^{[J/I]*}(\underset{I}{\boxtimes} C(A)_{X^{J_i}}) \xrightarrow{\sim} j^{[J/I]*} C(A)_{X^J}. \tag{3.4.11.5}$$

The identifications c are mutually compatible: for $K \twoheadrightarrow J$ the isomorphism $c_{[K/J]}$ coincides with the composition $c_{[K/I]}(\boxtimes c_{[K_i/J_i]})$.

3.4.12. Lemma. *For a chiral algebra A one has $H^n C(A)_{X^I} = 0$ for $n \neq -|I|$.*

End of the proof of 3.4.9. Let us define the factorization algebra structure on A^ℓ. Set $A^\ell_{X^I} := H^{-|I|} C(A)^\ell_{X^I} = \mathrm{Ker}(d : C(A)^{-|I|}_{X^I} \to C(A)^{-|I|+1}_{X^I})^\ell$. By the lemma we can rewrite (3.4.10.3) as a canonical identification $\Delta^{(I/S)*} A^\ell_{X^I} = A^\ell_{X^S}$ (see 2.1.2). These identifications are obviously transitive, so the $A^\ell_{X^I}$'s form a left $\mathcal{D}$-module, hence an $\mathcal{O}$-module, on $\mathcal{R}(X)$. So[21] $A^\ell_{X^I} \subset C(A)^{0\ell}_{X^I} = j^{(I)}_* j^{(I)*} A^{\ell \boxtimes I}$. We leave it to the reader to check that this embedding defines a factorization algebra structure on our $\mathcal{O}_{\mathcal{R}(X)}$-module (with the unit equal to the unit in A). Notice that our factorization algebra came together with a $\mathcal{D}$-module structure, which obviously coincides with the canonical structure (see 3.4.7). The functor $\mathcal{FA}(X) \to \mathcal{CA}(X)$ we have defined is clearly inverse to that of 3.4.9. We are done. □

Proof of Lemma. The statement is clear if $|I| = 1$ (one has $C(A)_X = A[1]$). For $|I| > 1$ we use induction on $|I|$. The proof is in three steps:

(i) The module $H^a C(A)_{X^I}$ for $a \neq -|I|$ is supported on the diagonal $X \subset X^I$. This follows from (3.4.11.5) and the induction assumption.

Let $i : Y = X^{I'} \hookrightarrow X^I$ be a diagonal of codimension 1, $v : X^I \smallsetminus Y \hookrightarrow X^I$ the complementary embedding. Consider the canonical embedding $\xi : i_* C(A)_Y \hookrightarrow C(A)_{X^I}$ (see (3.4.11.4)).

[21] We use the identification $\omega_{X^I} = \omega_X^{\boxtimes I} \otimes \lambda_I$; see 3.1.4.

(ii) The morphism $H^a\xi : i_*H^aC(A)_Y = H^a i_*C(A)_Y \to H^aC(A)_{X^I}$ is surjective for $a \neq -|I|$. Indeed, the complex $v_*v^*C(A)_{X^I}$ is acyclic outside degree $-|I|$ by (i) (since v is affine), so our claim follows from the exact sequence $0 \to i_*C(A)_Y \to C(A)_{X^I} \to v_*v^*C(A)_{X^I} \to 0$ (see (3.4.11.4)).

(iii) It remains to show that $H^a\xi = 0$. If $a \neq 1-|I|$, this is clear because in this case $H^aC(A)_Y = 0$ by the induction assumption.

Now let $a = 1-|I|$. Choose a section $I' \to I$ of $I \twoheadrightarrow I'$. Then we have the decomposition $X^I = X \times X^{I'} = X \times Y$ and the embedding $i : Y \hookrightarrow X \times Y$. We want to show that $\xi(i_*Z^a_Y) \subset d(C^{a-1}(A)_{X^I})$ where $Z^a_Y := \mathrm{Ker}(d : C^a(A)_Y \to C^{a+1}(A)_Y)$. Let s be a local section of $i_*Z^a_Y = v_*v^*(\omega_X \boxtimes Z^a_Y)/\omega_X \boxtimes Z^a_Y$. Lift it to a local section $\tilde{s}$ of $v_*v^*(\omega_X \boxtimes Z^a_Y)$. Then $\xi(s) = d(h(\tilde{s}))$ where h is the composition

$$v_*v^*(\omega_X \boxtimes Z^a_Y) \xrightarrow{f} v_*v^*(A \boxtimes C^a(A)_Y) \hookrightarrow C^{a-1}(A)_{X^I}$$

and f comes from $1_A : \omega_X \to A$. □

3.4.13. Remarks. (i) For $n \geq 1$ an *n-truncated factorization algebra* is defined exactly as the usual factorization algebra but we take into consideration only the X^I's with $|I| \leq n$. Denote the corresponding categories by $\mathcal{FA}^{\leq n}(X)$. As follows from 3.4.9, for $n \geq 3$ the obvious functors $\mathcal{FA}(X) \to \mathcal{FA}^{\leq n}(X)$, $\mathcal{FA}^{\leq n+1}(X) \to \mathcal{FA}^{\leq n}(X)$ are equivalences of categories.[22]

(ii) Let B be a factorization algebra, B^r the corresponding unital chiral algebra. Then there is a canonical isomorphism of complexes of right $\mathcal{D}$-modules on $X^{\mathcal{S}}$ (see (3.4.10.6))

$$C(B^r) \xrightarrow{\sim} C(\omega) \otimes B \qquad (3.4.13.1)$$

which identifies the summand $j^{(I)}_* j^{(I)*} B^{r\boxtimes I} = (j^{(I)}_* j^{(I)*}\omega_X^{\boxtimes I}) \otimes B_X^{\boxtimes I}$ in $C(B^r)_{X^I}$ with the one $(j^{(I)}_* j^{(I)*}\omega_X^{\boxtimes I}) \otimes B_{X^I}$ in $(C(\omega)\otimes B)_{X^I}$ via the factorization isomorphism (see (3.4.4.2)).

3.4.14. Free factorization algebras. It is easy to see that the chiral algebra freely generated (in the obvious sense) by a given non-empty set of sections does not exist. The nuisance can be corrected as follows.

Suppose we have a pair (N,P) where N is a quasi-coherent $\mathcal{O}_X$-module and $P \subset j_*j^*N \boxtimes N$ is a quasi-coherent $\mathcal{O}_{X\times X}$-submodule such that $P|_U = N \boxtimes N|_U$. Consider a functor on the category $\mathcal{CA}(X)$ which assigns to a chiral algebra A the set of all $\mathcal{O}_X$-linear morphisms $\phi : N \to A$ such that the chiral product μ_A kills $\phi^{\boxtimes 2}(P) \subset j_*j^*A^{\boxtimes 2}$.

Theorem. *This functor is representable.*

We refer to the corresponding universal chiral algebra as the *chiral algebra freely generated by* (N,P).

Remarks. (i) From the point of view of chiral operations, P should be thought of as "generalized commutation relations". For example, if $P = N \boxtimes N$, then A is the commutative chiral algebra freely generated by $N_{\mathcal{D}}$. From the point of view of chiral algebras, P is the module of "degree 2 generators" (see below).

[22] They are fully faithful for $n = 2$ and faithful for $n = 1$.

(ii) If C is any chiral algebra and $N \subset C$ an $\mathcal{O}_X$-submodule that generates C, then C is a quotient of the chiral algebra freely generated by (N, P) where P is, say, the kernel of the map $j_*j^*N \boxtimes N \to j_*j^*A \xrightarrow{\mu_A} \Delta_*A$.

Proof of Theorem. It is more convenient to consider the setting of factorization algebras. Here the theorem asserts representability of the functor which assigns to a factorization algebra B the set of all $\mathcal{O}_X$-linear morphisms $\phi : N \to B$ such that $\phi^{\boxtimes 2}(P)$ is contained in $B_{X\times X} \subset j_*j^*B_X \boxtimes B_X$. We call the corresponding universal factorization algebra the *factorization algebra freely generated by* (N, P).

In the course of construction we use certain auxiliary objects:

(i) A *quasi-factorization algebra* in formed by the datum $(B_{X^I}, \nu^{(\pi)}, c_{[\pi]})$ from 3.4.5 subject to axioms (a)–(d) and (f) in loc. cit.

(ii) A *pre-factorization algebra* is defined in the same way as a quasi-factorization algebra except that the morphisms $\nu^{(\pi)}$ are defined only for *injective* maps $\pi : J \hookrightarrow I$. Apart from the (obvious modification of) axioms (a)–(d) and (f) in 3.4.5, we demand that for every $n \geq 2$ the next property is satisfied:

$(*)_n$ The action of the transposition $\sigma = \sigma_{1,2}$ on T_{X^n} induces the trivial automorphism of the pull-back of T_{X^n} to the diagonal $x_1 = x_2$.

EXAMPLE. $T(N)_{X^I} := j_*^{(I)} j^{(I)*}(N \oplus \mathcal{O}_X)^{\boxtimes I}$ together with evident structure morphisms ν and c form a pre-factorization algebra $T(N)$.

Pre- and quasi-factorization algebras form tensor categories which we denote by $\mathcal{PFA}(X)$ and $\mathcal{QFA}(X)$. One has evident tensor functors (in fact, fully faithful embeddings) $\mathcal{FA}(X) \to \mathcal{QFA}(X) \to \mathcal{PFA}(X)$.

LEMMA. *These functors admit left adjoints $\mathcal{FA}(X) \to \mathcal{PFA}(X) \to \mathcal{QFA}(X)$.*

Granted the lemma, let us construct the factorization algebra freely generated by (N, P).

Let $T(N, P) \subset T(N)$ be a pre-factorization subalgebra defined as follows.

(i) For $|I| \leq 1$ one has $T(N, P)_{X^I} = T(N)_{X^I}$.

(ii) $T(N, P)_{X^2}$ is the minimal σ-invariant $\mathcal{O}_{X^2}$-submodule of $T(N)_{X^2}$ which contains $P \oplus N \boxtimes \mathcal{O}_X \oplus \mathcal{O}_X \boxtimes N \oplus \mathcal{O}_X \boxtimes \mathcal{O}_X$, is preserved by the action of the transposition σ, and satisfies property $(*)_2$.[23]

(iii) If $|I| > 2$, then $T(N, P)_{X^I}$ is the maximal $\mathcal{O}_{X^I}$-submodule of $T(N)_{X^I}$ such that for every $i_1 \neq i_2 \in I$ its restriction to the complement of the union of all the diagonals except for $x_{i_1} = x_{i_2}$ is equal to $T(N, P)_{X^{\{i_1,i_2\}}} \boxtimes T(N)_{X^{\boxtimes I \smallsetminus \{i_1,i_2\}}}$.

It is easy to see that $T(N, P)$ is indeed a pre-factorization subalgebra of $T(N)$.[24]

Now for any factorization algebra B a morphism of pre-factorization algebras $T(N, P) \to B$ amounts to a morphism of $\mathcal{O}_X$-modules $\phi : N \to B$ such that $\phi^{\boxtimes 2}(P) \subset B_{X\times X}$. The promised factorization algebra freely generated by (N, P) is the image of $T(N, P)$ by the composition of the functors from the lemma.

Proof of Lemma. (i) Let T be any pre-factorization algebra. Let us construct the corresponding universal quasi-factorization algebra $B = B(T)$.

[23] Our $T(N, P)_{X^2}$ is well defined and is $\mathcal{O}_X$-quasi-coherent. Indeed, $T(N, P)_{X^2} = \bigcup T_i$ where $T_0 \subset T_1 \subset \cdots$ are the submodules of $j_*j^*T(N)_X^{\boxtimes 2}$ defined by induction as follows. One has $T_0 := (P + \sigma(P)) \oplus N \boxtimes \mathcal{O}_X \oplus \mathcal{O}_X \boxtimes N \oplus \mathcal{O}_X \boxtimes \mathcal{O}_X$, and T_{i+1} is obtained from T_i by adding all sections $(t_1 - t_2)^{-1}a$ where $a \in T_i$ is such that $a|_\Delta \in \Delta^* T_i$ is σ-anti-invariant.

[24] To check $(*)_n$, notice that the pull-back of $T(N, P)_{X^n}$ to the diagonal $x_1 = x_2$ has no non-trivial local sections supported at the diagonal divisor.

Notice that for any surjection $\pi : J \twoheadrightarrow I$ the natural action of the group $\mathrm{Aut}\,(J/I)$ on the $\mathcal{O}_{X^I}$-module $\Delta^{(J/I)*}T_{X^J}$ is trivial.[25] Let us define a natural morphism $\alpha^{(\pi)} = \alpha^{(J/I)} : T_{X^I} \to \Delta^{(J/I)*}T_{X^J}$. Choose a section $s : I \hookrightarrow J$ of π, so we have the structure morphism $\nu^{(s)} : \Delta^{(s)*}T_{X^I} \to T_{X^J}$. Now set $\alpha^{(J/I)} := \Delta^{(J/I)*}(\nu^{(s)})$. Our $\alpha^{(J/I)}$ does not depend on the auxiliary choice of s. The $\alpha^{(\pi)}$ are compatible with the composition of π's in an evident way.

Therefore for any $I \in \mathcal{S}$ we have a functor $(\mathcal{S}/I)^\circ \to \mathcal{M}_{\mathcal{O}}(X^I)$ which assigns to J/I the $\mathcal{O}_{X^I}$-module $\Delta^{(J/I)*}T_{X^J}$ and to an arrow $K \twoheadrightarrow J$ the morphism $\Delta^{(J/I)*}\alpha^{(K/J)}$. Let $B_{X^I} \in \mathcal{M}_{\mathcal{O}}(X^I)$ be the inductive limit.

Our B is a quasi-factorization algebra. Namely, the isomorphisms c for B (see (3.4.5.2)) come from the corresponding isomorphisms of T. For $f : I \to K$ the morphism $\nu^{(f)} : \Delta^{(f)*}B_{X^I} \to B_{X^K}$ (see (3.4.5.1)) comes from the functor $\mathcal{S}/I \to \mathcal{S}/K$, $J/I \mapsto J_f/K := J \sqcup (K \smallsetminus f(I))/K$, and the corresponding morphism of the inductive systems $\Delta^{(f)*}\Delta^{(J/I)*}T_{X^J} = \Delta^{(J_f/K)*}\Delta^{(J/J_f)*}T_{X^J} \to \Delta^{(J_f/K)*}T_{X^{J_f}}$ where the last arrow is the $\Delta^{(J_f/K)}$-pull-back of the structure morphism $\nu^{(J/J_f)}$ for T. We leave it to the reader to check axioms (a)–(d) and (f) in 3.4.5. The universality property of B is evident.

(ii) Let B be a quasi-factorization algebra B. The corresponding universal factorization algebra $\bar{B}$ is the maximal factorization algebra quotient of B. To construct it, consider first the quotient qB of B where qB_{X^I} is the quotient of B_{X^I} modulo the $\mathcal{O}_{X^I}$-submodule generated by all sections of type $\nu^{(J/I)}(a)$ where $J \twoheadrightarrow I$ and $a \in B_{X^J}$ is a local section supported at the diagonal divisor of X^J. We leave it to the reader to check that qB is indeed a quasi-factorization algebra. We get a system of quotients $B \twoheadrightarrow qB \twoheadrightarrow q^2B := q(qB) \twoheadrightarrow \cdots$. Now $\bar{B} = \varinjlim q^i B$. $\square$

3.4.15. Tensor products. We know that $\mathcal{FA}(X)$ is a tensor category (see 3.4.4); hence so is $\mathcal{CA}(X)$. Let us explain the meaning of the tensor product of chiral algebras in purely chiral terms.

Let A_α be a finite family of chiral algebras. Let $A := \otimes A_\alpha \in \mathcal{CA}(X)$ be the corresponding tensor product, so $A^\ell = \otimes A^\ell_\alpha$. By 3.4.4 we have canonical morphisms of chiral algebras $\nu_\alpha : A_\alpha \to A$, $a \mapsto a \otimes (\underset{\alpha' \neq \alpha}{\otimes} 1_{A_{\alpha'}})$.

Let $\varphi_\alpha : A_\alpha \to C$ be morphisms of chiral algebras. We say that φ_α *mutually commute* if $[\varphi_{\alpha'}, \varphi_{\alpha''}]_C \in P^*_2(\{A_{\alpha'}, A_{\alpha''}\}, C)$ vanishes for every $\alpha' \neq \alpha''$. Denote by $\mathrm{Hom}^{com}_I(\{A_\alpha\}, C)$ the set of all mutually commuting (φ_α)'s.

PROPOSITION. *$\{\nu_\alpha\}$ is a universal family of mutually commuting morphisms: for every $C \in \mathcal{CA}(X)$ the map $\mathrm{Hom}(\otimes A_\alpha, C) \to \mathrm{Hom}^{com}_I(\{A_\alpha\}, C)$, $\psi \mapsto (\psi\nu_\alpha)$, is bijective.*

Proof. Let us translate our proposition back to the setting of factorization algebras. Assume for simplicity of notation that we play with two algebras A_1, A_2. Notice that φ_1, φ_2 as above mutually commute if and only if for every $a_1 \in A^\ell_1, a_2 \in A^\ell_2$ the section $\varphi_1(a_1) \boxtimes \varphi_2(a_2)$ of $j_*j^*C^\ell_{X\times X}$ lies in $C^\ell_{X\times X}$. One has $\nu_1(a_1) \boxtimes \nu_2(a_2) = (a_1 \boxtimes 1_{A_1}) \otimes (1_{A_2} \boxtimes a_2) \in A^\ell_{1X\times X} \otimes A^\ell_{2X\times X} = (A^\ell_1 \otimes A^\ell_2)_{X\times X}$, so our $\nu_{1,2}$ mutually commute. For $\varphi_{1,2} \in \mathrm{Hom}^{com}_I(\{A_1, A_2\}, C)$ the Ith component of the corresponding morphism $\psi : A^\ell_1 \otimes A^\ell_2 \to C^\ell$ sends $m_1 \otimes m_2$, where $m_1 \in A^\ell_{1X^I}, m_2 \in A^\ell_{2X^I}$, to the

[25] It suffices to check that the transposition of two elements of some $J_i \subset J$ acts trivially. By $(*)_{|J|}$ it acts trivially already on the pull-back of T_{X^J} to the corresponding codimension 1 diagonal.

pull-back of[26] $\varphi_1(m_1)\boxtimes\varphi_2(m_2)$ by the diagonal map $X^I \hookrightarrow X^I\times X^I$. The details are left to the reader. □

REMARKS. (i) Let $N_\alpha\subset A_\alpha$ be a sub-$\mathcal{O}_X$-module that generates A_α as a chiral algebra. Then $\varphi_{\alpha'}$, $\varphi_{\alpha''}$ mutually commute if (and only if) their restrictions to the N's commute, i.e., the image of $[\varphi_{\alpha'},\varphi_{\alpha''}]_C$ in $P_2^*(\{N_{\alpha'},N_{\alpha''}\},C)$ vanishes.

(ii) Suppose that A_α is freely generated by (N_α,P_α) (see 3.4.14). Then $\otimes A_\alpha$ is freely generated by $(\oplus N_\alpha,(\sum P_\alpha)+\sum\limits_{\alpha'\neq\alpha''}N_{\alpha'}\boxtimes N_{\alpha''})$.

Let M be a $\mathcal{D}_X$-module and $\mu_\alpha\in P_2^{ch}(\{A_\alpha,M\},M)$ are chiral A_α-actions on M. We say that the μ_α *mutually commute* if for every $\alpha\neq\alpha'$ the restriction of $\mu_\alpha(id_{A_\alpha},\mu_{\alpha'})-\mu_{\alpha'}(id_{A_{\alpha'}},\mu_\alpha)\in P_3^{ch}(\{A_\alpha,A_{\alpha'},M\},M)$ to the localization of $A_\alpha\boxtimes A_{\alpha'}\boxtimes M$ with respect to the diagonals $x_1=x_3$ and $x_2=x_3$ vanishes.

EXERCISES. (i) Show that for every $\otimes A_\alpha$-action on M the corresponding A_α-actions mutually commute, and this provides 1-1 correspondence between the sets of $\otimes A_\alpha$-module structures and collections of mutually commuting A_α-actions.

(ii) Let M_α be A_α-modules. Then the A_α-actions on $\otimes M_\alpha$ mutually commute, so $\otimes M_\alpha$ is a $\otimes A_\alpha$-module.[27]

3.4.16. Hopf algebras. A *Hopf chiral algebra* is a coassociative coalgebra object in the tensor category $\mathcal{CA}(X)$, i.e., a chiral algebra F equipped with a coassociative morphism of chiral algebras $\delta=\delta_F:F\to F\otimes F$. As a part of general tensor category story, we know what a counit $\epsilon_F:F\to\omega$ is, and what it means for F to be cocommutative. We also know what a coaction of F on any chiral algebra A is (this is a morphism $\delta=\delta_A:A\to F\otimes A$ satisfying a certain compatibility property with δ_F), when such a coaction is counital and when it is trivial.

For F, A as above set $A^F:=\{a\in A:\delta(a)=1_F\otimes a\}$. This is the maximal chiral subalgebra of A on which the coaction of F is trivial.

We denote by $\mathcal{CA}(X)^F$ the category of chiral algebras A equipped with a coaction of F; let $\mathcal{CA}(X)^F_{fl}\subset\mathcal{CA}(X)^F$ be the full subcategory of $\mathcal{O}_X$-flat algebras. If F is counital, then $\mathcal{CA}(X)^F_{cu}\subset\mathcal{CA}(X)^F$ is the full subcategory of those A that δ_A is counital and $\mathcal{CA}(X)^F_{cu,fl}:=\mathcal{CA}(X)^F_{cu}\cap\mathcal{CA}(X)^F_{fl}$.

The tensor product of Hopf chiral algebras is naturally a Hopf chiral algebra. If F_α are Hopf chiral algebras and $A_\alpha\in\mathcal{CA}(X)^{F_\alpha}$, then $\otimes A_\alpha\in\mathcal{CA}(X)^{\otimes F_\alpha}$.

REMARK. If M, N are F-modules, then $M\otimes N$ is again an F-module via δ_F and Exercise (ii) in 3.4.15. Thus $\mathcal{M}(X,F)$ is a monoidal category; it is a tensor category if F is cocommutative.

Suppose F is an $\mathcal{O}_X$-flat cocommutative Hopf chiral algebra. Then $\mathcal{CA}(X)^F_{fl}$ is a tensor category. Namely, for $A_\alpha\in\mathcal{CA}(X)^F_{fl}$ their tensor product $\overset{F}{\otimes}A_\alpha$ is the chiral subalgebra of $\otimes A_\alpha$ that consists of those sections a of $\otimes A_\alpha$ that all $\delta_\alpha(a):=(\delta_{A_\alpha}\otimes(\underset{\alpha'\neq\alpha}{\otimes}id_{A_{\alpha'}}))(a)\in F\otimes(\otimes A_\alpha)$ coincide. If F is counital, then $\mathcal{CA}(X)^F_{cu,fl}$ is a tensor subcategory of $\mathcal{CA}(X)^F_{fl}$, and F is naturally a unit object in $\mathcal{CA}(X)^F_{cu,fl}$. Invertible objects of $\mathcal{CA}(X)^F_{cu,fl}$ are called F-*cotorsors*; they form a Picard groupoid $\mathcal{P}(F)=\mathcal{P}(X,F)$.

For examples see 3.7.12, 3.7.20, and 3.10.

[26] This section belongs to $C^\ell_{X^I\times X^I}$: use 3.4.4 and the fact that $\varphi_{1,2}$ mutually commute.

[27] This also follows from the interpretation of A-modules in factorization terms, see 3.4.19.

3.4.17. Twists. In the following constructions we use only tensor products of an arbitrary chiral algebra and a commutative chiral algebra, so they can also be performed in the context of 3.3.10 (see Exercise in 3.4.20 below).

Suppose that a Hopf chiral algebra F is commutative (as a chiral algebra). Such F is the same as a semigroup $\mathcal{D}_X$-scheme $G = \operatorname{Spec} F$ affine over X. We call the F-coaction on A a G-action and write $A^G := A^F$. If A itself is commutative, then a G-action on A is the same as a G-action on $\operatorname{Spec} A^\ell$ in the category of $\mathcal{D}_X$-schemes, i.e., a scheme G-action compatible with connections. If G acts on two chiral algebras, then it acts naturally on their tensor product (via the diagonal embedding $G \hookrightarrow G \times G$), so the category of chiral algebras equipped with a G-action is a tensor category.

Suppose G is a *group* $\mathcal{D}_X$-scheme. Let $P = \operatorname{Spec} R^\ell$ be a $\mathcal{D}_X$-scheme G-torsor (i.e., a G-torsor equipped with a connection compatible with the G-action). For a chiral algebra A equipped with a G-action its *P-twist* is the chiral algebra $A(P) := (R \otimes A)^G$.

REMARKS. (i) If G acts on A trivially, then $A(P) = A$.

(ii) As a plain $\mathcal{D}$-module, $A(P)$ is the P-twist of the $\mathcal{D}$-module A; in particular, any section $s : X \to P$ defines an isomorphism of $\mathcal{O}$-modules $i_s : A \xrightarrow{\sim} A(P)$. If s is horizontal, then i_s is an isomorphism of chiral algebras.

(iii) Denote by G^∇ the sheaf of horizontal sections of G; this is a sheaf of groups on $X_{\acute{e}t}$. Suppose that P admits locally a horizontal section; then the sheaf of horizontal sections P^∇ is a G^∇-torsor. Notice that G^∇ acts on A, so we have the corresponding P^∇-twisted chiral algebra $A(P^\nabla)$. The isomorphisms i_s, $s \in P^\nabla$, provide a canonical identification of chiral algebras

$$A(P^\nabla) \xrightarrow{\sim} A(P). \tag{3.4.17.1}$$

(iv) Let H be a group X-scheme, and suppose that $G = \mathcal{J}H$ (the group jet $\mathcal{D}_X$-scheme; see 2.3.2). There is a canonical equivalence of groupoids

$$\{H\text{-torsors}\} \xrightarrow{\sim} \{\mathcal{D}_X\text{-scheme } G\text{-torsors}\} \tag{3.4.17.2}$$

which assigns to an H-torsor Q on X the $\mathcal{D}_X$-scheme G-torsor of jets $P = \mathcal{J}Q$; its inverse is the push-out functor for the canonical homomorphism of the group X-schemes $G = \mathcal{J}H \to H$. Notice that the canonical projections $G \to H$, $P \to Q$ identify the sheaves of horizontal sections of left-hand sides with those of arbitrary sections of right-hand sides, so, by (3.4.17.1), the P-twist can be interpreted as the plain twist by the torsor of sections of Q.

(v) If G is commutative, A and B are two chiral algebras equipped with G-actions, then the tensor product $A \overset{F}{\otimes} B$ from 3.4.16 coincides with invariants in $A \otimes B$ of the action of the anti-diagonal subgroup $G \hookrightarrow G \times G$, $g \mapsto (g, g^{-1})$.

3.4.18. Factorization modules. Let us describe modules over chiral algebras in factorization terms. Below for a finite set I we set $\tilde{I} := I \sqcup \cdot$, so $X^{\tilde{I}} = X \times X^I$.

Let B be a factorization algebra. A *factorization B-module* is a triple $B(M) = (B(M), \tilde{\nu}, \tilde{c})$ where:

(i) $B(M)$ is a rule that assigns to every finite set I a left $\mathcal{D}$-module $B(M)_{X^{\tilde{I}}}$ on $X^I \times X$. We demand that $B(M)_{X^{\tilde{I}}}$ have no sections supported at the diagonal divisors. Set $M := B(M)_X$.

(ii) $\tilde{\nu}$ assigns to every $\pi : \widetilde{J} \twoheadrightarrow \widetilde{I}$ preserving the $\cdot$'s an identification

$$\tilde{\nu}^{(\pi)} : \Delta^{(\pi)*}B(M)_{X^{\tilde{J}}} \xrightarrow{\sim} B(M)_{X^{\tilde{I}}}. \tag{3.4.18.1}$$

We demand that the $\tilde{\nu}^{(\pi)}$'s are compatible with the composition of the π's.

(iii) Consider a surjection $\widetilde{J} \twoheadrightarrow \widetilde{I}$ preserving the $\cdot$'s. Then $\widetilde{J}$ is the disjoint union of the preimage subsets $\widetilde{J}.$ and J_i, $i \in I$. Our $\tilde{c}$ assigns to such a surjection an isomorphism of $\mathcal{D}_{U[\tilde{J}/\tilde{I}]}$-modules[28]

$$\tilde{c}_{[\widetilde{J}/\widetilde{I}]} : j^{[\widetilde{J}/\widetilde{I}]*}((\underset{I}{\boxtimes} B_{X^{J_i}})) \boxtimes B(M)_{X^{\widetilde{J}.}}) \xrightarrow{\sim} j^{[\widetilde{J}/\widetilde{I}]*}B(M)_{X^J}. \tag{3.4.18.2}$$

We demand that the $\tilde{c}$'s be mutually compatible: for $\tilde{K} \twoheadrightarrow \widetilde{J}$ the isomorphism $\tilde{c}_{[\tilde{K}/\tilde{J}]}$ coincides with the composition $\tilde{c}_{[\tilde{K}/\tilde{I}]}((\boxtimes c_{[K_i/J_i]}) \boxtimes \tilde{c}_{\tilde{K}./\tilde{J}.})$ where c are the factorization isomorphisms for B. They should also be compatible with the isomorphisms $\tilde{\nu}$ (and the isomorphisms ν for B) in the obvious way.[29] Finally, for every $m \in M$ the section $\tilde{c}(1 \boxtimes m) \in j_*j^*B_X \boxtimes M$ should belong to $B(M)_{X \times X}$ and one should have $\tilde{\nu}((1 \boxtimes m)|_\Delta) = m$.

As in 3.4.4, the above $\tilde{c}_{[\widetilde{I}/\cdot]}$ define an embedding $B(M)_{X^{\tilde{I}}} \hookrightarrow j_*^{(\widetilde{I})}j^{(\widetilde{I})*}(B_X^{\boxtimes I} \boxtimes M)$, so the functor $B(M) \mapsto M$ is fully faithful. We call $B(M)$ a *factorizatization B-module structure* on the $\mathcal{D}_X$-module M.

Remark. Let $\ell : V \hookrightarrow X^{\tilde{I}}$ be the complement to the diagonal strata of codimension 2. As in 3.4.3, one shows that $M_{X^{\tilde{I}}} \xrightarrow{\sim} \ell_*\ell^*M_{X^{\tilde{I}}}$.

3.4.19. We denote the category of factorization B-modules by $\mathcal{M}^\ell(X,B)$. This is an abelian k-category. As in 3.3.4, $\mathcal{M}(X,B^r)$ is the category of chiral B^r-modules.

Proposition. *There is a canonical equivalence of categories*

$$\mathcal{M}^\ell(X,B) \xrightarrow{\sim} \mathcal{M}(X,B^r). \tag{3.4.19.1}$$

Proof. Our functor assigns to $B(M) \in \mathcal{M}^\ell(X,B)$ the right $\mathcal{D}_X$-module M^r equipped with a chiral operation $\mu_M \in P_2^{ch}(\{B^r, M^r\}, M^r)$ defined as the composition $j_*j^*B^r \boxtimes M^r \xrightarrow{\sim} j_*j^*B(M)^r_{X\times X} \to j_*j^*B(M)^r_{X\times X}/B(M)^r_{X\times X} \xrightarrow{\sim} \Delta_*M^r$ where the first and the last arrow come from the appropriate isomorphisms $\tilde{c}$ and $\tilde{\nu}$. As in 3.4.8, one checks immediately that μ_M is a B^r-module structure on M^r.

Let us construct the functor in inverse direction. According to 3.3.5(i), for a chiral B^r-module M^r the $\mathcal{D}_X$-module $B^r \oplus M^r$ is naturally a chiral algebra; so $F := B \oplus M$ carries a factorization algebra structure. For a finite set I let $B(M)_{X^{\tilde{I}}} \subset F_{X^{\tilde{I}}}$ be the submodule of sections which belong to $j_*^{(\tilde{I})}j^{(\tilde{I})*}B^{\boxtimes I} \boxtimes M \subset j_*^{(\tilde{I})}j^{(\tilde{I})*}F_{X^{\tilde{I}}}$. The isomorphisms $\tilde{\nu}$ and $\tilde{c}$ come from the corresponding isomorphisms of the factorization structure on F. It is clear that $(B(M), \tilde{\nu}, \tilde{c})$ is a factorization B-module structure on M.

It is clear (use the remark in 3.4.18 and the construction of F in 3.4.12) that the above functors are mutually inverse, and we are done. □

[28]See 3.4.4 for notation.

[29]I.e., in the same way as the c are compatible with ν; see 3.4.4.

REMARK. The definition of 3.4.19 makes sense if we take for an $M^\ell_{X^{\bar I}}$ an $\mathcal{O}_X \boxtimes \mathcal{D}_{X^I}$-module instead of a $\mathcal{D}_{X^{\bar I}}$-module. The above argument shows that the corresponding category is equivalent to $\mathcal{M}_{\mathcal{O}}(X, B^r)$ (see 3.3.5(ii)).

Let us describe chiral B^r-operations (see 3.3.4) in factorization terms.

Suppose we have a finite family of chiral B^r-modules $\{M_i^r\}$, $i \in I$. Then, as in 3.3.5(i), $B^r \oplus (\oplus M_i^r)$ is a chiral algebra; let G be the corresponding factorization algebra. For a finite set J let $B(\{M_i\})_{X^{J\sqcup I}} \subset G_{X^{J\sqcup I}}$ be the submodule of sections whose restriction to $U^{(J\sqcup I)}$ belongs to $B^J \boxtimes (\boxtimes M_i)$. Equivalently, $B(\{M_i\})_{X^{J\sqcup I}}$ is the submodule of $j_*^{(J\sqcup I)} j^{(J\sqcup I)*} B^{\boxtimes J} \boxtimes (\boxtimes M_i)$ which consists of sections killed by the morphisms $\mu_{j,i} : j_*^{(J\sqcup I)} j^{(J\sqcup I)*} B^{\boxtimes J} \boxtimes (\boxtimes M_i) \to \Delta_*^{j=i} j_*^{(\bar J\sqcup I)} j^{(\bar J\sqcup I)*} B^{\boxtimes \bar J} \boxtimes (\boxtimes M_i)$ for all $j \in J$ and $i \in I$; here $\bar J := J \setminus \{j\}$ and $\mu_{j,i}$ is the chiral B^r-action on M_i tensored by the identity map at the variables $\neq j, i$. So $B(\{M_i\})_{X^I} = j_*^{(I)} j^{(I)*} \boxtimes M_i$, etc. This module satisfies the usual factorization properties (inherited from G). In particular, let $j^{[J,I]} : U^{[J,I]} \hookrightarrow X^{J\sqcup I}$ be the complement to all the diagonals $x_j = x_i$, $j \in J$, $i \in I$; then $j^{[J,I]*} B(\{M_i\})_{X^{J\sqcup I}} = j^{[J,I]*}(B_{X^J} \boxtimes j_*^{(I)} j^{(I)*} \boxtimes M_i)$, so

$$B(\{M_i\})_{X^{J\sqcup I}} \hookrightarrow j_*^{[J,I]} j^{[J,I]*}(B_{X^J} \boxtimes j_*^{(I)} j^{(I)*} \boxtimes M_i). \tag{3.4.19.2}$$

Let N^r be another B^r-module and $\varphi \in P_I^{ch}(\{M_i^r\}, N^r)$ be a chiral operation. For any J set $\varphi_J := id_{B_{X^J}} \boxtimes \varphi : B_{X^J} \boxtimes j_*^{(I)} j^{(I)*} \boxtimes M_i \to B_{X^J} \boxtimes \Delta_*^{(I)} N$.

LEMMA. *If φ is a chiral B^r-operation, then for every finite set J the morphism φ_J sends $B(\{M_i\})_{X^{J\sqcup I}}$ to $(id_{X^J} \times \Delta^{(I)})_* B(N)_{X^{\bar J}} \subset j_*^{[J,I]} j^{[J,I]*} B_{X^J} \boxtimes \Delta_*^{(I)} N$. Conversely, if this condition is satisfied for $J = \cdot$, then φ is a chiral B^r-operation.* $\square$

3.4.20. We are going to describe the factorization algebra that corresponds to a commutative chiral algebra in geometric terms (see 3.4.22). We start with some technical preliminaries.

A factorization algebra is said to be *commutative* if the corresponding chiral algebra is commutative.

PROPOSITION. (i) *The next properties of a factorization algebra B are equivalent:*

(a) *B is commutative;*

(b) *for every $J \twoheadrightarrow I$ the isomorphism (3.4.4.1) extends to a morphism*

$$\underset{I}{\boxtimes} B_{X^{J_i}} \to B_{X^J}; \tag{3.4.20.1}$$

(c) *same as (b) but in the particular case $J = \{1,2\}$, $I = \{1\}$; i.e., the factorization isomorphism $j^*(B_X \boxtimes B_X) \xrightarrow{\sim} j^* B_{X^2}$ extends to a morphism*

$$B_X \boxtimes B_X \to B_{X^2}; \tag{3.4.20.2}$$

(d) *there exists a morphism of factorization algebras $m : B \otimes B \to B$ whose restrictions to $B \otimes 1 \subset B \otimes B$ and $1 \otimes B \subset B \otimes B$ are the identity morphisms.*

(ii) *The morphism m mentioned in (d) is unique: the corresponding morphism $m_X : B_X \otimes B_X \to B_X$ is the multiplication on B_X corresponding to the commutative chiral structure on $B_X^r = B_X \otimes \omega_X$. For every I the morphism $m_{X^I} : B_{X^I} \otimes B_{X^I} \to B_{X^I}$ defines a commutative $\mathcal{D}_{X^I}$-algebra structure on B_{X^I}.*

(iii) *If B is a commutative factorization algebra, then the multiplication morphism $B_X \otimes B_X \to B_X$ corresponding to the $\mathcal{D}_X$-algebra structure on B_X is the restriction of (3.4.20.2) to the diagonal $X \hookrightarrow X^2$.*

Proof. Clearly (b)⇒(c)⇔(a). To see that (c)⇒(b), use 3.4.3(ii). To prove that (d)⇒(c), define the morphism (3.4.20.2) to be the composition of $f : B_X \boxtimes B_X \to B_{X^2} \otimes B_{X^2}$ and $m_{X^2} : B_{X^2} \otimes B_{X^2} \to B_{X^2}$, where $f(b_1 \otimes b_2) := (b_1 \boxtimes 1) \otimes (1 \boxtimes b_2)$.

Statement (iii) follows from the definitions. Now let us show that (a)⇒(d). If B is commutative, then $B \otimes B$ is commutative (use the equivalence (a)⇔(c)). Applying (iii) to $B \otimes B$, we see that the multiplication in $B_X \otimes B_X = (B \otimes B)_X$ coming from the factorization structure on $B \otimes B$ is the usual tensor product multiplication (i.e., $(b_1 \otimes b_2)(b_3 \otimes b_4) = b_1 b_3 \otimes b_2 b_4$). So we have the $\mathcal{D}_X$-algebra morphism $B_X \otimes B_X \to B_X$, $b_1 \otimes b_2 \mapsto b_1 b_2$. The corresponding morphism of factorization algebras $m : B \otimes B \to B$ has the property mentioned in (d).

To prove (ii) consider any $m : B \otimes B \to B$ satisfying the property mentioned in (d). Then $m_X : B_X \otimes B_X \to B_X$ is a $\mathcal{D}_X$-algebra morphism such that $b \otimes 1 \mapsto b$ and $1 \otimes b \mapsto b$. So there is only one possibility for m_X and therefore for m. The rest of (ii) is obvious. □

Exercise. According to 3.3.10 (or, equivalently, by 3.1.3 and (3.3.3.1)) we know what the tensor product of any chiral algebra and a *commutative* chiral algebra is. Show that it coincides with the tensor product of these chiral algebras in the sense of 3.4.15.

3.4.21. If B is a commutative factorization algebra, then B_{X^I} is a commutative $\mathcal{D}_{X^I}$-algebra for every I (see (ii) in the proposition in 3.4.20). By 3.4.9, B_{X^I} can be uniquely reconstructed from B_X. We are going to explain the geometric meaning of the $\mathcal{D}_{X^I}$-scheme $\operatorname{Spec} B_{X^I}$ in terms of the $\mathcal{D}_X$-scheme $Y := \operatorname{Spec} B_X$. We will show that $\operatorname{Spec} B_{X^I}$ is the scheme Y_{X^I} of horizontal infinite multijets of Y.

First we have to define Y_{X^I}. A k-point of Y_{X^I} is a pair consisting of a point $x = (x_i)_{i \in I} \in X^I$ and a horizontal section $s : \hat{X}_x \to Y$, where $\hat{X}_x$ is the formal completion of X at the subset $\{x_i\} \subset X$. One defines R-points of Y_{X^I} quite similarly (if $x = (x_i)_{i \in I} \in X^I(R)$, then $\hat{X}_x$ is the formal completion of $X \otimes R$ along the union of the graphs of $x_i : \operatorname{Spec} R \to X$). Therefore we have a functor $\{k\text{-algebras}\} \to \{\text{sets}\}$, and we claim that it is representable by a scheme Y_{X^I} affine over X^I. It is enough to prove this if $Y = \operatorname{Spec} \operatorname{Sym}(\mathcal{D}_X \otimes V)$ where V is a vector space. In this case $Y_{X^I} = \operatorname{Spec} \operatorname{Sym}(L \otimes V)$, $L := \bigcup_N (\pi_* \mathcal{O}_{X \times X^I} / \mathcal{J}^N)^*$, where $\pi : X \times X^I \to X^I$ is the projection to the second factor and $\mathcal{J} \subset \mathcal{O}_{X \times X^I}$ is the product of the ideals of the subschemes $x = x_i$ of $X \times X^I$, $i \in I$ (we use the fact that the sheaves $\pi_* \mathcal{O}_{X \times X^I} / \mathcal{J}^N$ are locally free[30]).

Remarks. (to be used in 3.4.22) (i) We have shown that if $Y = \operatorname{Spec} \operatorname{Sym}(M)$ and M is a free $\mathcal{D}_X$-module, then Y_{X^I} is flat over X^I.

(ii) Suppose that a group scheme G over k acts on the $\mathcal{D}_X$-scheme Y. Then it acts on Y_{X^n}. Let $Y^G \subset Y$ be the subscheme of fixed points of G. Applying the above multijet construction to Y^G instead of Y, we get a $\mathcal{D}_{X^n}$-scheme $(Y^G)_{X^n}$. Then the morphism $(Y^G)_{X^n} \to Y_{X^n}$ induces an isomorphism $(Y^G)_{X^n} \to (Y_{X^n})^G$ (this follows immediately from the definitions).

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[30]This is not true if $\dim X > 1$, so it is not clear if our multijet functor is representable in this case.

(iii) The above construction has étale local origin with respect to Y_X, so it makes sense for an arbitrary (not necessary affine) algebraic $\mathcal{D}_X$-space Y_X.

The X^I-scheme Y_{X^I} has a natural structure of the $\mathcal{D}_{X^I}$-scheme. Indeed, the fibers of Y_{X^I} over infinitely close R-points $x, \tilde{x}$ of X^I are naturally identified because the completions $\hat{X}_x$ and $\hat{X}_{\tilde{x}}$ coincide ("infinitely close" means that the restrictions of $x, \tilde{x} : \operatorname{Spec} R \to X$ to the reduced part of $\operatorname{Spec} R$ coincide). So $Y_{X^I} = \operatorname{Spec} A_{X^I}$ for some $\mathcal{D}_{X^I}$-algebra A_{X^I}.

Notice that $Y_X = Y$ and therefore $A_X = B_X$. We will show that $A_{X^I} = B_{X^I}$. As a first step, let us show that A has a structure of the quasi-factorization algebra (see 3.4.14). Recall that this structure consists of morphisms $\nu^{(\pi)}$ and $c_{[\pi]}$ of (3.4.5.1) and (3.4.5.2) subject to axioms (a)–(d) and (f) from 3.4.5.

Suppose we have $\pi : J \to I$. It yields a map $\Delta^{(\pi)} : X^I \to X^J$ and a $\mathcal{D}_{X^I}$-morphism $Y_{X^I} \to Y_{X^J} \times_{X^J} X^I$ (restricting a horizontal section of Y over a formal neighbourhood of a finite subscheme to a formal neighbourhood of a smaller subscheme). We get the morphism $\nu^{(\pi)}$ of (3.4.5.1). We also have a similarly defined $\mathcal{D}_{X^J}$-morphism $Y_{X^J} \to \prod_I Y_{X^{J_i}}$ which is an isomorphism over $U^{[\pi]}$, and therefore the isomorphism $c_{[\pi]}$ of (3.4.5.1). The axioms (a)–(d) and (f) from 3.4.5 are evident, so A is a quasi-factorization algebra.

3.4.22. THEOREM. *There is a (unique) isomorphism of quasi-factorization algebras $A \xrightarrow{\sim} B$ such that the corresponding isomorphism $A_X \xrightarrow{\sim} B_X$ is the identity.*

Proof. (i) If we know that A is a factorization algebra (i.e., that A_{X^I} has no non-zero sections supported at the diagonal divisor), then $A = B$. Indeed, A is equipped with a morphism $m : A \otimes A \to A$ satisfying the conditions (i)(d) of the proposition in 3.4.20 and such that $m_X : A_X \otimes A_X \to A_X$ is the multiplication in $B_X = A_X$. So, by (ii) in the proposition in 3.4.20, the identity map $A_X \xrightarrow{\sim} B_X$ is an isomorphism of $\mathcal{D}_X$-algebras. Therefore it extends in a unique way to an isomorphism $A \xrightarrow{\sim} B$ of factorization algebras.

(ii) If B_X is a free commutative $\mathcal{D}_X$-algebra, then A_{X^I} has no non-zero sections supported at the diagonal divisor (see Remark (i) from 3.4.21) and therefore $A = B$.

(iii) Everything said above in 3.4.20–3.4.22 renders itself to the super setting and to the DG setting. So if B_X is a free commutative differential graded $\mathcal{D}_X$-algebra, then $A = B$ (as usual, "free" means "free as a $\mathbb{Z}$-graded commutative super $\mathcal{D}_X$-algebra").

(iv) The statement of the theorem is local, so we can assume that X is affine. Then B_X has a free DG resolution; i.e., there is a free commutative differential graded $\mathcal{D}_X$-algebra $\tilde{B}_X$ placed in degrees ≤ 0 equipped with a quasi-isomorphism $\tilde{B}_X \to B_X$ (we consider B_X as a DG algebra placed in degree 0). Applying the DG version of the theorem in 3.4.9, we see that $\tilde{B}_X$ extends to a factorization DG algebra $\tilde{B} = \{\tilde{B}_{X^I}\}_{I \in \mathcal{S}}$. Applying to $\tilde{B}_X$ the multijet construction from 3.4.21, we get a quasi-factorization DG algebra $\tilde{A}$ equipped with a morphism $\tilde{A} \to A$. As explained in (iii), $\tilde{A} = \tilde{B}$. For every I the morphism $\tilde{B}_{X^I} \to B_{X^I}$ is a quasi-isomorphism (see Remark (ii) in 3.4.4). So it remains to show that the morphism

$$H^0(\tilde{A}_{X^I}) \to A_{X^I} \tag{3.4.22.1}$$

is an isomorphism. Indeed, according to 1.1.16 a DG algebra is the same as a super algebra with an action of the semi-direct product G of $\mathbb{G}_m$ and the super group H

with Lie algebra $k[-1]$ ($k[-1]$ is odd and $\lambda \in \mathbb{G}_m$ acts on $k[-1]$ as multiplication by λ); the action of G defines the grading and the action of H defines the differential. Since A_{X^I} is placed in non-positive degrees, $\operatorname{Spec} H^0(\tilde{A}_{X^I}) = (\operatorname{Spec} \tilde{A}_{X^I})^G$. So the super version of Remark (ii) from 3.4.21 shows that (3.4.22.1) is indeed an isomorphism. □

3.4.23. Let M_X^ℓ be a left $\mathcal{D}_X$-module. Then $B_X := \operatorname{Sym} M_X^\ell$ is a commutative $\mathcal{D}_X$-algebra. We will give an explicit description of the corresponding factorization algebra $B = \{B_{X^I}\}_{I \in \mathcal{S}}$.

Let $Z_I \subset X \times X^I$ be the union of the subschemes $x = x_i$, $i \in I$ (notice that Z_I is singular if $|I| > 1$). We have the projections $\pi_I : Z_I \to X$ and $p_I : Z \to X^I$. Put

$$M_{X^I} := p_{I*}\pi_I^\dagger M_X = (p_I)_* \pi_I^! M_X[1-n], \quad n := |I|. \tag{3.4.23.1}$$

Here we use the notion of a $\mathcal{D}$-module on a singular scheme (see Remark in 2.1.3) and notation from 2.1.4. Notice that since Z_I *naturally* appears as a subscheme of the smooth scheme $X \times X^I$, one can easily reformulate the definition of M_{X^I} using only $\mathcal{D}$-modules on smooth schemes. Namely, one has

$$M_{X^I} = p_*(j_* j^*(M_X \boxtimes \omega_{X^I})/(M_X \boxtimes \omega_{X^I})) \tag{3.4.23.2}$$

where $j : X \times X^I \smallsetminus Z_I \hookrightarrow X \times X^I$ and $p : X \times X^I \to X^I$ is the projection.

According to the proposition in 2.1.4, $\pi_I^\dagger M_X$ is a $\mathcal{D}$-module (not merely a complex of $\mathcal{D}$-modules). Since π_I is finite, M_{X^I} is also a $\mathcal{D}$-module. On the complement $U^{(I)}$ of the diagonal divisor of X^I our M_{X^I} is canonically isomorphic to $\sum \operatorname{pr}_i^\dagger M_X$, where $\operatorname{pr}_i : X^I \to X$ is the ith projection. So on $U^{(I)}$ we have a canonical isomorphism between $\operatorname{Sym} M_{X^I}^\ell$ and $B_X^{\boxtimes I}$.

Proposition. *This isomorphism comes from a unique isomorphism of $\mathcal{D}_{X^I}$-modules*

$$B_{X^I} = \operatorname{Sym}(M_{X^I}^\ell). \tag{3.4.23.3}$$

Proof. We use notation from 3.4.5. For any $f : J \to I$ we have $\Delta^{(f)} : X^I \to X^J$. Then $\Delta^{(f)!} M_{X^J} = (p_I)_* \alpha_* \alpha^! \pi_I^! M_X[1-n]$, $n := |J|$, where α is the embedding $Z_{I'} \times X^{I \setminus I'} \hookrightarrow X^I$ and $I' := f(I)$. So we get a canonical morphism

$$\nu^{(f)} : \Delta^{(f)*} M_{X^J}^\ell \to M_{X^I}^\ell. \tag{3.4.23.4}$$

We also have a canonical morphism $\Sigma pr_i^* M_{X^{J_i}}^\ell \to M_{X^J}^\ell$ where $\operatorname{pr}_i : X^J \to X^{J_i}$ is the projection. Its restriction to $U^{[f]}$ (see 3.4.5) is an isomorphism

$$c_{[f]} : j^{[f]*} \sum \operatorname{pr}_i^* M_{X^{J_i}}^\ell \xrightarrow{\sim} j^{[f]*} M_{X^J}^\ell. \tag{3.4.23.5}$$

Set $\tilde{B}_{X^I} := \operatorname{Sym} M_{X^I}^\ell$. The (3.4.23.4) and (3.4.23.5) yield morphisms (3.4.5.1) and (3.4.5.2) for $\tilde{B}$ which evidently satisfy properties (a)–(f) in 3.4.5.[31] So $\tilde{B}$ is a commutative factorization algebra. By (ii) in the proposition in 3.4.20 the corresponding commutative $\mathcal{D}_X$-algebra is $B_X := \operatorname{Sym} M_X^\ell$ equipped with the usual multiplication. □

[31] Axiom (b) holds since (3.4.23.4) for surjective f is an isomorphism; axiom (e) holds since $M_{X^I}^\ell$ has no non-zero sections supported at the diagonal divisor.

3.4.24. Let M_X be a $\mathcal{D}_X$-module, M_{X^I} the $\mathcal{D}_{X^I}$-module defined by (3.4.23.1). Consider the $\mathcal{D}_X$-scheme $Y := \mathbb{V}(M_X^\ell) := \operatorname{Spec}\operatorname{Sym}(M_X^\ell)$ and denote by Y_{X^I} the corresponding multijet $\mathcal{D}_{X^I}$-scheme defined in 3.4.21.

PROPOSITION. *One has $Y_{X^I} = \mathbb{V}(M_{X^I}^\ell)$. More precisely, the canonical isomorphism $Y_{X^I} \times_{X^I} U^{(I)} \xrightarrow{\sim} (Y_X)^I \times_{X^I} U^{(I)} \xrightarrow{\sim} \mathbb{V}(M_{X^I}^\ell) \times_{X^I} U^{(I)}$ over the complement $U^{(I)} \subset X^I$ of the diagonal divisor extends (uniquely) to an isomorphism $Y_{X^I} \xrightarrow{\sim} \mathbb{V}(M_{X^I}^\ell)$.*

Proof. This is an immediate consequence of 3.4.22 and 3.4.23. Here is a direct proof.

An R-point of Y_{X^I} is a pair consisting of $x = (x_i) \in X(R)^I$ and an $R \otimes \mathcal{D}_X$-linear morphism $R \otimes M_X^\ell \to \varprojlim \mathcal{O}_{X\otimes R}/\mathcal{O}_{X\otimes R}(-nD_x)$, where D_x is the sum of the graphs of $x_i : \operatorname{Spec} R \to X$ considered as divisors on $X \otimes R$. Put $L_x := (j_x)_* j_x^*(\omega_X \otimes R)/(\omega_X \otimes R)$, where j_x is the embedding $(X \otimes R) \setminus D_x \hookrightarrow X \otimes R$. An $\mathcal{O}_X \otimes R$-morphism $f : R \otimes M_X^\ell \to \varprojlim \mathcal{O}_{X\otimes R}/\mathcal{O}_{X\otimes R}(-nD_x)$ is the same as an R-morphism $\varphi : (p_R)_\cdot(L_x \otimes_{\mathcal{O}_X} M_X^\ell) \to R$ where $p_R : X \otimes R \to \operatorname{Spec} R$ is the projection and $(p_R)_\cdot$ is the sheaf-theoretic direct image (φ is the composition of the morphism $(p_R)_\cdot(L_x \otimes_{\mathcal{O}_X} M_X^\ell) \to (p_R)_\cdot L_x$ induced by f and the "sum of residues" morphism $(p_R)_\cdot L_x \to R$). An $\mathcal{O}_X \otimes R$-linear morphism $f : R \otimes M_X^\ell \to \varprojlim \mathcal{O}_{X\otimes R}/\mathcal{O}_{X\otimes R}(-nD_x)$ is $\mathcal{D}_X \otimes R$-linear if and only if the corresponding $\varphi : (p_R)_\cdot(L_x \otimes_{\mathcal{O}_X} M_X^\ell) \to R$ factors through $(p_R)_\cdot(L_x \otimes_{\mathcal{D}_X} M_X^\ell)$.

So R-points of Y_{X^I} bijectively correspond to R-morphisms $P_x \to R$, where $P_x := (p_R)_\cdot(L_x \otimes_{\mathcal{D}_X} M_X^\ell)$; i.e.,

$$P_x = (p_R)_\cdot(\omega_X \otimes_{\mathcal{D}_X} N_x^\ell), \quad N_x^\ell := j_{x*} j_x^*(M_X^\ell \otimes R)/(M_X^\ell \otimes R). \tag{3.4.24.1}$$

On the other hand, an R-point of $\mathbb{V}(M_{X^I}^\ell)$ is a pair consisting of $x = (x_i) \in X(R)^I$ and an R-linear morphism $M_x^\ell \to R$ where M_x^ℓ is the x-pull-back of the $\mathcal{O}_{X^I}$-module $M_{X^I}^\ell$ to $\operatorname{Spec} R$. We claim that $M_x^\ell = P_x$. To show this, notice that D_x is the x-pull-back of the closed subscheme $Z_I \subset X \times X^I$ from 3.4.23. According to (3.4.23.2) one has

$$M_{X^I}^\ell = p_* N^\ell = p_\cdot(\omega_X \otimes_{\mathcal{D}_X} N^\ell), \quad N^\ell := j_* j^*(M_X^\ell \boxtimes \mathcal{O}_{X^I})/(M_X^\ell \boxtimes \mathcal{O}_{X^I}).$$

Comparing this description of $M_{X^I}^\ell$ with (3.4.24.1), we see that $M_x^\ell = P_x$.

So we have constructed an X-isomorphism $Y_{X^I} \xrightarrow{\sim} \mathbb{V}(M_{X^I}^\ell)$. It is easy to see that it is a $\mathcal{D}_{X^I}$-isomorphism and that it induces the required isomorphism over the complement of the diagonal divisor. □

3.5. Operator product expansions

In this section we identify a chiral algebra structure on a $\mathcal{D}$-module with a commutative and associative ope product. Here ope stands for "operator product expansion". This shows that our chiral algebras are related to their namesake from mathematical physics and - in the translation equivariant setting - amount to vertex algebras (see 0.15). This approach uses non-quasi-coherent $\mathcal{D}$-modules (referred to as "$\mathcal{D}$-sheaves" below), which requires some care. The ope set-up is convenient for writing formulas and can be considered as a direct generalization of the notion of the $\mathcal{D}_X$-algebra (see 2.3). It is also very convenient in the $(\mathfrak{g}, K)$-modules setting (see 3.5.15).

3.5.1. Let us begin with some preliminary considerations. For a smooth variety P a $\mathcal{D}_P$*-sheaf* is any (not necessarily quasi-coherent) sheaf of left $\mathcal{D}$-modules on $P_{\acute{e}t}$. We denote the category of $\mathcal{D}_P$-sheaves by $\bar{\mathcal{M}}^\ell(P)$, so $\mathcal{M}^\ell(P)$ is a full subcategory of $\bar{\mathcal{M}}^\ell(P)$. The $\mathcal{O}_P$-tensor product defines on $\bar{\mathcal{M}}^\ell(P)$ a structure of the tensor category. Let $i: Z \hookrightarrow P$ be a closed smooth subvariety and $\mathcal{I} \subset \mathcal{O}_P$ the corresponding ideal. If F is a $\mathcal{D}_P$-sheaf, then $i^*F := i^\cdot(F/\mathcal{I}F)$ is a $\mathcal{D}_Z$-sheaf in the usual way.

3.5.2. Lemma. (i) *The functor* $i^* : \bar{\mathcal{M}}^\ell(P) \to \bar{\mathcal{M}}^\ell(Z)$ *admits a right adjoint functor* $i_*^{\hat{}} : \bar{\mathcal{M}}^\ell(Z) \to \bar{\mathcal{M}}^\ell(P)$ *which is exact and fully faithful. It identifies* $\bar{\mathcal{M}}^\ell(Z)$ *with the full subcategory of those* $\mathcal{D}_P$*-sheaves* F *for which* $F = \varprojlim F/\mathcal{I}^n F$.

(ii) *For a* $\mathcal{D}_Z$*-sheaf* G *set* $i_*G := (i_*\omega_Z)^\ell \otimes i_*^{\hat{}}G$.[32] *The functor* $i_* : \bar{\mathcal{M}}^\ell(Z) \to \bar{\mathcal{M}}^\ell(P)$ *is exact, fully faithful, and right inverse to* i^*. *A* $\mathcal{D}_P$*-sheaf belongs to its image if and only if everyone of its local sections is killed by some power of* $\mathcal{I}$.

(iii) *The above* i_* *coincides on* $\mathcal{M}^\ell(Z)$ *with the direct image functor from 2.1.3: for* $N \in \mathcal{M}(Z)$ *one has* $i_*(N^\ell) = (i_*N)^\ell$.

Sketch of a proof. (i) The existence of $i_*^{\hat{}}$ follows from right exactness of i^*, as well as the fact that $i_*^{\hat{}}$ has the local nature. One can describe $i_*^{\hat{}}G$ explicitly: for an open U one has $\Gamma(U, i_*^{\hat{}}G) = \mathrm{Hom}(i^*\mathcal{D}_U, G_U)$ (the morphisms of $\mathcal{D}_{U\cap Z}$-sheaves). This easily implies all the assertions in (i) and also (ii) and (iii) (use Kashiwara's lemma; see 2.1.3). □

Remark. The functor i^* commutes with tensor products. Therefore $i_*^{\hat{}}$ is naturally a pseudo-tensor functor adjoint to i^* as a pseudo-tensor functor (see 1.1.5). Namely, for $F_i \in \bar{\mathcal{M}}^\ell(P)$ one has the identification $\mathrm{Hom}(\otimes i^*F_i, G) = \mathrm{Hom}(i^* \otimes F_i, G) = \mathrm{Hom}(\otimes F_i, i_*^{\hat{}}G)$. So if A is a $\mathcal{D}_Z$-algebra, then $i_*^{\hat{}}A$ is a $\mathcal{D}_P$-algebra, and if N is an A-module, then $i_*^{\hat{}}N$ and i_*N are $i_*^{\hat{}}A$-modules.

3.5.3. Suppose Z is a divisor; let $j : U := P \smallsetminus Z \hookrightarrow P$ be the complementary open embedding. For a $\mathcal{D}_Z$-sheaf G set

$$i_*^{\tilde{}}G := (i_*^{\hat{}}G) \otimes j_*\mathcal{O}_U \in \bar{\mathcal{M}}^\ell(P). \tag{3.5.3.1}$$

It is easy to see that the canonical map $i_*^{\hat{}}G \to i_*^{\tilde{}}G$ is injective. Since $(i_*\omega_Z)^\ell = j_*\mathcal{O}_U/\mathcal{O}_U$, we get a canonical exact sequence

$$0 \to i_*^{\hat{}}G \to i_*^{\tilde{}}G \to i_*G \to 0. \tag{3.5.3.2}$$

Remark. Since $\cdot \otimes j_*\mathcal{O}_U$ is a tensor functor, the remark in 3.5.2 shows that $i_*^{\tilde{}}$ is a pseudo-tensor functor.

3.5.4. Lemma. (i) *The projection* $i_*^{\tilde{}}G \to i_*G$ *is a universal morphism from a* $\mathcal{D}_P$*-sheaf which is a* $j_*\mathcal{O}_U$*-module to* i_*G. *The same is true for* $\mathcal{O}_P$*-module morphisms.*

(ii) *For any* $\mathcal{D}_P$*-sheaf* F *one has*

$$\mathrm{Hom}(F, i_*^{\tilde{}}G) \xleftarrow{\sim} \mathrm{Hom}(F \otimes j_*\mathcal{O}_U, i_*^{\tilde{}}G) \xrightarrow{\sim} \mathrm{Hom}(F \otimes j_*\mathcal{O}_U, i_*G). \tag{3.5.4.1}$$

Proof. (ii) follows from (i), and (i) essentially says that one recovers $i_*^{\tilde{}}G$ from i_*G by the "Tate module" construction. Namely, for a $\mathcal{D}_P$- (or $\mathcal{O}_P$-) sheaf Q which is a $j_*\mathcal{O}_U$-module one lifts a morphism $\alpha : Q \to i_*G$ to $\tilde{\alpha} : Q \to i_*^{\tilde{}}G$ by the formula $\tilde{\alpha}(f) = \varprojlim q^n\alpha(q^{-n}f)$. Here $f \in F \otimes j_*\mathcal{O}_U$, $q \in \mathcal{O}_P$ an equation of Z, so $q^n\alpha(q^{-n}f)$ is a well-defined element of $i_*^{\tilde{}}G/\mathcal{I}^n i_*^{\hat{}}G$. □

[32] Here $(i_*\omega_Z)^\ell \in \mathcal{M}^\ell(P)$; see 2.1.1 and 2.1.3.

REMARK. We also have the canonical isomorphisms

$$\mathrm{Hom}(F, i_*^{\sim}G) \xleftarrow{\sim} \varinjlim \mathrm{Hom}(F_\xi, i_*^{\wedge}G) \xrightarrow{\sim} \varinjlim \mathrm{Hom}(i^*F_\xi, G), \tag{3.5.4.2}$$

where F_ξ runs the set of $\mathcal{D}_P$-subsheaves of $F\otimes j_*\mathcal{O}_U$ such that $F_\xi \cdot j_*\mathcal{O}_U = F\otimes j_*\mathcal{O}_U$. The first arrow assigns to $F_\xi \to i_*^{\wedge}G$ its tensor product with $j_*\mathcal{O}_U$; the second arrow is the adjunction isomorphism. The inverse map to the first arrow sends $\varphi : F\otimes j_*\mathcal{O}_U \to i_*^{\sim}G$ to $\varphi_\xi : F_\xi \to i_*^{\wedge}G$ where $F_\xi := \varphi^{-1}(i_*^{\wedge}G)$, φ_ξ is the restriction of φ to F_ξ.

Notice that if both F and G are quasi-coherent, then, as follows from the above lemma, we may assume in (3.5.4.2) that all the F_ξ's are quasi-coherent.

3.5.5. Let us return to our situation, so X is a curve. For a finite non-empty I and a $\mathcal{D}_X$-sheaf G set $\hat{\Delta}_*^{(I)}G := \Delta^{(I)}{}_*^{\wedge}G$ and

$$\tilde{\Delta}_*^{(I)}G := (\hat{\Delta}_*^{(I)}G)\otimes j_*^{(I)}\mathcal{O}_{U^{(I)}} \in \bar{\mathcal{M}}(X^I). \tag{3.5.5.1}$$

The obvious morphism $\hat{\Delta}_*^{(I)}G \to \tilde{\Delta}_*^{(I)}G$ is injective. Note that $\hat{\Delta}_*^{(I)}G$ has the following explicit description. For $0\in I$ consider a $\mathcal{D}_{X^I}$-sheaf $pr_0^*G := G\boxtimes\mathcal{O}_X^{\boxtimes I\smallsetminus\{0\}}$. Then $\Delta^{(I)\sim}pr_0^*G = G$; hence $\hat{\Delta}_*^{(I)}G = \varprojlim pr_0^*G/\mathcal{J}_\Delta^n pr_0^*G = \varprojlim \mathcal{O}_{X^I}/\mathcal{J}_\Delta^n \underset{\mathcal{O}_X}{\otimes} G$ where $\mathcal{J}_\Delta \subset \mathcal{O}_{X^I}$ is the ideal of the diagonal $X\subset X^I$.

EXAMPLE. Let t be a local coordinate on X, $I=\{1,2\}$. Then $\hat{\Delta}_*^{(I)}G = G_1[[t_1-t_2]] = G_2[[t_1-t_2]]$ and $\tilde{\Delta}_*^{(I)}G = G_1((t_1-t_2)) = G_2((t_1-t_2))$ where G_1, resp. G_2, is a copy of G considered as a $\mathcal{D}$-sheaf along the first, resp. second, variable in $X\times X$.

More generally, for $S\in Q(I)$ let $\Delta^{(I/S)}: X^S \hookrightarrow X^I$ be the diagonal embedding and $j^{(I/S)}: U^{(I/S)} \hookrightarrow X^I$ the complement to the diagonals containing X^S. For $F\in \bar{\mathcal{M}}(X^S)$ set

$$\tilde{\Delta}_*^{(I/S)}F := (\hat{\Delta}_*^{(I/S)}F)\otimes j_*^{(I/S)}\mathcal{O}_{U^{(I/S)}} \in \bar{\mathcal{M}}(X^I). \tag{3.5.5.2}$$

Note that if F is a $j_*^{(S)}\mathcal{O}_{U^{(S)}}$-module, then $\tilde{\Delta}_*^{(I/S)}F$ is a $j_*^{(I)}\mathcal{O}_{U^{(I)}}$-module.

For an interval $\mathcal{I} = \{I_1 < \cdots < I_n = I\}\subset Q(I)$ and $G\in\bar{\mathcal{M}}(X)$ set

$$\tilde{\Delta}_*^{(\mathcal{I})}G := \tilde{\Delta}_*^{(I_n/I_{n-1})}\cdot\cdots\tilde{\Delta}_*^{(I_2/I_1)}\tilde{\Delta}_*^{(I_1)}G \in \bar{\mathcal{M}}(X^I). \tag{3.5.5.3}$$

This is a $j_*^{(I)}\mathcal{O}_{U^{(I)}}$-module. For $\mathcal{I}'\subset\mathcal{I}$ one has $\tilde{\Delta}_*^{(\mathcal{I}')}G\subset\tilde{\Delta}_*^{(\mathcal{I})}G$.

EXAMPLES. One has $\tilde{\Delta}_*^{(\{I\})}G = \tilde{\Delta}_*^{(I)}G$. If $I=\{1,\cdots,n\}$ and I_a is its quotient modulo the relation $a=a+1=\cdots=n$, then $\tilde{\Delta}_*^{(\mathcal{I})}G = G_i((t_1-t_2))\cdots((t_{n-1}-t_n))$. Here i is any element in $[1,n]$ and G_i is a copy of G considered as a $\mathcal{D}$-sheaf along the ith variable in X^n.

3.5.6. Suppose $\mathcal{I}$ is a maximal interval; i.e., $|I_a| = a+1$. We have a natural projection $\pi_{\mathcal{I}} : \tilde{\Delta}_*^{(\mathcal{I})}G \to \Delta_*^{(\mathcal{I})}G$ defined as the composition of maps $\Delta_*^{(I/I_k)}\tilde{\Delta}_*^{(\mathcal{I}_{\leq k})}G \to \Delta_*^{(I/I_{k-1})}\tilde{\Delta}_*^{(\mathcal{I}_{\leq k-1})}G$ coming from the projection $\tilde{\Delta}_*^{(I_k/I_{k-1})} \to \Delta_*^{(I_k/I_{k-1})}$. It is clear that any non-trivial $j_*^{(I)}\mathcal{O}_{U^{(I)}}$-submodule of $\tilde{\Delta}_*^{(\mathcal{I})}G$ projects non-trivially to $\Delta_*^{(\mathcal{I})}G$, so for any $\mathcal{D}_{X^I}$-sheaf N we have $\mathrm{Hom}(N,\tilde{\Delta}_*^{(\mathcal{I})}G) \xleftarrow{\sim} \mathrm{Hom}(N\otimes j_*^{(I)}\mathcal{O}_{U^{(I)}},\tilde{\Delta}_*^{(\mathcal{I})}G) \hookrightarrow$

$\mathrm{Hom}(N\otimes j_*^{(I)}\mathcal{O}_{U^{(I)}},\Delta_*^{(\mathcal{J})}G)$ where the right arrow comes from $\pi_{\mathcal{J}}$. In particular, for any $L_i, M\in\mathcal{M}(X)$ one has a canonical embedding (here λ_I comes from $\omega_{X^I}\otimes\lambda_I=\omega_X^{\boxtimes I}$, see 3.1.4, and we use 3.5.2(iii))

$$\mathrm{Hom}(\boxtimes L_i^{\ell},\tilde\Delta_*^{(I)}M)\hookrightarrow P_I^{ch}(\{L_i\},M)\otimes\lambda_I. \tag{3.5.6.1}$$

3.5.7. We will need a technical lemma. Consider the Cousin complex for $C(\omega)_{X^I}$ from the proof in 3.1.5. Tensoring it by $(\hat\Delta_*^{(I)}G)\omega_{X^I}^{-1}[-|I|]$, we get a complex $C_I(G)$ of $\mathcal{D}_{X^I}$-sheaves with terms $C_I(G)^m:=\underset{T\in Q(I,|I|-m)}{\oplus}\Delta_*^{(I/T)}\tilde\Delta_*^{(T)}G$ (in particular, $C_I(G)^0=\tilde\Delta_*^{(I)}G$) which is a resolution of $\hat\Delta_*^{(I)}G$.

Lemma. *Suppose that $N\in\bar{\mathcal{M}}(X^I)$ is a $j_*^{(I)}\mathcal{O}_{U^{(I)}}$-module. Then the sequence*

$$0\to\mathrm{Hom}(N,C_I(G)^0)\to\mathrm{Hom}(N,C_I(G)^1)\to\mathrm{Hom}(N,C_I(G)^2)$$

is exact.

Proof. The statement is X-local, so let us choose an equation $q=0$ of the diagonal divisor; then $\tilde\Delta_*^{(I)}G=\varprojlim\tilde\Delta_*^{(I)}G/q^n\hat\Delta_*^{(I)}G$. Now use the fact that for any n the sequence $0\to\tilde\Delta_*^{(I)}G/q^n\hat\Delta_*^{(I)}G\to C_I^0\xrightarrow{q^{-n}d^0}C_I(G)^1\xrightarrow{d^1}C_I(G)^2$ is exact. □

3.5.8. For an I-family of $\mathcal{D}_X$-sheaves set

$$O_I(\{F_i\},G):=\mathrm{Hom}(\boxtimes F_i,\tilde\Delta_*^{(I)}G)=\mathrm{Hom}((\boxtimes F_i)\otimes j_*^{(I)}\mathcal{O}_{U^{(I)}},\tilde\Delta_*^{(I)}G). \tag{3.5.8.1}$$

This is the vector space of *I-ope's*. They have a local nature with respect to the étale topology of X.

For example, in the notation of Example in 3.5.5, a binary ope $\circ: F\boxtimes F'\to\tilde\Delta_*^{(I)}G$ is a series $f,f'\mapsto\sum(t_1-t_2)^i(f\underset{i}{\circ}f')_2$ where $f\underset{i}{\circ}f'\in G$ and $(f\underset{i}{\circ}f')_2$ is the corresponding section of G_2.

Ope's are *not* operations in the sense of 1.1 for the composition of ope's need not be an ope: it belongs to a larger vector space. Namely, for $J\twoheadrightarrow I$ and a J-family $\{E_j\}$ of $\mathcal{D}_X$-sheaves the *ope composition map*

$$O_I(\{F_i\},G)\otimes(\underset{I}{\otimes}O_{J_i}(\{E_j\},F_i))\longrightarrow\mathrm{Hom}(\boxtimes E_j,\tilde\Delta_*^{(\{I,J\})}G) \tag{3.5.8.2}$$

sends $\gamma\otimes(\otimes\delta_i)$ to $\gamma(\delta_i)$ defined as the composition $\underset{J}{\boxtimes}E_j\xrightarrow{\boxtimes\delta_i}\underset{I}{\boxtimes}\tilde\Delta_*^{(J_i)}F_i\longrightarrow\tilde\Delta_*^{(J/I)}(\boxtimes F_i)\longrightarrow\tilde\Delta_*^{(J/I)}((\boxtimes F_i)\otimes j_*^{(I)}\mathcal{O}_{U^{(I)}})\xrightarrow{\tilde\Delta_*^{(J/I)}(\gamma)}\tilde\Delta_*^{(\{I,J\})}G$.

We say that ope's $(\gamma,\{\delta_i\})$ *compose nicely* if $\gamma(\delta_i)$ takes values in $\tilde\Delta_*^{(J)}G\subset\tilde\Delta_*^{(\{I,J\})}G$, so $\gamma(\delta_i)\in O_J(\{E_j\},G)$.

Remark. The canonical identification $\mathrm{Hom}(\boxtimes F_i,\hat\Delta_*^{(I)}G)=\mathrm{Hom}(\otimes F_i,G)$ (see 3.5.2(i)) and the embedding $\hat\Delta_*^{(I)}G\hookrightarrow\tilde\Delta_*^{(I)}G$ yield an embedding

$$\mathrm{Hom}(\otimes F_i,G)\subset O_I(\{F_i\},G). \tag{3.5.8.3}$$

Ope's from the left-hand side always compose nicely, and their ope composition coincides with the composition in $\mathcal{M}(X)^!$.

The composition of ope's is associative in the following sense. Let $K\twoheadrightarrow J$ be another surjection. For a K-family $\{C_k\}$ of $\mathcal{D}_X$-sheaves and ope's $\varepsilon\in O_{K_j}(\{C_k\},E_j)$

the double composition morphisms $(\gamma(\delta_i))(\varepsilon_j), \gamma(\delta_i(\varepsilon_j)) \in \mathrm{Hom}(\boxtimes C_k, \tilde{\Delta}_*^{(\{I,J,K\})}G)$ are defined, respectively, as compositions

$$\boxtimes C_k \xrightarrow{\boxtimes\varepsilon_j} \boxtimes\tilde{\Delta}_*^{(K_j)}E_j \longrightarrow \tilde{\Delta}_*^{(K/J)}(\boxtimes E_j) \xrightarrow{\tilde{\Delta}_*^{(K/J)}(\gamma(\delta_i))} \tilde{\Delta}_*^{(\{I,J,K\})}G,$$

$$\boxtimes C_k \xrightarrow{\boxtimes\delta_i(\varepsilon_j)} \boxtimes\tilde{\Delta}_*^{(\{J_i,K_i\})}F_i \longrightarrow \tilde{\Delta}_*^{(K/J)}\tilde{\Delta}_*^{(J/I)}(\boxtimes F_i) \xrightarrow{\tilde{\Delta}_*^{(K/J)}\tilde{\Delta}_*^{(J/I)}(\gamma)} \tilde{\Delta}_*^{(\{I,J,K\})}G.$$

The associativity property says that they coincide.

3.5.9. Let $\circ \in O_2(\{G,G\},G)$ be a binary ope, $a \boxtimes b \mapsto a \circ b \in \tilde{\Delta}_*G$. We say that $\circ$ is *associative* if both $(\circ, \{\circ, id_G\})$, $(\circ, \{id_G, \circ\})$ compose nicely and the composition ope's $\in O_3(\{G,G,G\},G)$ coincide, and we say $\circ$ is *commutative* if it is fixed by an obvious "transposition of coordinates" symmetry of $O_2(\{G,G\},G)$.

We call $\circ$ an *ope algebra product* on G and $(G,\circ)$ an *ope algebra*.

A *unit* for $\circ$ is a horizontal section $1 \in G$ such that for every $a \in G$ one has $a\circ 1, 1\circ a \in \hat{\Delta}_*G \subset \tilde{\Delta}_*G$ and modulo $\mathcal{I}_\Delta\hat{\Delta}_*G$ both $a\circ 1$ and $1\circ a$ equal $a \in G$. A unit is unique, if it exists.

Remarks. (i) For $\circ \in \mathrm{Hom}(G\otimes G, G) \subset O_2(\{G,G\},G)$ (see (3.5.8.3)) the above notions of associativity and commutativity are the same as the usual notions in $\mathcal{M}(X)^!$.

(ii) If $\circ$ is associative and commutative, then the triple product $\circ_3 := \circ(\{\circ, id_G\})$ is fixed by the action of the symmetric group of three variables.

3.5.10. According to (3.5.4.1) the canonical embedding (3.5.6.1) is an isomorphism for $|I| = 2$, so for any $A \in \mathcal{M}(X)$ one has a canonical bijection

$$P_2^{ch}(\{A,A\},A)\otimes\lambda_{\{1,2\}} \xrightarrow{\sim} O_2(\{A^\ell,A^\ell\},A^\ell), \qquad \mu\mapsto\circ_\mu. \tag{3.5.10.1}$$

Now we can state the main result of this section:

Theorem. *The above bijection identifies the set of chiral Lie brackets with the set of commutative and associative ope algebra products. A horizontal section $1 \in A^\ell$ is a unit for μ if and only if it is a unit for $\circ_\mu$.*

Proof. We will prove the first statement; the second one is left to the reader.

Since (3.5.10.1) commutes with transposition of coordinates, skew-commutative μ correspond to commutative $\circ_\mu$. Take such μ. Let us show that the Jacobi identity for μ amounts to the associativity of $\circ = \circ_\mu$.

Set $R := j_*^{(I)}j^{(I)*}(A^{\ell\boxtimes I})$. By 3.5.4 for every $S \in Q(I,|I|-1)$ we have isomorphisms

$$\mathrm{Hom}(A^{\ell\boxtimes I}, \tilde{\Delta}_*^{(\{S,I\})}A^\ell) \xleftarrow{\sim} \mathrm{Hom}(R, \tilde{\Delta}_*^{(I/S)}\tilde{\Delta}_*^{(S)}A^\ell) \xrightarrow{\sim} \mathrm{Hom}(R, \Delta_*^{(I/S)}\tilde{\Delta}_*^{(S)}A^\ell).$$

Since $\mathrm{Hom}(R, \tilde{\Delta}_*^{(I)}A^\ell) = \mathrm{Hom}(A^{\ell\boxtimes I}, \tilde{\Delta}_*^{(I)}A^\ell)$, the exact sequence from 3.5.7 looks like

$$0 \to \mathrm{Hom}(A^{\ell\boxtimes I}, \tilde{\Delta}_*^{(I)}A^\ell) \xrightarrow{d^0} \bigoplus_{S\in Q(I,|I|-1)} \mathrm{Hom}(A^{\ell\boxtimes I}, \tilde{\Delta}_*^{(\{S,I\})}A^\ell)$$
$$\xrightarrow{d^1} \bigoplus_{T\in Q(I,|I|-2)} \mathrm{Hom}(R, \Delta_*^{(I/T)}\tilde{\Delta}_*^{(T)}A^\ell).$$

The symmetric group of I indices acts on it in the obvious way.

Now suppose that $I = \{1,2,3\}$. The third vector space in the above exact sequence is $\mathrm{Hom}(j_*^{(3)} j^{(3)*}(A^{\ell\boxtimes 3}), \Delta_*^{(3)} A^\ell) = P_3^{ch}(\{A,A,A\},A)\otimes\lambda_{\{1,2,3\}}$ (use the identification $\omega_X^{\boxtimes 3} = \omega_{X^3}\otimes\lambda_{\{1,2,3\}}$). Our $\circ$ is commutative, so for every $S\in Q(I,2)$ we have the corresponding iterated product $\bar\circ_S \in \mathrm{Hom}(A^{\ell\boxtimes 3}, \tilde\Delta_*^{(\{S,I\})}A^\ell)$. For example, if S identifies $1,2\in I$, then $\bar\circ_S = \circ(\circ, id)$. The element $\bar\circ_3 := \Sigma\bar\circ_S$ of the middle term of the above exact sequence is invariant with respect to the action of the symmetric group. As follows from the construction, one has $d^1(\circ(\circ,id)) = \mu(\mu, id_A)$. Therefore $d^1(\bar\circ_3)$ is exactly the sum of terms of the Jacobi identity, so $d^1(\bar\circ_3)=0$ if and only if μ is a Lie bracket. On the other hand, by the very definition, $\circ$ is associative if and only if $\bar\circ_3$ lies in the image of d^0. Since our sequence is exact, we are done. □

3.5.11. From the point of view of factorization algebras, for a chiral algebra A the corresponding ope $\circ_A := \circ_{\mu_A}$ is the glueing datum for $A^\ell_{X\times X}$. Namely, for $a,b\in A^\ell$ the ope $a\circ_A b$ is the restriction of $a\boxtimes b$, considered as a section of $A^\ell_{X^2}$ on the complement to the diagonal, to the punctured formal neighbourhood of the diagonal.

This implies, in particular, that $\circ_A$ is compatible with the tensor product of chiral algebras (see 3.4.15). Namely, the morphism $\otimes(\tilde\Delta_* A^\ell_\alpha)\to\tilde\Delta_*(\otimes A^\ell_\alpha)$ (see the remark in 3.5.3) sends $\otimes\circ_{A_\alpha}$ to $\circ_{\otimes A_\alpha}$.

3.5.12. LEMMA. *Let $(A^\ell,\circ)$ be a commutative and associative ope algebra. Then for any finite non-empty I one has a canonical I-fold ope product operation $\circ_I \in O_I(\{A^\ell\},A^\ell)$. These operations are uniquely determined by the following properties:*

(i) *If $|I|=1$, then $\circ_I = id_{A^\ell}$; if $|I|=2$, then $\circ_I=\circ$.*

(ii) *For every $J\twoheadrightarrow I$ the operations $(\circ_I,\{\circ_{J_i}\})$ compose nicely; the composition is $\circ_J$.*

Proof. We define $\circ_I$ by induction. We know $\circ_I$ if $|I|\le 3$. Now let I be a finite set of order $n>3$, and assume that we know $\circ_{I'}$ for $|I'|<n$ so that (i), (ii) hold. Consider the exact sequence from the proof in 3.5.10. For $S\in Q(I,n-1)$ set $\bar\circ_S := \circ_S(\circ_{\{a,b\}}, id_{A_i^\ell})_{i\neq a,b} \in \mathrm{Hom}(A^{\ell\boxtimes I}, \tilde\Delta^{(I,S)}A^\ell)$. Here $a,b\in I$ are the two distinct S-equivalent elements. Set $\bar\circ_I := \Sigma\bar\circ_S \in \underset{S\in Q(I,n-1)}{\oplus}\mathrm{Hom}(A^{\ell\boxtimes I},\tilde\Delta^{(I,S)}A^\ell)$. The associativity of composition (see 3.5.8) and the existence of $\circ_3$ imply that $d^1(\bar\circ_I)=0$. Hence $\bar\circ_I = d^0(\circ_I)$ for certain $\circ_I$ uniquely determined by this condition. Property (ii) follows from associativity of composition. □

REMARK. Let us describe $\circ_I$ from the point of view of factorization algebras. Consider a factorization algebra that corresponds to our ope algebra. In particular, we have a $\mathcal{D}_{X^I}$-module $A^\ell_{X^I}$ whose pull-back to the diagonal equals A^ℓ. Thus its formal completion at the diagonal equals $\hat\Delta_* A^\ell$ (see 3.5.2 and 3.5.5). Now $\circ_I$ is the composition $A^{\ell\boxtimes I}\to j_*^{(I)}j^{(I)*}A^{\ell\boxtimes I}\xrightarrow{\sim} j_*^{(I)}j^{(I)*}A^\ell_{X^I}\to\tilde\Delta_*^{(I)}A^\ell$.

3.5.13. Let us translate some of the definitions from 3.3 in the ope language. Let (A,μ) be a chiral algebra, and let $(A^\ell,\circ)$ be the corresponding ope algebra.

By definition, the Lie* bracket $[\ \ ]_\mu : A\boxtimes A\to\Delta_* A$ coincides with the composition $A^\ell\boxtimes A^\ell\xrightarrow{\circ}\tilde\Delta_* A^\ell\to\Delta_* A^\ell$ tensored by $id_{\omega_{X^2}}$; hence $[\ \ ]_\mu$ is the *singular part* of $\circ$. Therefore our chiral algebra is commutative if and only if $\circ$ takes values in

$\hat{\Delta}_* A^\ell \subset \tilde{\Delta}_* A^\ell$; i.e., $\circ \in \mathrm{Hom}(A^{\ell \otimes 2}, A^\ell) \subset O_2(\{A^\ell, A^\ell\}, A^\ell)$ (see (3.5.8.3)). Notice that Remark (i) from 3.5.9 provides another proof of (3.3.2.1), (3.3.3.1).

3.5.14. If $(G, \circ)$ is an ope algebra, then a *G-module* is a $\mathcal{D}_X$-sheaf N together with an ope pairing $\circ_N \in O_2(\{G, N\}, N)$ such that both $\circ_N(\circ, id_N)$ and $\circ_N(id_G, \circ_N)$ compose nicely and the compositions $\in O_3(\{G, G, N\}, N)$ coincide. If G has unit, then one defines a unital G-module in the obvious manner.

Let A be a chiral algebra as above, M an A-module, so we have the chiral action $\mu_M \in P_2^{ch}(\{A, M\}, M)$. Let $\circ_M \in O_2(\{A^\ell, M^\ell\}, M^\ell)$ be the corresponding ope pairing (see 3.5.10). This is an $(A^\ell, \circ)$-module structure on M. This identifies the category of (A, μ)-modules with that of (quasi-coherent) $(A^\ell, \circ)$-modules. This follows directly from 3.5.10 (use 3.3.5(i)).

As in 3.5.8 one can write $\circ_M$ in terms of a local coordinate t as a morphism $A^\ell \boxtimes M \to M_2((t_1 - t_2))$, $a, m \mapsto \sum (t_1 - t_2)^i (a \underset{i}{\circ} m)_2$. For fixed a the map $M \to M_2((t_1 - t_2))$, $m \mapsto a \circ_M m$, is the *vertex operator* corresponding to a.

3.5.15. The ope approach is quite convenient in the setting of $(\mathfrak{g}, K)$-modules (see 3.1.16; we follow the notation of loc. cit. and 2.9.7). Namely, for M_i $N \in \mathcal{M}(\mathfrak{g}, K)$ one defines $O_I(\{M_i\}, N)$ as $\mathrm{Hom}_{G^I}(\otimes M_i, \tilde{\Delta}_*^{(I)} N)$ where we set $\tilde{\Delta}_*^{(I)} N := \tilde{F}^{(I)} \underset{F^{(I)}}{\otimes} \hat{\Delta}_*^{(I)} N$. With this definition the results of this section (in particular, the theorem in 3.5.10) remain valid for $(\mathfrak{g}, K)$-modules. For $\mathfrak{g} = k$ an ope algebra is the same as a vertex algebra in the sense of [**B1**] or [**K**] 1.3. An ope algebra for $G = \mathrm{Aut}\, k[\![t]\!]$ (see 2.9.9) is the same as a quasi-conformal vertex algebra in the sense of [**FBZ**] 5.2.4. The functors defined by $(\mathfrak{g}, K)$-structures transform ope algebras to ope algebras; this is the same as the old transformation of chiral algebras (see 3.3.14). In particular, quasi-conformal vertex algebras define universal chiral algebras on any curve, which is [**FBZ**] 18.3.3.

3.6. From chiral algebras to associative algebras

In this section we consider some functors which assign to a chiral algebra A on X certain associative "algebras of observables." In a local situation, fixing a point $x \in X$, we define two associative (closely related) algebras: a topological algebra A_x^{as} which governs the category of A-modules supported at x and a filtered *Wick* algebra A_x^w that encodes the standard rules of thumb for manipulating vertex operators (e.g., the expression of the coefficients of the operator product expansion as normally ordered products). If A is commutative, then A_x^{as} is the algebra of functions on the space of horizontal sections of $\mathrm{Spec}\, A^\ell$ over the formal punctured disc at x.

The above associative algebras are omnipresent in the vertex algebra literature as algebras of operators acting on concrete modules. The reader who wants to acquire some computational skills is advised to look into [**K**], [**FBZ**].

In 3.6.1 we introduce "normally ordered" tensor product $\vec{\otimes}$ of topological vector spaces, which is a general monoidal category context for the topological associative algebras we will consider. The topological algebra A_x^{as} is studied in 3.6.2–3.6.10. We define A_x^{as} in 3.6.2, describe it as a certain completion of $h(j_{x*} j_x^* A)_x$ in 3.6.4–3.6.7, check compatibility with tensor products in 3.6.8, and compare it with a construction from [**FBZ**] 4.1.4, 4.1.5 in 3.6.9–3.6.10. The Wick algebra A_x^w is defined in 3.6.11; it is mapped into A_x^{as} in 3.6.12. The situation when x varies is considered in

3.6.13–3.6.14. We describe arbitrary A-modules as modules over a sheaf of topological associative $\mathcal{D}_X$-algebras A^{as} in 3.6.15–3.6.17. If A is commutative, then A_x^{as} coincides with the same-noted algebra from 2.4. We look at the situation when x varies and show in 3.6.19 that the functor $A \mapsto A_x^{as}$ preserves formally smooth and formally étale morphisms. Finally 3.6.20–3.6.21 contains some remarks about a global version of the Wick algebra.

A natural problem is to assign (in the analytic setting) to any oriented loop γ on X an associative algebra A_γ^{as}; the algebra A_x^{as} should correspond to the infinitely small loop around the punctured point x. The construction of the global Wick algebra has to do with this question.

3.6.1. A digression on $\vec{\otimes}$. The reader can skip this subsection at the moment, returning when necessary.

Below, "topological k-vector space" means a k-vector space equipped with a complete and separated linear topology. The category of topological vector spaces is a monoidal k-category with respect to the "normally ordered" tensor product $\vec{\otimes}$ defined as follows. For $V_1, \dots , V_n$ the tensor product $V_1 \vec{\otimes} \cdots \vec{\otimes} V_n$ is the completion of the plain tensor product $V_1 \otimes \cdots \otimes V_n$ with respect to a topology in which a vector subspace U is open if and only if for every i, $1 \leq i \leq n$, and vectors $v_{i+1} \in V_{i+1}, \dots , v_n \in V_n$ there exists an open subspace $P \subset V_i$ such that $U \supset V_1 \otimes \cdots \otimes V_{i-1} \otimes P \otimes v_{i+1} \otimes \cdots \otimes v_n$. The tensor product $\vec{\otimes}$ is associative but *not* commutative. The unit object is k.

EXAMPLES. Consider $k((t))$ as a topological k-vector space equipped with the usual "t-adic" topology. Then for every topological k-vector space V there is an obvious canonical identification $V \vec{\otimes} k((t)) = V((t))$. One has $(\mathrm{End}V)((t)) \subset \mathrm{End}(V \vec{\otimes} k((t)))$. Thus for every endomorphism g of V the endomorphism $1 - g \vec{\otimes} t$ of $V \vec{\otimes} k((t))$ is invertible; if g is invertible, then so is $g \vec{\otimes} 1 - 1 \vec{\otimes} t$.

In particular, we see that

$$k((t_1)) \vec{\otimes} \cdots \vec{\otimes} k((t_n)) = k((t_1)) \cdots ((t_n)). \tag{3.6.1.1}$$

Any $k((t))$-vector space V is naturally a topological k-vector space: a subspace $U \subset V$ is open if its intersection with every $k((t))$-line in V is "t-adically" open. Suppose that V_i, $i = 1, \dots , n$, are $k((t_i))$-vector spaces. Then (3.6.1.1) yields a canonical map

$$(V_1 \otimes \cdots \otimes V_n) \underset{k((t_1)) \otimes \cdots \otimes k((t_n))}{\otimes} k((t_1)) \cdots ((t_n)) \to V_1 \vec{\otimes} \cdots \vec{\otimes} V_n \tag{3.6.1.2}$$

which is an isomorphism if the V_i's have finite $k((t_i))$-dimension.

An associative algebra in our monoidal category is the same as a topological associative algebra whose topology has a base formed by left ideals. Below we call such an R simply a *topological associative algebra*;[33] we always assume it to be unital. For a topological vector space V the free topological associative algebra generated by V is denoted by $\vec{T}V$, so $\vec{T}V = k \oplus V \oplus V^{\vec{\otimes} 2} \oplus \cdots$ (equipped with the direct limit topology).

For a k-vector space M considered as a discrete topological vector space a left unital R-action on M in the sense of our monoidal category is the same as a

[33] So the ring of Laurent formal power series $k((t))$ with its usual topology is *not* a topological algebra in this sense.

continuous left unital R-action (which means that the annihilator of every $m \in M$ is an open ideal). M equipped with such an action is called a *discrete R-module*. The category of discrete R-modules is denoted by Rmod.

REMARKS. (i) Our R reconstructs from Rmod and the obvious forgetful functor from Rmod to (discrete) vector spaces as the (topological) endomorphism algebra of this functor.

(ii) For a discrete vector space V the algebra $\mathrm{End}(V)$ equipped with the weakest topology such that its action on V is continuous is a topological associative algebra. The category of discrete $\mathrm{End}(V)$-modules is semisimple; each irreducible object is isomorphic to V.

(iii) A right R-action on M in our monoidal category is the same as a plain right R-action such that the annihilator of M is open in R; i.e., the action factors through R/I where I is an open two-sided ideal. We will not consider such senseless objects.

The category of topological vector spaces also carries a symmetric monoidal structure defined by the usual completed tensor product $\hat{\otimes} V_\alpha := \varprojlim \otimes (V_\alpha / V_{\alpha\xi})$, where $V_{\alpha\xi}$ runs the set all open subspaces in V_α. So we have a natural continuous morphism $V_1 \vec{\otimes} \cdots \vec{\otimes} V_n \to V_1 \hat{\otimes} \cdots \hat{\otimes} V_n$. The tensor products $\vec{\otimes}$ and $\hat{\otimes}$ mutually commute: for a collection $\{V_{\alpha i}\}$ of topological vector spaces bi-indexed by $\{\alpha\}$ and $i = 1, \ldots, n$ one has $\hat{\otimes}(V_{\alpha 1} \vec{\otimes} \cdots \vec{\otimes} V_{\alpha n}) = (\hat{\otimes} V_{\alpha 1}) \vec{\otimes} \cdots \vec{\otimes} (\hat{\otimes} V_{\alpha n})$. Therefore if the R_α are topological associative (unital) algebras, then so is $\hat{\otimes} R_\alpha$; if the V_α are discrete (unital) R_α-modules, then $\otimes V_\alpha$ is a discrete (unital) $\hat{\otimes} R_\alpha$-module.

REMARKS. (i) A topological associative algebra which is commutative (as an abstract algebra) is the same as a commutative algebra with respect to $\hat{\otimes}$.

(ii) The topological associative algebra $\mathrm{End}(V)$ (see the previous Remark (ii)) is *not* an algebra in the sense of $\hat{\otimes}$ and the action morphism $\mathrm{End}(V) \vec{\otimes} V \to V$ does *not* extend to $\mathrm{End}(V) \hat{\otimes} V$ unless $\dim V < \infty$.

3.6.2. The first definition of A_x^{as}. Let $x \in X$ be a closed point and $i_x : \{x\} \hookrightarrow X$, $j_x : U_x \hookrightarrow X$ the complementary embeddings. Denote by O_x the formal completion of the local ring at x, K_x its quotient field. Below, t is a formal parameter at x (we fix it for mere notational convenience), so $O_x = k[[t]]$, $K_x = k((t))$.

Let A be a chiral algebra on X. We will assign to it a topological associative algebra A_x^{as}. Here is a quick definition. Denote by $\mathcal{M}(X, A)_x$ the category of A-modules supported at x. We have a faithful exact functor $i_x^! = h$ on $\mathcal{M}(X, A)_x$ with values in the category of vector spaces. Our A_x^{as} is the algebra of endomorphisms of this functor equipped with the standard topology (its base is formed by annihilators of elements of hM, $M \in \mathcal{M}(X, A)_x$). Unfortunately this definition says little about the structure of A_x^{as} (to the extent that it smells of set-theoretic problems). So we will give another "concrete" definition of A_x^{as} comparing it with the above definition later in 3.6.6.

REMARK. The reader may prefer to consider instead of $\mathcal{M}(X, A)_x$ the subcategory $\mathcal{M}_{\mathcal{O}}(X, A)_x$ of $\mathcal{M}_{\mathcal{O}}(X, A)$ (see 3.3.5(ii)) which consists of modules supported scheme-theoretically at x.[34] The induction functor (3.3.5.1) identifies it with $\mathcal{M}(X, A)_x$.

[34] I.e., killed by the maximal ideal $\mathfrak{m}_x \subset \mathcal{O}_x$.

3.6.3. Lemma. *The functor* $\mathcal{M}(X, j_{x*}j_x^*A)_x \to \mathcal{M}(X, A)_x$ *defined by the obvious morphism of chiral algebras* $A \to j_{x*}j_x^*A$ *is an equivalence of categories.*

Proof. Let M be a $\mathcal{D}$-module supported at x. We want to show that every A-action on M extends uniquely to a $j_{x*}j_x^*A$-action, and this extension is compatible with the morphisms of the M's. Notice that for every finite family of $\mathcal{D}_X$-modules A_i, $i \in I$, the obvious morphism of $\mathcal{D}_{X^{\tilde{I}}}$-modules[35] $j_*^{(\tilde{I})}j^{(\tilde{I})*}((\boxtimes_I A_i)\boxtimes M) \to j_*^{(\tilde{I})}j^{(\tilde{I})*}((\boxtimes_I j_{x*}j_x^*A_i) \boxtimes M)$ is an isomorphism. Thus $P_{\tilde{I}}^{ch}(\{j_{x*}j_x^*A_i, M\}, M) \xrightarrow{\sim} P_{\tilde{I}}^{ch}(\{A_i, M\}, M)$. This implies the desired statement (take $A_i = A$). □

Remark. Suppose M is an A-module supported at x. Let us write the A-action as the ope $\circ_M$ (see 3.5.14). By Kashiwara's lemma we can replace M by $i_x^! M = hM$, i.e., consider $\circ_M$ as a morphism[36] $A_{O_x}^{\ell} \to \mathrm{Hom}(hM, hM((t)))$, $a \mapsto (m \mapsto a \circ_M m)$, compatible with the action of differential operators.[37] Then the ope for the $j_{x*}j_x^*A$-action on M is simply the K_x-linear extension $A_{K_x}^{\ell} \to \mathrm{Hom}(hM, hM((t)))$ of $\circ_M$.

3.6.4. The second definition of A_x^{as}. We change the notation: from now till 3.6.13 our A is a chiral algebra on U_x. Set $\mathcal{M}(X, A)_x := \mathcal{M}(X, j_{x*}A)_x$.

Denote by Ξ_x^{as} the set of chiral subalgebras $A_\xi \subset j_{x*}A$ which coincide with A over U_x. If $A_\xi, A_{\xi'}$ are in Ξ_x^{as}, then so is $A_\xi \cap A_{\xi'}$, so Ξ_x^{as} is a topology on $j_{x*}A$ at x (see 2.1.13 for terminology).

Remark. As in Remark (i) in 2.1.13 the map $A_\xi \mapsto A_{\xi O_x}$ identifies Ξ_x^{as} with the set of chiral subalgebras of $A_{K_x} = (j_{x*}A)_{O_x} = A_{O_x} \otimes K_x$ with torsion quotient, and we can consider Ξ_x^{as} as a topology on A_{K_x} or on $h(A_{K_x})$.

The fibers $A_{\xi x}^{\ell} := i_x^* A_\xi^{\ell} = i_x^!(j_{x*}A/A_\xi)$ form a Ξ_x^{as}-projective system of vector spaces connected by surjective morphisms. We denote by A_x^{as} its projective limit. Equivalently (see 2.1.13), A_x^{as} is the completion of $h(j_{x*}A)_x$ or $h(A_{K_x})$ with respect to the Ξ_x^{as}-topology. So every $A_{\xi x}^{\ell}$ is a quotient of A_x^{as} modulo an open submodule I_ξ.

We are going to define on A_x^{as} a canonical structure of the associative algebra such that the I_ξ become left ideals. To do this, we need an auxiliary lemma. For $\xi \in \Xi_x^{as}$ let $1_{\xi x} \in A_\xi^{as}/I_\xi = i_x^* A_\xi^{\ell} = i_x^!(j_{x*}A/A_\xi)$ be the value at x of the unit section of A_ξ^{ℓ}.

3.6.5. Lemma. *For every* $\xi \in \Xi_x^{as}$ *and* $M \in \mathcal{M}(X, A_\xi)$ *the map* $\varphi \mapsto \varphi(1_{\xi x})$ *identifies* $\mathrm{Hom}(j_{x*}A/A_\xi, M)$ *with the space of* A_ξ*-central sections of* M *supported scheme-theoretically at* x.

Proof (cf. Exercise (ii) in 3.3.7). It is clear that $1_{\xi x}$ is a central section. Since it generates $j_{x*}A/A_\xi$, our map is injective. The inverse map is constructed as follows. Assume we have an A_ξ-central section m of M supported scheme-theoretically at x; let us define the corresponding morphism ψ_m. We can assume that M is generated by m; hence M is supported at x. Thus, by 3.6.3, M is a $j_{x*}A$-module. Consider the chiral action morphism $j_{x*}A \boxtimes M \to \Delta_* M$. Pulling it back by $(id_X \times i_x)^!$, we

[35] Here $\tilde{I} := I \sqcup \cdot$.

[36] Recall that $A_{O_x} := A \underset{\mathcal{O}_x}{\otimes} O_x$, $A_{K_x} := A \underset{\mathcal{O}_x}{\otimes} K_x$.

[37] $\mathcal{D}_X$ acts on the right-hand side as on the Laurent series $k((t)) = K_x$.

get a morphism of $\mathcal{D}_X$-modules $\psi : j_{x*}A \otimes i_x^! M \to M$. Since m is A_ξ-central, the morphism $j_{x*}A \to M$, $b \mapsto \psi(b \otimes m)$, vanishes on A_ξ, so we have $\psi_m : j_{x*}A/A_\xi \to M$. One checks immediately that it is a morphism of A_ξ-modules. This is the morphism we want. □

3.6.6. Set $\Phi_\xi := j_{x*}A/A_\xi$. This is an A_ξ-module supported at x, hence a $j_{x*}A$-module by 3.6.3.

Notice that the projection $j_{x*}A \to \Phi_\xi$ is *not* a morphism of $j_{x*}A$-modules unless $\Phi_\xi = 0$.

For $A_{\xi'} \subset A_\xi$ the canonical projection $\Phi_{\xi'} \twoheadrightarrow \Phi_\xi$ is a morphism of $A_{\xi'}$-modules, hence, by 3.6.3, that of $j_{x*}A$-modules. We have defined a Ξ_x^{as}-projective system $\Phi = \{\Phi_\xi\}$ in $\mathcal{M}(X,A)_x$. One has $i_x^! \Phi = A_x^{as}$. Set $1_x := \varprojlim 1_{\xi x} \in A_x^{as}$.

Take any $M \in \mathcal{M}(X,A)_x$. For every $m \in i_x^! M$ its centralizer (see 3.3.7; here we consider m as a section of M) belongs to Ξ_x^{as}. Therefore, by 3.6.5, we have

$$\text{(3.6.6.1)} \qquad \operatorname{Hom}(\Phi, M) := \bigcup \operatorname{Hom}(\Phi_\xi, M) \xrightarrow{\sim} i_x^! M = h(M), \quad \psi \mapsto \psi(1_x).$$

In particular, $A_x^{as} = \operatorname{End}\Phi$. We define on A_x^{as} the structure of an associative algebra so that this identification becomes an anti-isomorphism of algebras (so Φ is a right A_x^{as}-module). Therefore A_x^{as} is the algebra of endomorphisms of the functor h on $\mathcal{M}(X,A)_x$, as was promised in 3.6.2. The element 1_x is the unit of our associative algebra. We see that A_x^{as} is a topological associative unital algebra in the sense of 3.6.1 and every $i_x^! M$ is a discrete unital A_x^{as}-module.

Remarks. (i) Consider A as a Lie* algebra. Thus $h(A_{K_x})$ is a Lie algebra and the map

$$\text{(3.6.6.2)} \qquad h(A_{K_x}) \to A_x^{as}$$

is obviously a morphism of Lie algebras. Notice that the Ξ_x^{as}-topology is weaker than the Ξ_x^{Lie}-topology (see 2.5.12), so (3.6.6.2) extends by continuity to the Ξ_x^{Lie}-completion.

(ii) Let us spell out the definition of the A_x^{as} product more concretely. Take any $a, b \in A_x^{as}$. Choose any $A_\xi \in \Xi_x^{as}$; let us compute $(ab)_\xi \in A_x^{as}/I_\xi = A_{\xi x}^\ell$. Consider the ope product $\circ_\xi : A_\xi^\ell \boxtimes A_\xi^\ell \to (A_\xi^\ell)_2((t_1 - t_2))$. Restrict $\circ_\xi$ to $X \times \{x\}$ and fix the second argument to be $b_\xi \in A_{\xi x}^\ell$; we get a morphism of $\mathcal{D}$-modules $A_\xi^\ell \to A_{\xi x}^\ell((t))$, $c \mapsto c \circ_\xi b_\xi$. Let $A_{\xi'}^\ell \subset A_\xi^\ell$ be the preimage of $A_{\xi x}^\ell[\![t]\!] \subset A_{\xi x}^\ell(\!(t)\!)$. Then $A_{\xi'} \in \Xi_x^{as}$ and we have a morphism of $\mathcal{D}$-modules $A_{\xi'}^\ell \to A_\xi^\ell[\![t]\!]$, hence the map $A_{\xi'}^\ell \to A_\xi^\ell$ of fibers at x. Our $(ab)_\xi$ is the image of $a_{\xi'}$.

3.6.7. Lemma. *The functor $i_x^! = h : \mathcal{M}(X,A)_x \to A_x^{as}\text{mod}$ is an equivalence of categories.*

Proof. Let us define the inverse functor. Recall that by Kashiwara's lemma (see 2.1.3) the functor $i_x^!$ is an equivalence between the category of right $\mathcal{D}_X$-modules supported at x and that of vector spaces; the inverse functor sends a vector space V to $i_{x*}V$.

Now assume that V is a discrete A_x^{as}-module. Then $i_{x*}V$ is a chiral A-module. Indeed, by Kashiwara's lemma, one has $i_{x*}A_x^{as} = \Phi$. So, since i_{x*} commutes with direct limits, we have $i_{x*}V = \Phi \underset{A_x^{as}}{\otimes} V$ which yields the promised A-module structure. The functor $i_{x*} : A_x^{as}\text{mod} \to \mathcal{M}(X,A)_x$ is obviously inverse to $i_x^!$. □

3.6.8. The functor $A \mapsto A_x^{as}$ is compatible with tensor products:[38]

LEMMA. *There is a canonical isomorphism of topological associative algebras*

$$\hat{\otimes} A_{\alpha x}^{as} \xrightarrow{\sim} (\otimes A_\alpha)_x^{as}. \tag{3.6.8.1}$$

Proof. It is clear that any subalgebra $(\otimes A_\alpha)_\xi$ contains a subalgebra of type $\otimes(A_{\alpha\xi_\alpha})$.[39] This yields an isomorphism of topological vector spaces $\hat{\otimes} A_{\alpha x}^{as} := \varprojlim \otimes A_{\alpha\xi_\alpha x}^{\ell} \xrightarrow{\sim} (\otimes A_\alpha)_x^{as}$ which is obviously compatible with the products. □

COROLLARY. *If A is a Hopf chiral algebra, then A_x^{as} is a topological Hopf algebra.*

3.6.9. The third definition of A_x^{as}. Here is another, less economic, construction of A_x^{as} (cf. [**FBZ**] 4.1.5). Denote by $\hat{h}A_{K_x}$ the completion of hA_{K_x} or $h_x j_{x*}A$ with respect to the Ξ_x-topology (see 2.1.13). The map $\hat{h}A_{K_x} \to A_x^{as}$ yields a morphism of topological algebras (see 3.6.1)

$$\pi : \vec{T}\hat{h}A_{K_x} \to A_x^{as}. \tag{3.6.9.1}$$

It appears that π identifies A_x^{as} with a quotient of $\vec{T}\hat{h}A_{K_x}$ modulo explicit quadratic relations we are going to define.

Consider A_{K_x} itself as a topological k-vector space: we declare a subspace $P \subset A_{K_x}$ to be open if it intersects every K_x-line by a subspace open in the "t-adic" topology (see 3.6.1). Then the obvious map $\zeta : A_{K_x} \to \hat{h}A_{K_x}$ is continuous.

Denote by $K_x^{(2)}$ the localization of $k[\![t_1, t_2]\!]$ with respect to t_1, t_2 and $t_1 - t_2$; let σ be an involution of $K_x^{(2)}$ which interchanges t_i. We have an embedding $K_x^{(2)} \hookrightarrow k(\!(t_1)\!)(\!(t_2)\!)$.

Now take any $a, b \in A_{K_x}$ and $f = f(t_1, t_2) \in K_x^{(2)}$. Consider f as an iterated Laurent power series $f = \sum f_{ij} t_1^i t_2^j \in k(\!(t_1)\!)(\!(t_2)\!)$. The series $\sum f_{ij}(t_1^i a) \otimes (t_2^j b)$ converges in $A_{K_x}^{\vec{\otimes}2}$, so we have $\zeta(f, a, b) := \sum f_{ij} \zeta(t_1^i a) \otimes \zeta(t_2^j b) \in (\hat{h}A_{K_x})^{\vec{\otimes}2}$.

Set $r(f, a, b) := \zeta(f, a, b) - \zeta(\sigma(f), b, a) - \mu(f, a, b) \in (\hat{h}A_{K_x})^{\vec{\otimes}2} \oplus \hat{h}A_{K_x} \subset \vec{T}\hat{h}A_{K_x}$. Here $\mu(f, a, b) \in \hat{h}A_{K_x}$ is the image of $fa \boxtimes b$ by the chiral product map composed with the projection $\Delta_* A_{K_x} \to h\Delta_* A_{K_x} = hA_{K_x} \to \hat{h}A_{K_x}$.

Let $A_x^{as'}$ be the topological quotient algebra of $\vec{T}\hat{h}A_{K_x}$ modulo the relations $r(f, a, b) = 0$ for all f, a, b as above and an extra relation $\zeta(1_A dt/t) = 1$.

3.6.10. LEMMA. *The morphism π from (3.6.9.1) yields an isomorphism of topological algebras*

$$A_x^{as'} \xrightarrow{\sim} A_x^{as}. \tag{3.6.10.1}$$

Proof. Consider the functor $A_x^{as}\text{mod} \to \vec{T}\hat{h}A_{K_x}\text{mod}$ defined by π. Our statement amounts to the fact that it is a fully faithful embedding, and a discrete $\vec{T}\hat{h}A_{K_x}$-module M comes from an A_x^{as}-module if and only if the $\vec{T}\hat{h}A_{K_x}$-action on M kills our relations.

For a vector space M a $\vec{T}\hat{h}A_{K_x}$-action on it amounts to a morphism $(\hat{h}A_{K_x})\vec{\otimes}M \to M$ which is the same as a morphism of $\mathcal{D}_X$-modules $j_{x*}A \otimes M \to i_{x*}M$, or, by

[38] See 3.4.15 and 3.6.1.

[39] Take for $A_{\alpha\xi_\alpha}$ the preimage of $(\otimes A_\alpha)_\xi$ by the morphism $A_\alpha \to \otimes A_\alpha$.

Kashiwara's lemma applied to $X \times \{x\} \subset X \times X$, that of $\mathcal{D}_{X^2}$-modules $j_{x*}A \boxtimes i_{x*}M \to \Delta_* i_{x*}M$. Since $j_{x*}A \boxtimes i_{x*}M = j_* j^*(j_{x*}A \boxtimes i_{x*}M)$, this is the same as a chiral operation $\mu \in P_2^{ch}(\{j_{x*}A, i_{x*}M\}, i_{x*}M)$. Therefore i_{x*} identifies $\vec{T}\hat{h}A_{K_x}$mod with the category of pairs (P, μ) where $P \in \mathcal{M}(X)_x$ and $\mu \in P_2^{ch}(\{j_{x*}A, P\}, P)$. Our relations just mean that μ is a chiral unital action of $j_{x*}A$ on $i_{x*}M$. We are done by 3.6.7. □

3.6.11. The Wick algebra. A better way to appreciate the construction of 3.6.9 and the like is to consider the *Wick algebra* A_x^w of A. Here is a definition.

For $n \geq 1$ let $O_x^{(n)}$ be the formal completion of the local ring of $(x, \dots, x) \in X^n$ and let $K_x^{(n)}$ be the localization of $O_x^{(n)}$ with respect to the equations of the diagonal divisor and the divisors $x_i = x$, $i = 1, \dots, n$. Set $T_x^{w\,n}A := A^{\boxtimes n} \underset{\mathcal{O}_{(x,\dots,x)}}{\otimes} K_x^{(n)} = A_{K_x}^{\otimes n} \underset{K_x^{\otimes n}}{\otimes} K_x^{(n)}$ and[40] $\bar{T}_x^{w\,n}A := h(T_x^{w\,n}A)$. Then $T_x^w A := \oplus T_x^{w\,n}A$ and its quotient $\bar{T}_x^w A := \oplus \bar{T}_x^{w\,n}A$ are $\mathbb{Z}_{\geq 0}$-graded associative algebras with respect to the exterior tensor product.

For every $n \geq 2$ the action of the symmetric group Σ_n on X^n and $A^{\boxtimes n}$ provides a Σ_n-action on the nth component of $T_x^{w\,n}A$ and $\bar{T}_x^{w\,n}A$. For $i = 1, \dots, n-1$ let $\sigma_i \in \Sigma_n$ be the transposition of $i, i+1$ and let $\mu_i : \bar{T}^{w\,n}A(X) \to \bar{T}^{w\,n-1}A(X)$ be the map induced by the chiral product μ_A at $i, i+1$ variables. One has $\mu_i\sigma_i = -\mu_i$.

The span of elements $r(\bar{a}) := \bar{a} - \sigma_i\bar{a} - \mu_i\bar{a}$ for $\bar{a} \in \bar{T}_x^{w\,n}A$, n, i as above, is a two-sided ideal $\bar{I}_x^w A$. We define the *Wick algebra* A_x^w of A at x as the quotient of $\bar{T}_x^w A$ modulo $\bar{I}_x^w A$ and an extra (central) relation $1_A \cdot \bar{\nu} = \mathrm{Res}_x \bar{\nu}$ for $\bar{\nu} \in h(\omega_{K_x})$.[41]

Our A_x^w carries a *commutative* filtration $A_{x0}^w \subset A_{x1}^w \subset \cdots$, $A_{xn}^w :=$ the image of $\bar{T}_x^{w\,\leq n}A$. There is a canonical morphism of Lie algebras

$$hA_{K_x} \to A_{1x}^w. \tag{3.6.11.1}$$

Let us define a canonical morphism of associative algebras

$$\delta : A_x^w \to A_x^{as} \tag{3.6.11.2}$$

compatible with the standard Lie algebra maps from $h(A_{K_x})$.

3.6.12. Lemma. *There is a commutative diagram of morphisms of associative algebras*

$$\begin{array}{ccc} T_x^w A & \xrightarrow{\tilde{\delta}} & \vec{T}A_{K_x} \\ \downarrow & & \downarrow \vec{T}\zeta \\ \bar{T}_x^w A & \xrightarrow{\bar{\delta}} & \vec{T}\hat{h}A_{K_x} \\ \downarrow & & \downarrow \pi \\ A_x^w & \xrightarrow{\delta} & A_x^{as} \end{array} \tag{3.6.12.1}$$

[40]As usual, h means coinvariants of the Lie algebra of vector fields acting on our right $\mathcal{D}_{K_x^{(n)}}$-module; if t is a local coordinate at x, these coinvariants coincide with the coinvariants of the vector fields $\partial_{t_1}, \dots, \partial_{t_n}$.

[41]Here $1_A \cdot \bar{\nu} \in h(A_{K_x}) = \bar{T}_x^{w\,1}A$.

in which the top horizontal arrow $\tilde{\delta}$ is a morphism of $\mathbb{Z}_{\geq 0}$-graded algebras such that every component $\tilde{\delta}^n$ is $O_x^{(n)}$-linear and $\tilde{\delta}^1 = id_{A_{K_x}}$. The left vertical arrows are the projections from 3.6.11; for the right ones see 3.6.9. Such a diagram is unique.

Proof. According to 3.6.1 every $A_{K_x}^{\vec{\otimes}n}$ is an $O_x^{(n)}$-module and the multiplication by $t_i - t_j$, $i \neq j$, on it is invertible. Therefore the morphism of graded algebras $TA_{K_x} \to \vec{T}A_{K_x}$ which is $\mathrm{id}_{A_{K_x}}$ in degree 1 extends uniquely to a morphism of graded algebras $\tilde{\delta} : T_x^w A \to \vec{T}A_{K_x}$ such that every $\tilde{\delta}^n$ is $O_x^{(n)}$-linear.

The left vertical arrows in (3.6.12.1) are surjections, so $\tilde{\delta}$ determines the other horizontal arrows uniquely. It remains to show that they are well defined.

Consider $\bar{\delta}$ first. We need to check that the composition $T_x^{w\,n} A_{K_x} \xrightarrow{\tilde{\delta}} A_{K_x}^{\vec{\otimes}n} \to (\hat{h}A_{K_x})^{\vec{\otimes}n}$ vanishes on the image of every operator ∂_{t_i}, $1 \leq i \leq n$. Notice that ∂_t is a continuous operator on A_{K_x}. Define $\partial_{t_i} \in \mathrm{End}\, A_{K_x}^{\vec{\otimes}n}$ as the tensor product of ∂_t at the ith place and $\mathrm{id}_{A_{K_x}}$ at other places. Then $\tilde{\delta}$ commutes with ∂_{t_i}. The composition $A_{K_x}^{\vec{\otimes}n} \xrightarrow{\partial_{t_i}} A_{K_x}^{\vec{\otimes}n} \to \hat{h}(A_{K_x})^{\vec{\otimes}n}$ vanishes, and we are done.

To see that δ is well defined, we have to check that $\bar{T}_x^w A \xrightarrow{\bar{\delta}} \vec{T}\hat{h}(A_{K_x}) \xrightarrow{\pi} A_x^{as}$ kills $r(\bar{a}) := \bar{a} - \sigma_i \bar{a} - \mu_i \bar{a}$ for every $a \in T_x^{w\,n} A$, $n \geq 2$, $i = 1, \dots, n-1$.[42] For $n = 2$ this follows from the Jacobi identity for the chiral bracket.[43] For n, i arbitrary we have $a = f(b \otimes c \otimes d)$ where f is in $O_x^{(n)}$ localized with respect to all the $t_a - t_b$'s except $t_i - t_{i+1}$, $b \in A_{K_x}^{\otimes i-1}$, $c \in A_{K_x}^{\otimes 2}[(t_i - t_{i+1})^{-1}]$, $d \in A_{K_x}^{\otimes n-i-1}$. Consider f as an element of the ring $R_{n,i} := k(\!(t_1)\!) \cdots (\!(t_{i-1})\!)[\![t_i, t_{i+1}]\!][t_i^{-1}, t_{i+1}^{-1}](\!(t_{i+2})\!) \cdots (\!(t_n)\!)$, so $f = \sum f_{\alpha_1 \cdots \alpha_n} t_1^{\alpha_1} \cdots t_n^{\alpha_n}$ where $f_{\alpha_1 \cdots \alpha_n} \in k$. Then $\bar{\delta} r(\bar{a})$ is the sum of a convergent series $\sum f_{\alpha_1 \cdots \alpha_n} (t_1^{\alpha_1} \cdots t_{i-1}^{\alpha_{i-1}} b) \otimes r(t_i^{\alpha_i} t_{i+1}^{\alpha_{i+1}} c) \otimes (t_{i+2}^{\alpha_{i+2}} \cdots t_n^{\alpha_n} d)$. Since π is a continuous morphism of algebras which kills every $r(t_i^{\alpha_i} t_{i+1}^{\alpha_{i+1}} c)$, one has $\pi \bar{\delta} r(\bar{a}) = 0$, and we are done. □

REMARK. By 3.6.10, the map π identifies A_x^{as} with the topological algebra quotient of $\vec{T}\hat{h}A_{K_x}$ modulo the ideal generated by $\bar{\delta}(\mathrm{Ker}(\bar{T}_x^w A \to A_x^w))$.

3.6.13. The construction of A_x^{as} generalizes immediately to the situation when x depends on parameters. Namely, assume we are in the situation of 2.1.16, so we have a quasi-compact and quasi-separated scheme Y and a Y-point $x \in X(Y)$. Let $i_x : Y \hookrightarrow X \times Y$ be the graph of x and let $j_x : U_x \hookrightarrow X \times Y$ be its complement.

For a chiral $\mathcal{O}_Y$-algebra $\mathcal{A}$ on X (see 3.3.10) $j_{x*} j_x^* \mathcal{A}$ is again a chiral $\mathcal{O}_Y$-algebra on X. The lemma in 3.6.3 (together with its proof) remains valid in the present setting: we have an equivalence of categories

$$\mathcal{M}(X \times Y, j_{x*} j_x^* \mathcal{A})_x \xrightarrow{\sim} \mathcal{M}(X \times Y, \mathcal{A})_x \qquad (3.6.13.1)$$

where the lower index x means the full subcategory of modules supported (set-theoretically) at the graph of x (see 3.3.10 for notation).

Denote by Ξ_x^{as} the set of chiral $\mathcal{O}_Y$-subalgebras $\mathcal{A}_\xi \subset j_{x*} j_x^* \mathcal{A}$ which coincide with $\mathcal{A}$ over U_x. This is a topology on $j_{x*} j_x^* \mathcal{A}$ at x (we use terminology from 2.1.13 in the version of 2.1.16). The fibers $\mathcal{A}_{\xi x}^{\ell} := i_x^* \mathcal{A}_\xi^{\ell} = i_x^!(j_{x*} j_x^* \mathcal{A})/\mathcal{A}_\xi)$ form a Ξ_x^{as}-projective system of $\mathcal{O}_Y$-modules connected by surjective morphisms. We

[42] The extra relation $1_A \cdot \bar{\nu} = \mathrm{Res}_x \bar{\nu}$ for $\bar{\nu} \in h(\omega_{K_x})$ holds by the definition of A_x^{as}.
[43] Cf. the proof in 3.6.10.

denote by $\mathcal{A}_x^{as}$ its projective limit. Equivalently (see 2.1.16), $\mathcal{A}_x^{as}$ is the completion of $i_x^{\cdot} h(j_{x*}j_x^*\mathcal{A})$ with respect to the Ξ_x^{as}-topology. So every $\mathcal{A}_{\xi x}^{\ell}$ is a quotient of $\mathcal{A}_x^{as}$ modulo an open submodule $\mathcal{I}_\xi$ and $\mathcal{A}^{as} = \varprojlim \mathcal{A}^{as}/\mathcal{I}_\xi$.

The lemma in 3.6.5 (together with its proof) remains valid, as well as the constructions and statements of 3.6.6. Therefore $\mathcal{A}_x^{as}$ is an associative unital topological $\mathcal{O}_Y$-algebra; every $\mathcal{I}_\xi$ is an open left ideal in $\mathcal{A}_x^{as}$. The lemma in 3.6.7 together with its proof[44] remain valid, so we have an equivalence of categories

$$i_x^! = h_x : \mathcal{M}(X\times Y, \mathcal{A})_x \xrightarrow{\sim} \mathcal{A}_x^{as}\text{mod} \tag{3.6.13.2}$$

where $\mathcal{A}_x^{as}$mod is the category of *discrete A_x^{as}-modules* := the sheaves of discrete left $\mathcal{A}_x$-modules on Y which are quasi-coherent as $\mathcal{O}_Y$-modules. The lemma in 3.6.8 together with the proof also remains valid.

EXERCISE. (cf. Remark (i) in 2.1.16). Check that the formation $\mathcal{A}_{xY}^{as}$ is compatible with the base change (i.e., for every $f : Y' \to Y$ the pro-$\mathcal{O}_{Y'}$-modules $f^*\mathcal{A}_{xY}^{as}$ and $\mathcal{A}_{xf\,Y'}^{as}$ coincide).

3.6.14. Suppose now that Y from 3.6.13 is another copy of X, $x = \mathrm{id}_X$ (so $i_x = \Delta$, $j_x = j : U \hookrightarrow X\times X$). Let A be a chiral algebra on X. Applying 3.6.13 to $\mathcal{A} = A_Y := A \boxtimes \mathcal{O}_Y$, we get an associative topological $\mathcal{O}_X$-algebra $A^{as} := \mathcal{A}_x^{as}$.

Let τ be any infinitesimal automorphism of X. It acts on $X\times X = X\times Y$ along the Y-copy of X preserving U. This action lifts in the obvious way to A_Y, hence to $j_*j^*A_Y$. Therefore[45] τ acts on A^{as} as on a topological $\mathcal{O}_X$-algebra in a natural way. Thus A^{as} carries a canonical flat connection; we denote it by ∇.

Let $\mathcal{I}_0 \subset A^{as}$ be the open left ideal that corresponds to $A_Y \in \Xi_x^{as}$. Then $\mathcal{I}_0$ is preserved by ∇ and one has a canonical identification of left $\mathcal{D}_X$-modules

$$A^{as}/\mathcal{I}_0 = A^{\ell}. \tag{3.6.14.1}$$

For a section a of $j_*j^*A\boxtimes\mathcal{O}_X$ we denote by $\mathcal{V}(a)$ the image of a in A^{as}. The map $\mathcal{V}$ commutes with the action of $\mathcal{D}_X$ along the second variable of the source, and is a differential operator with respect the $\mathcal{O}_X$-action along the first variable. For $b \in A$ the section $\mathcal{V}(b\boxtimes 1)$ belongs to $\mathcal{J}_0$, is ∇-horizontal, and depends only on the class $\bar{b} \in h(A)$. We denote it by $\bar{b}^{as}$. So one has a morphism of sheaves

$$h(A) \to \mathcal{I}_0^{\nabla} \subset (A^{as})^{\nabla},\ \bar{b} \mapsto \bar{b}^{as}. \tag{3.6.14.2}$$

3.6.15. Suppose M is an A-module. Then the $\mathcal{O}_X$-module M^ℓ carries a canonical A^{as}-action. Indeed, $M\boxtimes\mathcal{O}_Y$ is an A_Y-module, so $(j_*j^*M\boxtimes\mathcal{O}_X)/M\boxtimes\mathcal{O}_X$ is an A_Y-module supported at the diagonal. Since $M^\ell = \Delta^!((j_*j^*M\boxtimes\mathcal{O}_X)/M\boxtimes\mathcal{O}_X) = h((j_*j^*M\boxtimes\mathcal{O}_X)/M\boxtimes\mathcal{O}_X)$, it is an A^{as}-module by (3.6.13.2); here h is the integration along the X-multiple of $X\times Y$ (see 2.1.16). This picture is equivariant with respect to the action of infinitesimal symmetries of X acting along the Y-multiple,[46] so the A^{as}-action on M^ℓ is compatible with the connections.[47]

[44] Use a version of Kashiwara's lemma from 2.1.16.

[45] Use Exercise in 3.6.13.

[46] Use the h part of the above formula.

[47] M^ℓ carries a connection since it is a left $\mathcal{D}_X$-module.

EXAMPLE. If $M = A$, then the A^{as}-action on M^ℓ comes from (3.6.14.1).

Denote by $\mathcal{M}^\ell(X, A^{as})$ the category whose objects are discrete A^{as}-modules equipped with a connection (as $\mathcal{O}_X$-modules) such that the A^{as}-action is horizontal. To say it differently, the connection ∇ on A^{as} defines on $A^{as}[\mathcal{D}_X] := A^{as} \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X$ the structure of an associative algebra,[48] and $\mathcal{M}^\ell(X, A^{as})$ is the category of *discrete* $A^{as}[\mathcal{D}_X]$*-modules*, i.e., left unital $A^{as}[\mathcal{D}_X]$-modules which are discrete A^{as}-modules.

3.6.16. PROPOSITION. *There is a canonical equivalence of categories*

$$\mathcal{M}(X, A) \xrightarrow{\sim} \mathcal{M}^\ell(X, A^{as}). \tag{3.6.16.1}$$

Proof. We have seen in 3.6.15 that for $M \in \mathcal{M}(X, A)$ the left $\mathcal{D}_X$-module M^ℓ carries a canonical horizontal A^{as}-action, so we have the functor $\mathcal{M}(X, A) \to \mathcal{M}^\ell(X, A^{as})$. This is an equivalence of categories. Indeed, by Kashiwara's lemma, the functor $M \mapsto (j_*j^*M \boxtimes \mathcal{O}_Y)/M \boxtimes \mathcal{O}_Y$ identifies the category of A-modules with the category of A_Y-modules supported on the diagonal and equivariant with respect to the action of infinitesimal automorphisms of Y (acting along the Y-multiple; recall that Y is another copy of X). Now use (3.6.13.2) and the definition of the canonical connection ∇ on A^{as}. □

VARIANT. Forgetting about ∇, we see that the category of discrete A^{as}-modules is canonically equivalent to $\mathcal{M}_{\mathcal{O}}(X, A)$ (see 3.3.5(ii)).

REMARK. One can describe chiral A-operations in terms of the A^{as}-module structures using the lemma from 3.4.19.

3.6.17. For an $\mathcal{O}_X$-module M the above 1-1 correspondence between discrete A^{as}-actions and chiral A-actions on M can be rewritten as follows.

The (non-quasi-coherent) $\mathcal{O}_X$-module $A^{as} \otimes M$ carries a topology whose open subsheaves are $V \subset A^{as} \otimes M$ that satisfy the following property: for every local section $m \in M$ there exist an open ideal $I_\xi \subset A^{as}$ such that $I_\xi \otimes m \subset V$. It is clear that every open subsheaf contains a smaller open subsheaf V such that V is an $\mathcal{O}$-submodule and $A^{as} \otimes M/V$ is a quasi-coherent $\mathcal{O}_X$-module. Denote the completion by $A^{as} \hat{\otimes} M$.

The sheaf $j_*j^*A \boxtimes M$ carries a topology whose open subsheaves are $W \subset j_*j^*A \boxtimes M$ that satisfy the following property: for every local section $m \in M$ there exists $A_\xi \subset j_*j^*(A \boxtimes \mathcal{O}_X)$ in Ξ^{as}_Δ such that $A_\xi \otimes m \subset W$. Every open subsheaf contains a smaller open subsheaf W which is a quasi-coherent $\mathcal{D}_X \boxtimes \mathcal{O}_X$-module. The completion $(j_*j^*A \boxtimes M)\hat{}$ is a topological $\mathcal{D}_X \boxtimes \mathcal{O}_X$-module.

The identifications $j_*j^*A \boxtimes \mathcal{O}_X/A_\xi = \Delta_* A^\ell_{\xi\Delta} = \Delta_*(A^{as}/I_\xi)$ of $\mathcal{D}_X \boxtimes \mathcal{O}_X$-modules yield a canonical isomorphism of topological $\mathcal{D}_X \boxtimes \mathcal{O}_X$-modules

$$(j_*j^*A \boxtimes M)\hat{} \xrightarrow{\sim} \Delta_*(A^{as} \hat{\otimes} M). \tag{3.6.17.1}$$

Now every chiral A-action on M is automatically continuous in the above topology, so it can be considered as a morphism $(j_*j^*A \boxtimes M)\hat{} \to \Delta_* M$. Similarly, every discrete A^{as}-action on M can be cosidered as a morphism $A^{as} \hat{\otimes} M \to M$. A chiral A-action μ_M is identified with an A^{as}-action $\cdot_M$ if (3.6.17.1) identifies $\Delta_*(\cdot_M)$ with μ_M.

[48] It is characterized by the properties that the maps $A^{as}, \mathcal{D}_X \to A^{as}[\mathcal{D}_X]$, $a \mapsto a \otimes 1, \partial \mapsto 1 \otimes \partial$, are morphisms of algebras, and for $a \in A^{as}, \tau \in \Theta_X$ one has $\tau a - a\tau = \tau(a)$.

If M is a $\mathcal{D}_X$-module, then the $\mathcal{D}_X$-action on $A^{as} \otimes M$ extends by continuity to $A^{as}\hat{\otimes} M$, $(j_*j^*A \boxtimes M)\hat{}$ is a topological $\mathcal{D}_{X\times X}$-module, and (3.6.17.1) is an isomorphism of topological $\mathcal{D}_{X\times X}$-modules. Therefore μ_M is horizontal if and only if $\cdot_M$ is.

3.6.18. The commutative case. Suppose now that our A is commutative. Then the algebras A_x^{as}, A^{as} are also commutative. We already met A_x^{as} in the situation when x is a single point, $x \in X(k)$ (see 2.4.8). The "geometric" description of A_x^{as} from 2.4.9 generalizes immediately to the general situation. Namely, for $x \in X(Y)$ as in 3.6.13 and a commutative quasi-coherent $\mathcal{O}_Y$-algebra F we define a *non-quasi-coherent* $\mathcal{O}_{X\times Y}$-algebra $O_x\hat{\otimes} F$ as the formal completion of $\mathcal{O}_X \boxtimes F$ at (the graph) of x; let $K_x\hat{\otimes} F$ be its localization with respect to an equation of x. Our $O_x\hat{\otimes} F \subset K_x\hat{\otimes} F$ are $\mathcal{D}_{X\times Y/Y}$-algebras; i.e., they carry an obvious connection along X. Now one has a canonical identification

$$\text{(3.6.18.1)}\quad \operatorname{Hom}(A_x^{as}, F) = \operatorname{Hom}(A^\ell \boxtimes \mathcal{O}_Y, K_x\hat{\otimes} F) = \operatorname{Hom}(j_{x*}j_x^*(A^\ell \boxtimes \mathcal{O}_Y), K_x\hat{\otimes} F)$$

where the first Hom means continuous morphisms of $\mathcal{O}_Y$-algebras[49] and the next ones mean morphisms of $\mathcal{D}_{X\times Y/Y}$-algebras. For the construction see the proof in 2.4.9. As was mentioned in loc. cit., (3.6.18.1) is the subset of the algebra morphisms in (3.5.4.2).

3.6.19. LEMMA. *If $\varphi : A^\ell \to B^\ell$ is a formally smooth morphism of commutative $\mathcal{D}_X$-algebras, then the corresponding morphism of topological $\mathcal{O}_Y$-algebras $A_x^{as} \to B_x^{as}$ is formally smooth. The same is true if we replace "formally smooth" by "formally étale."*[50]

Proof. Assume that φ is formally smooth. Let F be a commutative k-algebra, $I \subset F$ an ideal such that $I^2 = 0$. Let $f : \operatorname{Spec} F \to \operatorname{Spf} A_x^{as}$, $g : \operatorname{Spec} F/I \to \operatorname{Spf} B_x^{as}$ be φ-compatible morphisms. We want to find a lifting $h : \operatorname{Spec} F \to \operatorname{Spf} B_x^{as}$ of f that extends g.

According to (3.6.18.1) we may rewrite f, g as morphisms of $\mathcal{D}_{X\times Y/Y}$-algebras $f' : A^\ell \boxtimes \mathcal{O}_Y \to K_x\hat{\otimes} F$, $g' : B^\ell \boxtimes \mathcal{O}_Y \to K_x\hat{\otimes} F/I$. Here $K_x\hat{\otimes} F$, $K_x\hat{\otimes} F/I$ are non-quasi-coherent sheaves of $\mathcal{D}_{X\times Y/Y}$-algebras.

Our problem is local (see [**RG**]) so we can assume that X and Y are affine. Denote by Γ the functor of global sections over $X \times Y$. So we have commutative $\mathcal{O}(X)$-algebras $\Gamma(K_x\hat{\otimes} F)$, $\Gamma(K_x\hat{\otimes} F/I)$ equipped with a connection along X. The second algebra is a quotient of the first one modulo an ideal of square 0.

Our f', g' amount to maps of commutative $\mathcal{O}(X)$-algebras $\Gamma f' : A^\ell(X) \to \Gamma(K_x\hat{\otimes} F)$, $\Gamma g' : B^\ell(X) \to \Gamma(K_x\hat{\otimes} F/I)$ compatible with connections along X. Since φ is a formally smooth morphism of $\mathcal{D}_X$-algebras, we can find a lifting $\Gamma h' : B^\ell(X) \to \Gamma(K_x\hat{\otimes} F)$ of $\Gamma f'$ that extends $\Gamma g'$. As above, $\Gamma h'$ is the same as a morphism $h : \operatorname{Spec} F \to \operatorname{Spf} B_x^{as}$ which is the desired lifting.

The case of formally étale φ is left to the reader. □

3.6.20. The global Wick algebra. In the rest of this section we make a few remarks about the global version $A^w(X)$ of the Wick algebra; they will not be used elsewhere in the book. We assume that X is an affine curve.

[49] F is assumed to carry the discrete topology.

[50] For the definition of formally smooth and formally étale morphisms of commutative $\mathcal{D}_X$-algebras (or, more generally, algebraic $\mathcal{D}_X$-spaces) see 2.3.16.

First we have a $\mathbb{Z}_{\geq 0}$-graded associative algebra $T^w A(X) = \oplus T^{w\,n} A(X)$ where (we use the notation of 3.1.1) $T^{w\,n}A(X) := \Gamma(U^{(n)}, A^{\boxtimes n})$ and the multiplication is the exterior tensor product. Passing to H^0_{DR}, we get the quotient algebra[51] $\bar{T}^w A(X) := \oplus H^0_{DR}(U^{(n)}, A^{\boxtimes n})$.

For every $n \geq 2$ the action of the symmetric group Σ_n on X^n and $A^{\boxtimes n}$ provides a Σ_n-action on the nth component of $T^{w\,n}A(X)$ and $\bar{T}^{w\,n}A(X)$. For $i = 1, \ldots, n-1$ let $\sigma_i \in \Sigma_n$ be the transposition of $i, i+1$ and let $\mu_i : \bar{T}^{w\,n}A(X) \to \bar{T}^{w\,n-1}A(X)$ be the map induced by the chiral product μ_A at $i, i+1$ variables. One has $\mu_i \sigma_i = -\mu_i$.

Let $\bar{I}^w A(X) \subset \bar{T}^w A(X)$ be the span of (non-homogeneous) elements $r(\bar{a}) := \bar{a} - \sigma_i \bar{a} - \mu_i \bar{a}$ for all $\bar{a} \in \bar{T}^{w\,n}A(X)$, n, i as above. This is a two-sided ideal in $\bar{T}^w A(X)$. Our $A^w(X)$ is the quotient algebra $\bar{T}^w A(X)/\bar{I}^w A(X)$. It carries an increasing filtration $A^w_n(X) :=$ the image of $\bar{T}^{w\,\leq n}A(X)$ in $A^w(X)$, $A^w(X) = \bigcup A^w_n(X)$. Thus $A^w_\cdot$ is a $\mathbb{Z}_{\geq 0}$-filtered associative unital algebra that depends on A in a functorial way.

Remarks. (i) The constructions are functorial with respect to the étale base change, so, replacing X by étale X-schemes, we get a presheaf A^w on $X_{\acute{e}t}$. There is an evident morphism of Lie algebras

$$h(A) \to A^w_1. \tag{3.6.20.1}$$

(ii) There is a (more local) variant of the above constructions with $T^{w\,n}A(X)$ replaced by $\Gamma(X^n, A^{\boxtimes n} \otimes \tilde{\Delta}^{(n)}\mathcal{O}_X)$ (see (3.5.5.1) for notation).

The following properties of A^w are immediate:

(i) $A^w_\cdot$ is a commutative filtration: one has $[A^w_m, A^w_n] \subset A^w_{m+n-1}$. So $\mathrm{gr}_\cdot A^w$ is a commutative algebra.

(ii) If the morphisms of chiral algebras $A, B \to C$ mutually commute (see 3.4.15), then the images of A^w, B^w in C^w mutually commute. In particular, if R is a commutative chiral algebra, then R^w is commutative; if A is a chiral R-algebra, then A^w is an R^w-algebra.

3.6.21. Let us compute the Wick algebra of the unit chiral algebra ω. Recall that $h(\omega)$ is the sheaf $V \mapsto H^1_{DR}(V)$.

Proposition. *There is a canonical isomorphism of filtered algebras*

$$\mathrm{Sym}\, h(\omega) \xrightarrow{\sim} \omega^w.$$

Proof. Since ω^w is commutative, (3.6.20.1) yields a morphism $\mathrm{Sym}\, h(\omega) \to \omega^w$. We want to prove that this is an isomorphism of filtered algebras or, equivalently, that the corresponding morphism of graded algebras $\mathrm{Sym}\, h(\omega) \to \mathrm{gr}\, \omega^w$ is an isomorphism. It suffices to show that the maps $\mathrm{Sym}^n H^1_{DR}(X) \to \mathrm{gr}_n \omega^w(X)$ are isomorphisms.

Surjectivity: Let K be the kernel of $H^n_{DR}(U^{(n)}) = \bar{T}^{w\,n}\omega(X) \twoheadrightarrow \mathrm{gr}_n \omega^w(X)$. We want to check that K together with the image of $\mathrm{Sym}^n H^1_{DR}(X) \subset H^1_{DR}(X)^{\otimes n} = H^n_{DR}(X^n)$ in $H^n_{DR}(U^{(n)})$ span $H^n_{DR}(U^{(n)})$. Notice that the action of the symmetric group Σ_n on $\bar{T}^{w\,n}\omega(X)$ is the obvious action on $H^n_{DR}(U^{(n)})$ multiplied by the sign character (see 3.1.4). Look at the Σ_n-action on the spectral sequence computing $H^\cdot_{DR}(U^{(n)})$ which corresponds to the filtration W on $j^{(n)}_* \omega_{U^{(n)}}$ (see 3.1.6 and 3.1.7).

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[51] Recall that the $A^{\boxtimes n}$ are *right* $\mathcal{D}$-modules, so H^0_{DR} is the "middle" cohomology; see 2.1.7.

It shows that the map $\mathrm{Sym}^n H^1_{DR}(X) \to H^n_{DR}(U^{(n)})$ is injective and its image is the subspace $H^n_{DR}(U^{(n)})^{sgn}$ of the elements that transform according to the sign character. As follows from the definition of $\bar{I}^w$, the subspace K contains every subspace $H^n_{DR}(U^{(n)})^{\sigma_i}$ of σ_i-invariants. We are done since for every Σ_n-module P one has $P = P^{sgn} + \Sigma P^{\sigma_i}$.[52]

Injectivity: Here is a topological argument. We can assume that $k = \mathbb{C}$, so the dual vector space to $H^1_{DR}(X)$ identifies with the singular homology group $H_1(X, \mathbb{C})$. We have proved that Spec $\mathrm{gr}\,\omega^w(X)$ is a closed subscheme of $H_1(X, \mathbb{C})$, and we want to show that it equals $H_1(X, \mathbb{C})$. It suffices to check that it contains every integral homology class γ. Thus we need to construct a morphism of $\mathbb{C}$-algebras $\gamma^w : \omega^w(X) \to \mathbb{C}$ whose composition with the canonical map $H^1_{DR}(X) \to \omega^w(X)$ is integration along γ. Let us represent γ by a union of several mutually non-intersecting oriented C^∞-loops γ_α each of which has no self-intersections (i.e., γ_α is an image of a regular embedding $S^1 \hookrightarrow X$). Let y_α be a normal coordinate function on a tubular neighbourhood of γ_α. We assume that the orientation of y_α, i.e., the sign of dy_α on $\gamma_\alpha = \{y_\alpha = 0\}$, is choosen in such a way that if γ_α is the unit circle $S^1 \subset \mathbb{C}$ with the standard orientation, then $y_\alpha(z) = \log|z|$ is well oriented. So for a small $\epsilon \in \mathbb{R}$ we have an oriented loop $\gamma_{\alpha\,\epsilon} := \{y_\alpha = \epsilon\}$. For every $n \geq 1$ consider a cycle $\gamma_n = \Sigma\gamma_{\alpha\,n}$ on $U^{(n)}$, $\gamma_{n\,\alpha} := \gamma_{\alpha\,\epsilon_1} \times \cdots \times \gamma_{\alpha\,\epsilon_n}$ where the ϵ_i are small and $\epsilon_1 > \cdots > \epsilon_n$. The functionals $\int\limits_{\gamma_n} : H^n_{DR}(U^{(n)}) \to \mathbb{C}$ form a morphism of $\mathbb{C}$-algebras $T^w\gamma : \bar{T}^w\omega(X) \to \mathbb{C}$. It kills the ideal $\bar{I}^w\omega(X)$, so we have defined our $\gamma^w : \omega^w(X) \to \mathbb{C}$. □

According to property (ii) in 3.6.20 and 3.6.21 the Wick algebra of any chiral algebra is, in fact, a filtered associative $\mathrm{Sym}\,h(\omega)$-algebra.

REMARK. For chiral algebras A, B a canonical morphism $A^w \otimes B^w \to (A \otimes B)^w$ defined by property (iii) in 3.6.20 comes from $A^w \underset{\mathrm{Sym}\,h(\omega)}{\otimes} B^w \to (A \otimes B)^w$. The latter arrow *need not* be surjective.

3.7. From Lie* algebras to chiral algebras

The rest of this chapter deals with some methods of constructing chiral algebras. This section treats chiral enveloping algebras. Different proofs of the theorems in 3.7.1 and 3.7.14 in the graded vertex algebra setting (see 0.15) can be found in [**FBZ**] and in the preprint version of [**GMS2**].

The observation that the vacuum representation of a Kac-Moody or Virasoro algebra carries a canonical structure of vertex algebra goes back to the first days of the vertex algebra theory and beyond. Needless to say, this fact was always known to mathematical physicists.

We begin in 3.7.1 with the existence theorem for chiral envelopes. A more general theorem which deals with arbitrary Jacobi type operads (instead of the Lie operad) is formulated in 3.7.2; the proof takes 3.7.3–3.7.4 (the results of 3.7.2–3.7.4 will not be used elsewhere in the book and the reader can skip them). An explicit factorization algebra construction of chiral envelopes is presented in 3.7.5–3.7.11;

[52]To see this, we can assume that P is irreducible and $P \neq P^{sgn}$. Then $P^{\sigma_i} \neq 0$ (by Young's table picture), so it suffices to show $P^s := \Sigma P^{\sigma_i} \subset P$ is preserved by the Σ_n-action, or, equivalently, P^s is preserved by every σ_i. Now σ_i preserves P^{σ_j} for $j \neq i \pm 1$ (since σ_i, σ_j commute), and it preserves both $P^{\sigma_i} + P^{\sigma_{i\pm 1}}$ (an exercise in representation theory of Σ_3).

the key tool is an auxiliary Lie algebra $L^\natural$ in the tensor category of the left $\mathcal{D}$-modules on $\mathcal{R}(X)$ (see 3.7.6). This construction implies immediately the Poincaré-Birkhoff-Witt theorem for chiral envelopes, see 3.7.14. We relate L- and $U(L)$-modules in 3.7.15–3.7.18, describe the associative algebra $U(L)^{as}_x$ in 3.7.19, and consider the twisted envelopes in 3.7.20–3.7.22. The enveloping algebra construction for a Lie* algebra acting on a chiral algebra is treated in 3.7.23–3.7.24. Finally, we discuss in 3.7.25 Virasoro vectors and the Sugawara construction.

3.7.1. Consider the forgetful functor $\mathcal{CA}(X) \to \mathcal{L}ie^*(X)$, $A \mapsto A^{Lie}$.

THEOREM. *This functor admits a left adjoint functor* $U : \mathcal{L}ie^*(X) \to \mathcal{CA}(X)$.

For a Lie* algebra L we call $U(L)$ the *chiral enveloping algebra*, or simply the *chiral envelope* of L.

Proof. This is an immediate corollary of the theorem in 3.4.14. Namely, let A be the chiral algebra freely generated by (L, P) where P is the image of $\mathrm{Ker}[\ ]_L \subset L \boxtimes L$ in $j_*j^*L \boxtimes L$. Then $U(L)$ is the quotient of A modulo the ideal generated by the relations saying that the morphism $L \to U(L)$ is a morphism of Lie* algebras. Precisely, let $i_A : L \to A$ be the universal morphism. Let A' be the quotient of A modulo the ideal generated by elements $i(\ell\tau) - i(\ell)\tau$ where $\ell \in L$ and $\tau \in \Theta_X$, so $i_{A'} : L \to A'$ is a morphism of $\mathcal{D}_X$-modules. Then $U(L)$ is the quotient of A' modulo the ideal generated by the image of $i_{A'}[\ ]_L - [\ ]_{A'}\, i_{A'}^{\boxtimes 2} : L \boxtimes L \to \Delta_* A'$. $\square$

3.7.2. The above theorem remains true for algebras over more general operads.

DEFINITION. A k-operad $\mathcal{B}$ is of *Jacobi type* if it is generated by $\mathcal{B}_1$ and $\mathcal{B}_2$, and for every $\alpha, \alpha' \in \mathcal{B}_2$ there exist $\beta, \beta', \gamma, \gamma' \in \mathcal{B}_2$ such that $\alpha(x_1, \alpha'(x_2, x_3)) = \beta(x_3, \beta'(x_1, x_2)) + \gamma(x_2, \gamma'(x_3, x_1))$.

For example, the operads $\mathcal{L}ie$, $\mathcal{P}oiss$, $\mathcal{C}om$, $\mathcal{A}ss$ are of Jacobi type.

THEOREM. *For any operad $\mathcal{B}$ of Jacobi type the functor $\beta^{\mathcal{B}}$ of (3.3.1.1) admits a left adjoint functor* $U_{\mathcal{B}} : \mathcal{B}^*(X) \to \mathcal{B}^{ch}(X)$.

Composing $U_{\mathcal{L}ie}$ with the "adding of unit" functor (see 3.3.3) one gets the functor U of 3.7.1.

Proof. Here is the idea. If the chiral pseudo-tensor structure were representable (see 1.1.3), then 3.7.2 would be immediate for arbitrary $\mathcal{B}$. Indeed, representability implies that for every L the free $\mathcal{B}^{ch}$ algebra generated by L is well defined. So, if L is a $\mathcal{B}^*$ algebra, then $U_{\mathcal{B}}(L)$ is a quotient of the free $\mathcal{B}^{ch}$ algebra modulo the obvious relations. Now the *binary* chiral pseudo-tensor product is representable up to the non-representability of the $*$ pseudo-tensor product. This shows that the *quadratic* part of $U_{\mathcal{B}}(L)$ is well defined. To get control of all of $U_{\mathcal{B}}(L)$, one needs the Jacobi property of $\mathcal{B}$.

Let us turn to the actual proof.

3.7.3. We begin with some complements to 3.4.10.

Let $\mathcal{M}(X^{\mathcal{S}})_\Delta \subset \mathcal{M}(X^{\mathcal{S}})$ be the full subcategory of those M for which every M_{X^I} is supported on the diagonal $X \subset X^I$; it is closed under subquotients and extensions. $\mathcal{M}(X^{\mathcal{S}})_\Delta$ is equivalent to the category $\mathcal{M}(X)^{\mathcal{S}}$ of all functors $\mathcal{S}^\circ \to \mathcal{M}(X)$ (one identifies $M \in \mathcal{M}(X^{\mathcal{S}})_\Delta$ with the functor $I \mapsto \Delta^{(I)!} M_{X^I}$). Notice that $\Delta_*^{(\mathcal{S})}$ identifies $\mathcal{M}(X)$ with the full subcategory of $\mathcal{M}(X^{\mathcal{S}})_\Delta$, and this embedding admits a left adjoint functor $\epsilon : \mathcal{M}(X^{\mathcal{S}})_\Delta \to \mathcal{M}(X)$, $M \mapsto \varinjlim \Delta^{(I)!} M_{X^I}$. It extends in the obvious way to pseudo-tensor functors

$$\mathcal{M}(X^{\mathcal{S}})^*_\Delta \to \mathcal{M}(X)^*, \quad \mathcal{M}(X^{\mathcal{S}})^{ch}_\Delta \to \mathcal{M}(X)^{ch} \tag{3.7.3.1}$$

which are left adjoint to the pseudo-tensor functors from (3.4.10.5).

For any k-operad $\mathcal{B}$ denote by $\mathcal{B}^*(X^{\mathcal{S}})$, $\mathcal{B}^{ch}(X^{\mathcal{S}})$ the categories of $\mathcal{B}$ algebras in $\mathcal{M}(X^{\mathcal{S}})^*$, $\mathcal{M}(X^{\mathcal{S}})^{ch}$ (we call them $\mathcal{B}^*$ and $\mathcal{B}^{ch}$ algebras on $X^{\mathcal{S}}$). Notice that by (3.4.10.4) every $\mathcal{B}$ algebra in $\mathcal{M}(X^{\mathcal{S}})^{ch}$ is automatically a $\mathcal{B}$ algebra in $\mathcal{M}(X^{\mathcal{S}})^*$; i.e., the pseudo-tensor functor $\beta^{\mathcal{S}}$ yields a functor $\mathcal{B}^{ch}(X^{\mathcal{S}}) \to \mathcal{B}^*(X^{\mathcal{S}})$. This functor admits a left adjoint functor

$$U^{\mathcal{S}}_{\mathcal{B}} : \mathcal{B}^*(X^{\mathcal{S}}) \to \mathcal{B}^{ch}(X^{\mathcal{S}}). \tag{3.7.3.2}$$

Indeed, for a $\mathcal{B}^*$ algebra L the corresponding $\mathcal{B}^{ch}$ algebra $U^{\mathcal{S}}_{\mathcal{B}}(L)$ is the quotient of the free $\mathcal{B}^{ch}$ algebra generated by L modulo the relations needed to assure that the canonical morphism $i^{\mathcal{S}} : L \to U^{\mathcal{S}}_{\mathcal{B}}(L)$ is a morphism of $\mathcal{B}^*$ algebras. Precisely, let Φ^*, Φ^{ch} be the free $\mathcal{B}^*$ and $\mathcal{B}^{ch}$ algebras generated by L. Since Φ^{ch} is a $\mathcal{B}^*$ algebra, the canonical morphism $i^{ch} : L \to \Phi^{ch}$ extends to a morphism of $\mathcal{B}^*$ algebras $\alpha : \Phi^* \to \Phi^{ch}$. Since L is a $\mathcal{B}^*$ algebra, the identity morphism id_L extends to a morphism of $\mathcal{B}^*$ algebras $\beta : \Phi^* \to L$. Now $U^{\mathcal{S}}_{\mathcal{B}}(L)$ is the quotient of Φ^{ch} modulo the $\mathcal{B}^{ch}$ ideal $\mathcal{J}$ generated by the image of $\alpha - i^{ch}\beta : \Phi^* \to \Phi^{ch}$.

3.7.4. We return to the proof of 3.7.2. Let L be a $\mathcal{B}^*$ algebra on X. We want to construct the corresponding universal $\mathcal{B}^{ch}$ algebra $U_{\mathcal{B}}(L)$ on X equipped with a morphism of $\mathcal{B}^*$ algebras $i : L \to U_{\mathcal{B}}(L)$.

According to (3.4.10.5), $\Delta^{(\mathcal{S})}_* L$ is a $\mathcal{B}^*$ algebra on $X^{\mathcal{S}}$. Set $U := U^{\mathcal{S}}_{\mathcal{B}}(\Delta^{(\mathcal{S})}_* L) \in \mathcal{B}^{ch}(X^{\mathcal{S}})$. In the next lemma we will show that $U \in \mathcal{M}(X^{\mathcal{S}})_\Delta$. Assuming this, set $U_{\mathcal{B}}(L) := \epsilon(U)$, $i := \epsilon(i^{\mathcal{S}}) : L \to U_{\mathcal{B}}(L)$. By (3.7.3.1), $U_{\mathcal{B}}(L)$ is a $\mathcal{B}$ algebra in $\mathcal{M}(X)^{ch}$ and i is a morphism of $\mathcal{B}^*$ algebras; by the adjunction property of ϵ the pair $(U_{\mathcal{B}}(L), i)$ satisfies the desired universal property. We are done.

It remains to prove the following lemma:

LEMMA. *If $\mathcal{B}$ is of Jacobi type, then $U \in \mathcal{M}(X^{\mathcal{S}})_\Delta$.*

Proof of Lemma. (a) For a finite set I and $\alpha, \beta \in I$, $\alpha \neq \beta$, let $\mathcal{B}^{\alpha\beta}_I \subset \mathcal{B}_I$ be the image of the composition map $\mathcal{B}_{I/\{\alpha,\beta\}} \otimes \mathcal{B}_{\{\alpha,\beta\}} \to \mathcal{B}_I$. A simple induction shows that $\mathcal{B}$ is of Jacobi type if and only if it satisfies the following property:

For any finite non-empty sets I_1, I_2 the vector space $\mathcal{B}_{I_1 \sqcup I_2}$ is generated by the subspaces $\mathcal{B}^{\alpha\beta}_{I_1 \sqcup I_2}$, $\alpha \in I_1$, $\beta \in I_2$.

(b) Let Φ^{ch} be the free $\mathcal{B}^{ch}$ algebra on $X^{\mathcal{S}}$ generated by L as in 3.7.3. It has a natural gradation $\Phi^{ch} = \bigoplus_{n \geq 1} \Phi^{ch}_n$ such that $\Phi^{ch}_n = (L^{\otimes^{ch} I} \otimes \mathcal{B}_I)_{\mathrm{Aut} I}$ where I is a set of order n. This gradation yields an increasing filtration on $U = \Phi^{ch}/\mathcal{J}$. We are going to show that $\mathrm{gr}\, U \in \mathcal{M}(X^{\mathcal{S}})_\Delta$; this will prove our lemma.

Clearly $\mathrm{Ker}(\Phi^{ch}_n \twoheadrightarrow \mathrm{gr}_n U) \supset N_{\mathrm{Aut}\, I}$ where $N := \sum_{\alpha \neq \beta \in I} K_{\alpha\beta} \otimes \mathcal{B}^{\alpha\beta}_I \subset L^{\otimes^{ch} I} \otimes \mathcal{B}_I$ and $K_{\alpha\beta}$ is the image of $(L \otimes^* L) \otimes^{ch} L^{\otimes^{ch}(I \smallsetminus \{\alpha,\beta\})}$ in $L^{\otimes^{ch} I}$. So it is enough to show that $M := (L^{\otimes^{ch} I} \otimes \mathcal{B}_I)/N$ belongs to $\mathcal{M}(X^{\mathcal{S}})_\Delta$.

Set $U_{\alpha\beta} := \{(x_i) \in X^I : x_i \neq x_j \text{ if } i \neq j, \{i,j\} \neq \{\alpha,\beta\}\}$; let $j_{\alpha\beta} : U_{\alpha\beta} \hookrightarrow X^I$ be the embedding. Set $F := j_*^{(I)} \mathcal{O}_{U^{(I)}} \otimes \mathcal{B}_I$, $G := \sum_{\alpha\neq\beta\in I} j_{\alpha\beta *}\mathcal{O}_{U_{\alpha\beta}} \otimes \mathcal{B}_I^{\alpha\beta} \subset F$, $H := F/G$. By definition, for every $J \in \mathcal{S}$ one has $M_{X^J} = \underset{J \twoheadrightarrow I}{\oplus} \Delta_*^{(J/I)}(L^{\boxtimes I} \otimes H)$. It remains to show that H is supported on the diagonal $X \subset X^I$.

(c) For $I = I_1 \sqcup I_2$ where $I_1, I_2 \neq \emptyset$ set $U_{I_1 I_2} := \{(x_i) \in X^I : x_i \neq x_j \text{ if } i \in I_1, j \in I_2\}$. On $U_{I_1 I_2}$ the sheaves $j_{\alpha\beta *}\mathcal{O}_{U_{\alpha\beta}}$ coincide with $j_*^{(I)}\mathcal{O}_{U^{(I)}}$. So (a) above implies that the restriction of H to $U_{I_1I_2}$ is zero. The union of all the $U_{I_1I_2}$'s is the complement to $X \subset X^I$, and we are done. □

3.7.5. Let us return to the setting of 3.7.1. Let L be a Lie* algebra. Below we will give an explicit construction of $U(L)$ as a factorization algebra (together with the canonical $\mathcal{D}$-module structure). Our problem is X-local, so we can assume that X is affine. The first step is to define an auxiliary Lie algebra $L^\natural$ in the tensor category of the left $\mathcal{D}$-modules on $\mathcal{R}(X)$ (see 3.7.6). The chiral envelope is constructed in 3.7.7 (it is denoted by V there). The universality property is checked in 3.7.11.

Remark. A slightly unpleasant point of our construction is that the auxiliary Lie algebra $L^\natural$ is a non-local object (while $U(L)$ is of course local). A remedy would be to consider in (3.7.6.1) instead of global cohomology the one of the formal neighbourhood of the divisor $\bigcup\{x = x_i\}$. The price is to deal with non quasi-coherent modules; we prefer not to do this.

In 3.7.6–3.7.11 we assume that X is affine.

3.7.6. Let I be a finite set. Denote by p_I the projection $X \times X^I \to X^I$, and let $j_I : V = V_{\tilde I} \hookrightarrow X \times X^I$ be the open subset of those $(x,(x_i))$ that $x \in V_{(x_i)} := X \smallsetminus \{x_i\}$. For $L \in \mathcal{M}^r(X)$[53] consider the de Rham complex $DR(L) = \mathcal{C}one(L \otimes \Theta_X \to L)$. Set

$$L^\natural_{X^I} := H^0(p_I j_I)_\cdot j_I^*(DR(L) \boxtimes \mathcal{O}_{X^I}). \tag{3.7.6.1}$$

This is an $\mathcal{O}_{X^I}$-module; its fiber at $(x_i) \in X^I$ equals $H^0_{DR}(V_{(x_i)}, L) = \Gamma(V_{(x_i)}, h(L))$. The obvious left $p_I^\cdot \mathcal{D}_X^I$-module structure on $DR(L) \boxtimes \mathcal{O}_{X^I}$ shows that $L^\natural_{X^I}$ is a left $\mathcal{D}_{X^I}$-module. In other words, $L^{\natural r}_{X^I} = H^0(p_I j_I)_* j_I^*(L \boxtimes \omega_{X^I})$.

Our $L^\natural$ carries a bunch of structures described in (i)–(iii) below that will be used in the construction of the chiral envelope:

(i) For every diagonal embedding $\Delta^{(I/S)} : X^S \hookrightarrow X^I$ we have an obvious identification of left $\mathcal{D}$-modules

$$\Delta^{(I/S)*} L^\natural_{X^I} \xrightarrow{\sim} L^\natural_{X^S}. \tag{3.7.6.2}$$

So the $L^\natural_{X^I}$ form a left $\mathcal{D}$-module on $\mathcal{R}(X)$ (see 3.4.2).

More generally, the $L^\natural_{X^I}$ behave nicely with respect to *arbitrary* standard morphisms between X^I's. Namely, if $\pi : J \to I$ is any map of finite sets, then the pull-back of $V_{\tilde J}$ by $\Delta^{(\pi)} : X^I \to X^J$ contains $V_{\tilde I}$. So we have the restriction map

$$\Delta^{(\pi)*} L^\natural_{X^J} \to L^\natural_{X^I} \tag{3.7.6.3}$$

[53]We do not assume that L is a Lie* algebra at the moment.

which coincides with (3.7.6.2) if π is surjective. These maps are injective morphisms of $\mathcal{D}$-modules; they are compatible with the composition of the π's.

For example, if $J = \emptyset$, then the corresponding vector space $L_0^\natural := L^\natural_{X^\emptyset}$ is $\Gamma(X, h(L))$ (see 2.5.3). We get a canonical embedding of $\mathcal{D}_{X^I}$-modules

$$L_0^\natural \otimes \mathcal{O}_{X^I} \hookrightarrow L^\natural_{X^I}. \tag{3.7.6.4}$$

(ii) Denote by $L^\ell_{X^I}$ the cokernel of (3.7.6.4). Since X is affine, one has $L^\ell_{X^I} = H^0 p_{I\cdot} F_{\tilde{X^I}}$ where $F_{\tilde{X^I}} := (j_{I\cdot} j_I^* DR(L)\boxtimes \mathcal{O}_{X^I})/(DR(L)\boxtimes\mathcal{O}_{X^I})$. The corresponding right $\mathcal{D}_{X^I}$-module equals $p_{I*}\Phi_{\tilde{X^I}}$ where $\Phi_{\tilde{X^I}} := j_{I*}j_I^*(L\boxtimes\omega_{X^I})/L\boxtimes\omega_{X^I}$. In particular, $L^\ell_X = L^\ell$.

Notice that $L^\ell_{X^I}$, unlike $L^\natural_{X^I}$, depends on L in a purely X-local way; i.e., $L^\ell_{(x_i)}$ depends only on the restriction of L to a neighbourhood of $\{x_i\}$.

For $S \in Q(I)$ consider the open subset $j^{[I/S]} : U^{[I/S]} \hookrightarrow X^I$ (see 3.4.4); for $s\in S$ let $p_s : X^I \to X^{I_s}$ be the projection and $\tilde{p_s} := id_X \times p_s$. Decomposing $F_{\tilde{X^I}}$ by connected components of its support, we get $j^{[I/S]*}\prod\limits_{s\in S}\tilde{p_s}^* F_{\tilde{X^{I_s}}} \xrightarrow{\sim} j^{[I/S]*}F_{\tilde{X^I}}$ hence the isomorphism

$$j^{[I/S]*}\prod_{s\in S} p_s^* L^\ell_{X^{I_s}} \xrightarrow{\sim} j^{[I/S]*}L^\ell_{X^I}. \tag{3.7.6.5}$$

In particular, one has $j^{(I)*}\prod p_i^* L^\ell \xrightarrow{\sim} j^{(I)*}L^\ell_{X^I}$.

Here is another interpretation of (3.7.6.5). For $s \in S$ consider the projection $q_s : X^I \to X^{I\smallsetminus I_s}$ and the coresponding embedding (3.7.6.3) of $\mathcal{D}$-modules $q_s^* L^\natural_{X^{I\smallsetminus I_s}} \hookrightarrow L^\natural_{X^I}$. We have a commutative diagram of embeddings of $\mathcal{D}_{X^I}$-modules

$$\begin{array}{ccccc}
\prod\limits_S p_s^* L^\natural_{X^{I_s}} & \longrightarrow & (L^\natural_{X^I})^S & \longleftarrow & L^\natural_{X^I} \\
\uparrow & & \uparrow & & \uparrow \\
(L_0^\natural\otimes\mathcal{O}_{X^I})^S & \longrightarrow & \prod\limits_S q_s^* L^\natural_{X^{I\smallsetminus I_s}} & \longleftarrow & L_0^\natural\otimes\mathcal{O}_{X^I}
\end{array} \tag{3.7.6.6}$$

The morphisms between the cokernels of vertical embeddings $\prod\limits_S (p_s^* L^\natural_{X^{I_s}}/L_0^\natural \otimes \mathcal{O}_{X^{I_s}}) \to \prod\limits_S L^\natural_{X^I}/L^\natural_{X^{I\smallsetminus I_s}} \leftarrow L^\natural_{X^I}/L_0^\natural\otimes\mathcal{O}_{X^I}$ being restricted to $U^{[I/S]}$ become isomorphisms; the composition is (3.7.6.5).

(iii) Assume that L is a Lie* algebra. Then $L^\natural_{X^I}$ is a Lie algebra in the tensor category $\mathcal{M}^\ell(X^I)$. One defines the Lie bracket on $L^\natural_{X^I}$ in the same way that we defined the bracket on $h(L)$. Namely, the $*$ bracket on L yields a morphism $DR(L)\boxtimes DR(L)\to DR(\Delta_*L)$ on $X\times X$. Take the exterior tensor product of this morphism with $\mathcal{O}_{X^I}$, restrict it to $V \underset{X^I}{\times} V \subset (X\times X)\times X^I$, and push forward to X^I. We get a morphism of complexes of the left $\mathcal{D}_{X^I}$-modules $((p_I j_I)_\cdot j_I^*(DR(L)\boxtimes\mathcal{O}_{X^I}))^{\otimes 2} \to (p_I j_I)_\cdot j_I^*(\Delta^\cdot DR(\Delta_* L)\boxtimes\mathcal{O}_{X^I})$. The second term is canonically quasi-isomorphic to $(p_I j_I)_\cdot j_I^*(DR(L)\boxtimes\mathcal{O}_{X^I})$ (see (b) in 2.1.7). Passing to H^0, we get a morphism $[\ \ ]^\natural : L^\natural_{X^I}\otimes L^\natural_{X^I}\to L^\natural_{X^I}$ which is the desired Lie bracket.

All the canonical morphisms in (i) and (ii) above are compatible with our Lie algebra structure. In particular, $L^\natural$ is a Lie algebra in the tensor category of left $\mathcal{D}$-modules on $\mathcal{R}(X)$.

3.7.7. Now we are ready to define our factorization algebra. Set

$$V_{X^I} := U(L^\natural_{X^I})/U(L^\natural_{X^I})L^\natural_0. \tag{3.7.7.1}$$

Here $U(L^\natural_{X^I})$ is the enveloping associative algebra of our Lie algebra $L^\natural_{X^I}$ in the abelian tensor k-category $\mathcal{M}^\ell(X^I)$. So V_{X^I} is the *vacuum representation* := the $L^\natural_{X^I}$-module induced from the trivial $L^\natural_0 \otimes \mathcal{O}_{X^I}$-module.

The construction of the enveloping algebra and induced module is compatible with pull-backs. Therefore $U(L^\natural_{X^I})$ and V_{X^I} form left $\mathcal{D}$-modules $U(L^\natural)$ and V on $\mathcal{R}(X)$.

Our V has a canonical factorization structure. To define factorization isomorphisms c (see 3.4.4.1), look at (3.7.6.6) and consider the corresponding morphisms of induced modules

$$\underset{S}{\boxtimes} V_{X^{I_s}} \longrightarrow \underset{S}{\otimes} U(L^\natural_{X^I})/U(L^\natural_{X^I})q_s^* L^\natural_{X^{I\smallsetminus I_s}} \longleftarrow V_{X^I}. \tag{3.7.7.2}$$

The following well-known lemma (together with (ii) in 3.7.6) shows that over $U^{[I/S]}$ both arrows in (3.7.7.2) are isomorphisms. This provides the factorization isomorphisms. The compatibilities from 3.4.4 obviously hold.

3.7.8. LEMMA. *Let $\phi : L_1 \to L_2$ be a morphism of Lie algebras over a commutative ring R, and let $N_1 \subset L_1$, $N_2 \subset L_2$ be Lie subalgebras such that $\phi(N_1) \subset N_2$ and $L_1/N_1 \xrightarrow{\sim} L_2/N_2$. Then the morphism $U(L_1)/U(L_1)N_1 \to U(L_2)/U(L_2)N_2$ is an isomorphism.*

REMARK. The following stronger statement holds: the map $U(L_1) \underset{U(N_1)}{\otimes} U(N_2) \to U(L_2)$ is an isomorphism. In fact, this statement follows from the lemma.[54]

We give two proofs of the lemma. The first one is elementary. The second proof is more "immediate" but it uses a machinery too heavy for the purpose.[55]

First Proof. Set $P_i := U(L_i)/U(L_i)N_i$; let $v_i \in P_i$ be the "vacuum vectors" and let $\pi : P_1 \to P_2$ be our morphism.

(i) π is surjective. Indeed, π is compatible with the standard filtrations on the P_i's. Since $\mathrm{gr}.P_i$ is a quotient of $\mathrm{Sym}^{\cdot}(L_i/N_i)$, $\mathrm{gr}\,\pi$ is surjective, and we are done.

(ii) So to prove our lemma, it suffices to construct a left inverse $P_2 \to P_1$ of π. To do this, we will define an L_2-module structure on P_1 such that:

(a) The obvious L_1-action on P_1 equals the one coming from the L_2-action via π.

(b) The action of $N_2 \subset L_2$ kills v_1.

As follows from (b), there is a unique morphism of L_2-modules $P_2 \to P_1$ which sends v_2 to v_1. By (a) it is left inverse to π.

(iii) It remains to construct the L_2-action. We use the following remark: for any L_1-modules Q, R one has $\mathrm{Hom}_{L_1}(Q \otimes P_1, R) = \mathrm{Hom}_{L_1}(P_1, \mathrm{Hom}(Q,R)) = \mathrm{Hom}_{N_1}(Q,R)$.

Taking $Q = L_2, R = P_1$, we see that there is a unique morphism of L_1-modules $a : L_2 \otimes P_1 \to P_1$ with the property $a(l_2 \otimes v_1) = l_1 v_1$ for every $l_i \in L_i$ such that $l_1 \,\mathrm{mod} N_1 = l_2 \mathrm{mod}\, N_2$.

---

[54]It suffices to consider the case $L_1 \subset L_2$. Then apply the lemma to $\tilde{L}_1 := L_1 \times N_2$, $\tilde{L}_2 := L_2 \times L_2$, $\tilde{N}_1 := N_1$, $\tilde{N}_2 := L_2$.

[55]It will be of use in the proof of the Poincaré-Birkhoff-Witt theorem 3.7.14 though.

Let us show that a is a Lie algebra action. We need to check that the two maps $\Lambda^2 L_2 \otimes P_1 \to P_1$ coincide. Both maps are L_1-equivariant, so by the above remark it suffices to show that their restrictions to $\Lambda^2 L_2 \otimes v_1$ are equal. This is clear since both maps vanish on $\Lambda^2 N_2 \otimes v_1$, and on the image of $L_1 \otimes L_2 \otimes v_1$ they coincide by the L_1-equivariance of a.

Property (a) follows from the case $Q = L_1, R = P_1$ in the above remark; property (b) holds by construction. We are done.

Second Proof. Consider the derived version of the induced module construction. Namely, for an emedding of Lie algebras $N \subset L$ one may find Lie DG algebras $L^{\sim}, N^{\sim}$ placed in degrees ≤ 0 whose components are flat (e. g. free) R-modules, together with projections $L^{\sim} \to L$, $N^{\sim} \to N$ which are quasi-isomorphisms, and morphism of Lie DG algebras $N^{\sim} \to L^{\sim}$ that lifts the embedding $N \hookrightarrow L$. Consider the enveloping associative DG algebra $U(L^{\sim})$ as an $(L^{\sim}, N^{\sim})$-bimodule (with respect to the left and right actions). Denote by C the homological Chevalley complex of $N^{\sim}$ with coefficients in $U(L^{\sim})$ (where $N^{\sim}$ acts on $U(L^{\sim})$ by right multiplication). This is a DG $L^{\sim}$-module ($L^{\sim}$ acts by left multiplication) of degrees ≤ 0. There is an obvious isomorphism $H^0 C \xrightarrow{\sim} U(L)/U(L)N$.

As a graded module, C equals $U(L^{\sim}) \otimes \mathrm{Sym}(N^{\sim}[1])$. It carries a filtration $C_0 \subset C_1 \subset \cdots$ equal to the tensor product of the standard filtration $U(L^{\sim})_\cdot$ and the filtration $\mathrm{Sym}^{\leq \cdot}(N^{\sim}[1])$. This filtration is stable with respect to the differential, and there is a canonical isomorphism $\mathrm{gr}_i C = \mathrm{Sym}^i(\mathcal{C}one(N^{\sim} \to L^{\sim}))$. Therefore, as an object of the derived category of R-modules, $\mathrm{gr}_i C$ equals $\mathbb{L}\mathrm{Sym}^i(L/N)$.

Let us return to the situation of our lemma. Choose resolutions as above for both (L_1, N_1), (L_2, N_2) and morphisms of Lie DG algebras $\phi_{\tilde{L}} : L_1^{\sim} \to L_2^{\sim}$, $\phi_{\tilde{N}} : N_1^{\sim} \to N_2^{\sim}$ that lift ϕ and for which the diagram

$$\begin{array}{ccc} L_1^{\sim} & \xrightarrow{\phi_{\tilde{L}}} & L_2^{\sim} \\ \uparrow & & \uparrow \\ N_1^{\sim} & \xrightarrow{\phi_{\tilde{N}}} & N_2^{\sim} \end{array}$$

commutes. We get a morphism of filtered complexes $\phi_C : C_1 \to C_2$. Notice that $\mathrm{gr}_i(\phi_C)$ is the ith symmetric power of $\phi^{\sim} : \mathcal{C}one(N_1^{\sim} \to L_1^{\sim}) \to \mathcal{C}one(N_2^{\sim} \to L_2^{\sim})$. The condition of 3.7.4 assures that this $\phi^{\sim}$, hence $\mathrm{Sym}^{\cdot}\phi^{\sim}$, is a quasi-isomorphism. Therefore ϕ_C is a quasi-isomorphism. Passing to H^0 we get our lemma. □

Remark. The above lemma also shows that V_X has the étale local nature. Namely, let $\pi : X' \to X$ be an étale map, $L_{X'}$ the π-pull-back of $L = L_X$, $V_{X'}$ the corresponding vacuum module. One has an obvious morphism of Lie algebras $\pi^* L_X^{\natural} \to L_{X'}^{\natural}$. It yields the morphism of vacuum modules $\pi^* V_X \to V_{X'}$ which is an isomorphism.

3.7.9. Lemma. *Our V is a factorization algebra. Its $\mathcal{D}$-module structure coincides with the canonical $\mathcal{D}$-module structure defined by the factorization structure (see 3.4.7).*

Proof. One needs only to check the flatness along codimension 1 diagonals. Indeed, the unit in V is the canonical generator 1 of the induced module V_X; since 1 is horizontal, our $\mathcal{D}$-module structure coincides with the canonical one (see 3.4.7).

Assume that $I = \{1, \ldots, n\}$ and the codimension 1 diagonal $X^{n-1} \hookrightarrow X^n$ we consider is given by the equation $x_{n-1} = x_n$. Let $j' : U' \hookrightarrow X^n$ be the complement to the union of diagonals $x_1 = x_n, \dots, x_{n-1} = x_n$, and the projections $q : X^n \to X^{n-1}$, $p : X^n \to X$ are, respectively, $(x_1, \dots, x_n) \mapsto (x_1, \dots, x_{n-1})$, $(x_1, \dots, x_n) \mapsto x_n$.

The embeddings $q^* L^\natural_{X^{n-1}} \hookrightarrow L^\natural_{X^n} \hookleftarrow p^* L^\natural_X$ (see (i) in 3.7.6) yield a short exact sequence

$$0 \to q^* L^\natural_{X^{n-1}} \to L^\natural_{X^n} \to j'_* j'^* p^* L^\ell_X \to 0. \tag{3.7.9.1}$$

Consider the canonical filtration on the enveloping algebra $U(L^\natural_{X^n})$. The Poincaré-Birkhoff-Witt theorem $\operatorname{gr} U = \operatorname{Sym}$ holds for Lie algebras in any abelian tensor $\mathbb{Q}$-category (see, e.g., [**DM**] 1.3.7), so $\operatorname{gr}_a U(L^\natural_{X^n}) = \operatorname{Sym}^a L^\natural_{X^n}$. Let us consider (3.7.9.1) as a two-step increasing filtration on $L^\natural_{X^n}$. Notice that the sub- and quotient modules in our filtration are Tor-independent. Therefore it defines on every $\operatorname{Sym}^a L^\natural_{X^n}$ an increasing filtration with successive quotients $q^*(\operatorname{Sym}^{a-m} L^\natural_{X^{n-1}}) \otimes j'_* j'^* p^* (\operatorname{Sym}^m L^\ell_X)$. Set $Q := U(L^\natural_{X^n})/q^* U(L^\natural_{X^{n-1}})$; we see that $Q = j'_* j'^* Q$.

The Lie algebra $L^\natural_0 \subset q^* L^\natural_{X^{n-1}} \subset L^\natural_{X^n}$ acts on $U(L^\natural_{X^n})$ by right multiplications, and V_{X^n} is the $\mathcal{D}_{X^n}$-module of coinvariants $U(L^\natural_{X^n})_{L^\natural_0}$. The short exact sequence of $L^\natural_0$-modules $0 \to q^* U(L^\natural_{X^{n-1}}) \to U(L^\natural_{X^n}) \to Q \to 0$ yields the short exact sequence of coinvariants

$$0 \to q^* V_{X^{n-1}} \to V_{X^n} \to Q_{L^\natural_0} \to 0. \tag{3.7.9.2}$$

Indeed, one need only check the exactness from the left which follows from the fact that every morphism from a $j'_* \mathcal{O}_{U'}$-module to $q^* V_{X^{n-1}}$ vanishes.

Both sub- and quotient modules in (3.7.9.2) are flat along our diagonal $X^{n-1} \subset X^n$ (recall that $Q_{L^\natural_0} = j'_* j'^* Q_{L^\natural_0}$), so the same is true for the middle term; q.e.d. □

3.7.10. Consider the chiral algebra V^r that corresponds to our factorization algebra V. So V^r is the right $\mathcal{D}$-module corresponding to the left $\mathcal{D}_X$-module $V_X = U(L^\natural_X)/U(L^\natural_X)L^\natural_0$. The chiral product $\mu : j_* j^* V_X \boxtimes V_X \to \Delta_* V_X$ may be described explicitly as follows. The exact sequence $0 \to p_2^* L^\natural_X \to L^\natural_{X\times X} \to j_* j^* p_1^* L^\ell_X \to 0$ (see (3.7.9.1)) shows that the embedding $p_1^* L^\natural_X \hookrightarrow L^\natural_{X\times X}$ yields an isomorphism between the induced modules $j^* p_1^* V_X \xrightarrow{\sim} j^* U(L^\natural_{X\times X})/U(L^\natural_{X\times X}) p_1^* L^\natural_X$. In particular, $L^\natural_{X\times X}$ acts on $j_* j^* p_1^* V_X$; the same is true for $j_* j^* p_2^* V_X$. Therefore the tensor product $j_* j^* V_X \boxtimes V_X$ is an $L^\natural_{X\times X}$-module. There is a canonical isomorphism of $L^\natural_{X\times X}$-modules $j_* j^* V_{X\times X} \xrightarrow{\sim} j_* j^* V_X \boxtimes V_X$ which sends the generator $1_{X\times X}$ to $1_X \boxtimes 1_X$. Our chiral product μ is the composition of the inverse to this isomorphism and the projection $j_* j^* V_{X\times X} \to (j_* j^* V_{X\times X})/V_{X\times X} = \Delta_* V_X$ where the latter identification is the structure isomorphism $\Delta^* V_{X\times X} = V_X$. Notice that μ is a morphism of $L^\natural_{X\times X}$-modules.

The morphism $L^\natural_X \to V_X$, $l \mapsto l \cdot 1$, kills $L^\natural_0$, so it yields a morphism of right $\mathcal{D}_X$-modules $i : L \to V^r$. The next proposition finishes the proof of 3.7.1:

3.7.11. Proposition. *The above $i : L \to V^r$ is a morphism of Lie* algebras. It satisfies the universality property: for every chiral algebra A the map $\operatorname{Hom}_{\mathcal{CA}(X)}(V^r, A) \to \operatorname{Hom}_{\mathcal{L}ie^*(X)}(L, A^{Lie})$, $\varphi \mapsto \varphi i$, is bijective.*

Proof. We begin with a useful general construction.

Let A be a chiral algebra, M an A-module. Consider the Lie algebra[56] $A_X^\natural := A_X^{Lie\natural}$ in the tensor category $\mathcal{M}^\ell(X)$ (see 3.7.6). One has $H^0 p_{2*} j_* j^* A \boxtimes M = H^0(p_{2*} j_* j^* A \boxtimes \omega_X) \otimes^! M = A_X^\natural \otimes M$; here $p_2 : X \times X \to X$ is the second projection. Therefore we get a canonical morphism

$$\bullet = \bullet_M := p_{2*}(\mu_M) : A_X^\natural \otimes M \to M. \tag{3.7.11.1}$$

Exercise: Check that this is a Lie algebra action of $A_X^\natural$ on the $\mathcal{D}$-module M. On the Lie subalgebra $A_0^\natural = H^0_{DR}(X, A) = \Gamma(X, h(A)) \subset A_X^\natural$ this action coincides with the usual $h(A)$-action for the A^{Lie}-module structure on M (see 2.5.4).

Notice that $\bullet_M$ determines μ_M uniquely (since p_{2*} is right exact and for $\mathcal{D}_{X\times X}$-modules supported on the diagonal it is fully faithful).

In particular, we have a canonical action of $A_X^\natural$ on A. The morphism $A_X^\natural \to A^\ell$, $a^\natural \mapsto a^\natural \bullet 1$, coincides with the standard projection $A_X^\natural \to A_X^\natural / A_0^\natural \otimes \mathcal{O}_X = A^\ell$ (see (ii) in 3.7.6). A morphism of chiral algebras $A \to A'$ is the same as a morphism of $\mathcal{D}$-modules $\varphi : A \to A'$ which is compatible with $\bullet$ actions (i.e., $\varphi \bullet_A = \bullet_{A'}(\varphi^\natural \otimes \varphi)$) and sends 1_A to $1_{A'}$.

Let us return to our chiral algebra V^r. Let $\bullet_L : L_X^\natural \otimes V_X \to V_X$ be the morphism $\bullet_{V^r}$ composed with $i^\natural : L_X^\natural \to V_X^{r\natural}$.

SUBLEMMA. *$\bullet_L$ coincides with our old canonical $L^\natural$-action on V_X.*

Proof of Sublemma. We know that the chiral product $\mu : j_* j^* V_X \boxtimes V_X \to \Delta_* V_X$ is a morphism of $L_{X\times X}^\natural$-modules (see 3.7.10). Restricting this action to $p_2^* L_X^\natural \subset L_{X\times X}^\natural$ and pushing μ forward by p_2, we see that $\bullet : V_X^{r\natural} \otimes V_X \to V_X$ is a morphism of $L_X^\natural$-modules. Here $L_X^\natural$ acts on $V_X^{r\natural} := H^0 p_{2*} j_* j^* p_1^* V_X$ via the embedding $p_2^* L_X^\natural \hookrightarrow L_{X\times X}^\natural$ and the $L_{X\times X}^\natural$-action on $j_* j^* p_1^* V_X = U(L_{X\times X}^\natural)/U(L_{X\times X}^\natural) p_2^* L_X^\natural$ (see 3.7.10), and on V_X in our old canonical way (as on the vacuum module). One checks easily that $i^\natural : L_X^\natural \to V_X^{r\natural}$ is a morphism of $L_X^\natural$-modules. So for $l, l' \in L_X^\natural$, $v \in V_X$ one has $l'(l \bullet_L v) = [l', l] \bullet_L v + l \bullet_L (l'v)$. Since $l \bullet_L 1 = l1$ and 1 generates V_X as an $L_X^\natural$-module, we are done. □

End of proof of Proposition. Take $l \in L_0^\natural \subset L_X^\natural$, $l' \in L$. By the sublemma, $l \bullet_L i(l') = i(ad_l l')$. This shows that i is a morphism of Lie* algebras (see 2.5.2). Our sublemma also implies that V^r is generated by $i(L)$ as a chiral algebra, so the map $\mathrm{Hom}_{\mathcal{CA}(X)}(V^r, A) \to \mathrm{Hom}_{\mathcal{L}ie^*(X)}(L, A^{Lie})$ is injective. Let us define its inverse. Let $\psi : L \to A^{Lie}$ be a morphism of Lie* algebras. It yields a morphism of Lie algebras $\psi^\natural : L_X^\natural \to A_X^\natural$; hence $\bullet_A$ defines an $L_X^\natural$-action on A. Since $L_0^\natural \subset L_X^\natural$ kills 1_A, we get a morphism of $L_X^\natural$-modules $\tilde{\psi} : V_X \to A^\ell$ which sends 1 to 1_A. One has $\tilde{\psi} i = \psi$, so it remains to check that $\tilde{\psi}$ is a morphism of chiral algebras. Since it is a morphism of $L_X^\natural$-modules, we know that $\mu_A(\tilde{\psi}^{\boxtimes 2})$ equals $\tilde{\psi}\mu_{V^r}$ on the image of $j_* j^* L \boxtimes V^r \to j_* j^* V^r \boxtimes V^r$. Since the image of L generates V^r as a chiral algebra, we are done. □

3.7.12. Let L_α be a family of Lie* algebras. Consider the Lie* algebra $\oplus L_\alpha$. It follows immediately from the universality property of $\otimes$ and U (see 3.4.15 and

[56] Specialists in vertex algebras sometimes call a similar object "the Lie algebra of local fields."

3.7.1) that one has a canonical isomorphism of chiral algebras

$$U(\prod L_\alpha) \xrightarrow{\sim} \otimes U(L_\alpha). \tag{3.7.12.1}$$

In particular, for any Lie* algebra L the diagonal map $L \to L \times L$ yields a coproduct morphism $U(L) \to U(L) \otimes U(L)$ which makes $U(L)$ a Hopf chiral algebra (see 3.4.16).

3.7.13. We are going to prove a chiral version of the Poincaré-Birkhoff-Witt theorem.

Let A be a chiral algebra equipped with a filtration $A_0 \subset A_1 \subset \cdots \subset A$ (see 3.3.12). Notice that A_0 is a chiral subalgebra of A and A_1 is an A_0-submodule. We say that our filtration $A.$ is *1-generated* if all the arrows $j_* j^* A_i \boxtimes A_j \to \Delta_* A_{i+j}$, $i, j \geq 0$, are surjective. If $R \subset A$ is a chiral subalgebra of A and $M \supset R$ is an R-submodule of A that generates A, then there is a unique 1-generated filtration $A.$ such that $A_0 = R$, $A_1 = M$. Namely, $A_{i+1} := \mu(j_* j^* A_1 \boxtimes A_i)$ for $i \geq 0$. We say that this filtration is generated by (R, M); if $R = \omega_X \cdot 1$, then we simply say that our filtration is generated by M. Assume that M is a Lie* subalgebra of A,[57] R is commutative, and M normalizes R. Then our filtration is commutative (see 3.3.12), and the obvious surjective morphism of $\mathbb{Z}$-graded commutative $\mathcal{D}_X$-algebras $\mathrm{Sym}^{\cdot} M^\ell \to \mathrm{gr}.A^\ell$ is compatible with the coisson brackets.

3.7.14. Let us return to our situation. Denote by L^{tor} the $\mathcal{O}_X$-torsion submodule of L. Since any binary chiral operation on an $\mathcal{O}$-torsion $\mathcal{D}$-module vanishes, the canonical morphism $L \to U(L)$ kills the commutator $[L^{tors}, L^{tors}]$ (which is a Lie ideal in L). Therefore $U(L) \xrightarrow{\sim} U(L/[L^{tor}, L^{tor}])$.

The chiral algebra $U(L)$ is generated by the image of L which is a Lie* subalgebra of $U(L)$. The corresponding filtration $U(L).$ is called the Poincaré-Birkhoff-Witt filtration. We have a canonical surjective morphism of coisson algebras

$$\mathrm{Sym}^{\cdot} L^\ell \twoheadrightarrow \mathrm{gr}.U(L)^\ell, \tag{3.7.14.1}$$

which factors through

$$\mathrm{Sym}^{\cdot}(L/[L^{tor}, L^{tor}])^\ell \twoheadrightarrow \mathrm{gr}.U(L)^\ell. \tag{3.7.14.2}$$

THEOREM. *The morphism (3.7.14.2) is an isomorphism. In particular, if L is $\mathcal{O}_X$-flat, then (3.7.14.1) is an isomorphism.*

Proof. We can assume that X is affine. Consider the projection $U(L_X^\natural) \to V_X = U(L)^\ell$. As follows from the proof of 3.7.11 (see the sublemma), our PBW filtration equals the image of the usual PBW filtration on the enveloping algebra $U(L_X^\natural)$.

Consider the filtered $\mathcal{O}_X$-complex C from the second proof of 3.7.8 for the pair $L_0^\natural \subset L_X^\natural$. Then $H^0 C$ equals $U(L)^\ell$ (as a filtered module) and $\mathrm{gr}_i C = \mathbb{L}\mathrm{Sym}^i_{\mathcal{O}_X}(L^\ell)$. So one has $H^0 \mathrm{gr}_i C = \mathrm{Sym}^i L^\ell$, $H^{-1}\mathrm{gr}_i C = \mathrm{Sym}^{i-2}(L/L^{tor})^\ell \otimes \mathrm{Tor}_1^{\mathcal{O}_X}(L^{tor\ell}, L^{tor\ell})^{\Sigma_2} = \mathrm{Sym}^{i-2}(L/L^{tor})^\ell \otimes \Delta^!(L^{tor} \boxtimes L^{tor})^{\ell\, sgn}$. The differential $d_1^{-2,1} : H^{-1}\mathrm{gr}_2 C \to H^0 \mathrm{gr}_1 C$ in the spectral sequence coincides with the Lie* bracket

[57] The next conditions hold automatically if $R = \omega_X \cdot 1$.

$\Delta^!(L^{tor} \boxtimes L^{tor})^{\ell\, sgn} \to L^{tor\ell} \subset L^\ell$.[58] Now recall that C is a module over some Lie algebra whose quotient is $L_X^\natural$. This implies that the image of the differential $d_1 : H^{-1}\mathrm{gr}\, C \to H^0 \mathrm{gr}\, C$ is the ideal generated by the image of $d_1^{-2,1}$, and all higher $d_r^{p,q}$'s for $p+q=-1$ vanish (since they all vanish on $\mathrm{Ker}\, d_1^{-2,1}$). We are done. □

3.7.15. We are going to describe $U(L)$-modules in more explicit terms (see 3.7.18). Notice that, contrary to the usual (non-chiral) setting, $U(L)$-modules are *not* the same as L-modules.

PROPOSITION. *The obvious functor $\mathcal{M}(X, U(L)) \to \mathcal{M}(X, L)$ admits a left adjoint induction functor*

$$Ind = Ind_L : \mathcal{M}(X, L) \to \mathcal{M}(X, U(L)). \tag{3.7.15.1}$$

Proof. For a $\mathcal{D}$-module M an L-action on it amounts to a Lie* algebra structure on $L \oplus M$ such that the bracket on M vanishes and $L \hookrightarrow L \oplus M$ is a morphism of Lie* algebras. So an L-module M yields a chiral algebra $U(L \oplus M)$. The $\mathbb{G}_m$-action on M by homotheties defines a $\mathbb{Z}$-grading on this chiral algebra. Its zero component is $U(L)$. The degree 1 component is $\mathrm{Ind}\, M$. To check the universality property use 3.3.5(i). □

The induction functor is compatible with the étale localization of X.

Let us describe the induced module $\mathrm{Ind}\, M$ explicitly. We can assume that X is affine. Consider the Lie algebra $L_X^\natural$ defined in (iii) in 3.7.6. The Lie* action of L on M yields an action on M of the Lie subalgebra $L_0^\natural \subset L_X^\natural$. Now one has a canonical isomorphism

$$\mathrm{Ind}\, M \xrightarrow{\sim} U(L_X^\natural) \underset{U(L_0^\natural)\otimes \mathcal{O}_X}{\otimes} M. \tag{3.7.15.2}$$

REMARKS. (i) The PBW filtration on $U(L)$ defines a filtration $(\mathrm{Ind}\, M)_i := U(L)_i M$ on $\mathrm{Ind}\, M$. By construction, we have a natural morphism $(\mathrm{Sym}\, L) \otimes M \twoheadrightarrow \mathrm{gr}\, \mathrm{Ind}\, M$ which is an isomorphism if L is $\mathcal{O}_X$-flat.[59]

(ii) Let A be any chiral algebra. The functor $\mathrm{Ind}_A : \mathcal{M}(X, A^{Lie}) \to \mathcal{M}(X, A)$ from 3.3.6 is the composition $\mathcal{M}(X, A^{Lie}) \xrightarrow{\mathrm{Ind}} \mathcal{M}(X, U(A^{Lie})) \to \mathcal{M}(X, A)$; the latter arrow is $N \mapsto N/\mathcal{J}N$ where $\mathcal{J} \subset U(A^{Lie})$ is the ideal such that $A = U(A^{Lie})/\mathcal{J}$.

(iii) The above constructions and statements remain valid if we consider $\mathcal{O}$-modules equipped with L- and $U(L)$-actions (see 2.5.2, 3.3.5(ii)) instead of $\mathcal{D}$-modules.

3.7.16. Define a *chiral L-module*, or *L^{ch}-module*, as a $\mathcal{D}_X$-module M equiped with a chiral pairing $\mu = \mu_{LM} \in P_2^{ch}(\{L, M\}, M)$ (called a *chiral action* of L on M) which satisfies the following "truncated" version of the Lie algebra action axiom. Let $j' : U' \hookrightarrow X \times X \times X$ be the complement to the diagonals $x_1 = x_3$ and $x_2 = x_3$. Consider a chiral operation $\mu_{L_1 M}(id_{L_1}, \mu_{L_2 M}) - \mu_{L_2 M}(id_{L_2}, \mu_{L_1 M}) \in$

[58] To check this, we can assume that $L = L^{tor}$ and it is supported at a single point $x \in X$, i.e., $L = i_{x*}F$ for some Lie algebra F. Then $L^\natural = j_{x*}(F \otimes \mathcal{O}_{U_x})$, $L_0^\natural = F \otimes \mathcal{O}_X$ and C is the Chevalley complex of F with coefficients in $U(F) \otimes j_{x*}\mathcal{O}_{U_x}$. Now the statement is clear.

[59] The latter assertion follows from 3.7.14 together with the construction of $\mathrm{Ind}\, M$ in the proof of the proposition.

$P_3^{ch}(\{L,L,M\},M)$ and its restriction $\nu : j'_* j'^*(L\boxtimes L\boxtimes M)\to \Delta^{(3)}M$. We demand that ν equals the composition

$$j'_* j'^*(L\boxtimes L\boxtimes M)\to \Delta_*^{1=2} j_* j^*(L\boxtimes M)\to \Delta_*^{(3)}M. \tag{3.7.16.1}$$

Here $\Delta^{1=2} : X\times X\hookrightarrow X\times X\times X$ is the embedding $(x,y)\mapsto(x,x,y)$, the first arrow is $[\quad]_L\boxtimes id_M$, and the second one is $\Delta_*^{1=2}(\mu_{LM})$.

EXAMPLE. Assume that M is supported at a point $x\in X$. Then a chiral L-action on M is the same as a $*$ action of the Lie* algebra $j_{x*}j_x^*L$ on M. Here $j_x : U_x\hookrightarrow X$ is the complement to x.

Chiral L-modules form an abelian category which we denote by $\mathcal{M}(X,L^{ch})$.

A chiral operation $\varphi\in P_I^{ch}(\{M_i\},N)$, where M_i, N are chiral L-modules, is said to be compatible with the chiral L-actions (or to be a *chiral L^{ch}-operation*) if $\mu_{LN}(id_L,\varphi)=\sum_{i\in I}\varphi(\mu_{LM_i}, id_{M_{i'}})_{i'\neq i}$. Let $P^{ch}_{L^{ch}I}(\{M_i\},N)\subset P_I^{ch}(\{M_i\},N)$ be the subspace of such an operations. They are closed under the composition, so we have defined a pseudo-tensor structure $\mathcal{M}(X,L^{ch})^{ch}$ on $\mathcal{M}(X,L^{ch})$.

3.7.17. If X is affine, then $\mathcal{M}(X,L^{ch})^{ch}$ can be described as follows. The proofs are left to the reader.

As in the beginning of the proof in 3.7.11, a chiral pairing μ_{LM} amounts to a morphism $\bullet_\mu : L_X^\natural\otimes M\to M$ such that its restriction to $L_0^\natural\otimes M=\Gamma(X,h(L))\otimes M$ comes from a $*$ operation $\in P_2^*(\{L,M\},M)$.

LEMMA. *μ_{LM} is a chiral action if and only if $\bullet_\mu$ is an action of the Lie algebra $L_X^\natural$ on M.* □

Let us describe chiral L^{ch}-operations in these terms.

Let M_i, $i\in I$, be chiral L-modules. Then $j_*^{(I)}j^{(I)*}\boxtimes M_i$ is naturally an $L^\natural_{X^I}$-module (recall that $L^\natural_{X^I}$ is a Lie algebra in the tensor category of left $\mathcal{D}_{X^I}$-modules, see (iii) in 3.7.6). To see this, consider for $i\in I$ the morphism of $\mathcal{D}_{X^{\tilde I}}$-modules $\mu_{LM_i}\boxtimes(\boxtimes_{i'\neq i} id_{M_{i'}}) : j_*^{(\tilde I)}j^{(\tilde I)*}L\boxtimes(\boxtimes M_i)\to \Delta_*^i j_*^{(I)}j^{(I)*}\boxtimes M_i$ where $\Delta^i : X^I\hookrightarrow X^{\tilde I}=X\times X^I$ is the diagonal $x=x_i$. Its push-forward by the projection to X^I is a morphism of $\mathcal{D}_{X^I}$-modules $a_i : L^\natural_{X^I}\otimes(j_*^{(I)}j^{(I)*}\boxtimes M_i)\to j_*^{(I)}j^{(I)*}\boxtimes M_i$. This is an $L^\natural_{X^I}$-action on $j_*^{(I)}j^{(I)*}\boxtimes M_i$, and the actions a_i for different $i\in I$ mutually commute. The promised $L^\natural_{X^I}$-module structure on $j_*^{(I)}j^{(I)*}\boxtimes M_i$ is $\sum_{i\in I} a_i$.

If N is another L^{ch}-module, then $\Delta_*^{(I)}N$ is an $L^\natural_{X^I}$-module: the action morphism is $L^\natural_{X^I}\otimes\Delta_*^{(I)}N=\Delta_*^{(I)}(L^\natural_{X^I}\otimes N)\xrightarrow{\Delta_*^{(I)}(\bullet_\mu)}\Delta_*^{(I)}N$.

LEMMA. *A chiral operation $\varphi : j_*^{(I)}j^{(I)*}\boxtimes M_i\to\Delta_*^{(I)}N$ is compatible with the chiral L-actions if and only if φ is a morphism of $L^\natural_{X^I}$-modules.*

3.7.18. A chiral $U(L)$-module structure on a $\mathcal{D}_X$-module M defines in the obvious manner a chiral L-action on M. A chiral $U(L)$-operation between $U(L)$-modules (see 3.3.4) is automatically an L^{ch}-operation. So we have a pseudo-tensor functor

$$\mathcal{M}(X,U(L))^{ch}\to\mathcal{M}(X,L^{ch})^{ch}. \tag{3.7.18.1}$$

PROPOSITION. *This is an equivalence of pseudo-tensor categories.*

Proof. Use 3.7.17 and (3.7.15.2). A chiral L^{ch}-operation is the same as a chiral $U(L)$-operation by Remark (iii) in 3.3.4 (applied to $P = L \subset U(L)$). □

COROLLARY. *The functor Ind from 3.7.15 extends naturally to a pseudo-tensor functor $\mathcal{M}(X,L)^{ch} \to \mathcal{M}(X,U(L))^{ch}$ left adjoint to the obvious pseudo-tensor functor $\mathcal{M}(X,U(L))^{ch} \to \mathcal{M}(X,L)^{ch}$.*

Proof. We can assume that X is affine. Let P_i, $i \in I$, be L-modules. According to (3.7.15.2), the $L^\natural_{X^I}$-module $j_*^{(I)} j^{(I)*} \boxtimes \mathrm{Ind} P_i$ is equal to the induced $L^\natural_{X^I}$-module $U(L^\natural_{X^I}) \underset{U(L_0^\natural)\otimes \mathcal{O}_{X^I}}{\otimes} j_*^{(I)} j^{(I)*} \boxtimes P_i$. We are done by the second lemma in 3.7.17. □

3.7.19. Let $x \in X$ be a point, $j_x : U_x \hookrightarrow X$ its complement. The topological associative algebra $U(L)_x^{as}$ (see 3.6.2–3.6.7) can be described as follows. This algebra depends only on the restriction of $U(L)$ to U_x, so it is convenient to change notation for the moment and assume that L is a Lie* algebra on U_x. Consider the topological Lie algebra $\hat{h}_x^{Lie}(L) := \hat{h}_x^{Lie}(j_{x*}L)$ (see 2.5.12). Denote by $U(\hat{h}_x^{Lie}(L))$ the topological enveloping algebra of $\hat{h}_x^{Lie}(L)$ defined as the completion of the plain enveloping algebra with respect to the topology whose base is formed by the left ideals generated by an open subspace in $\hat{h}_x^{Lie}(L)$.

PROPOSITION. *There is a canonical isomorphism of topological associative algebras*

$$U(\hat{h}_x^{Lie}(L)) \xrightarrow{\sim} U(L)_x^{as}. \tag{3.7.19.1}$$

Proof. $\hat{h}_x^{Lie}(L)$ and $U(L)_x^{as}$ are completions of, respectively, $h_x(j_{x*}L)$ and $h_x(j_{x*}U(L))$ with respect to certain topologies. It follows from the definitions that the canonical morphism $h_x(j_{x*}L) \to h_x(j_{x*}U(L))$ is continuous with respect to these topologies; thus we have a continuous morphism $\hat{h}_x^{Lie}(L) \to U(L)_x^{as}$. This is a morphism of Lie algebras (as follows from Remark (i) in 3.6.6). So it yields a morphism of topological associative algebras $U(\hat{h}_x^{Lie}(L)) \to U(L)_x^{as}$. According to 2.5.15, 3.7.18, 3.6.6, and Example in 3.7.16 this morphism yields an equivalence between the categories of discrete modules. Since our algebras are complete and separated, and their topologies are defined by systems of left ideals, this shows that our morphism of topological algebras is an isomorphism. □

3.7.20. Twisted enveloping algebras. Let $L^\flat$ be an (automatically central) ω-extension of a Lie* algebra L (see 2.5.8). Let $1^\flat$ be the corresponding horizontal section of $L^{\flat\ell}$ (so $L^\ell = L^{\flat\ell}/\mathcal{O}_X 1^\flat$). The $\flat$-*twisted* chiral enveloping algebra of L is the quotient $U(L)^\flat$ of $U(L^\flat)$ modulo the ideal generated by $1 - 1^\flat$. So for any chiral algebra A a morphism $U(L)^\flat \to A$ amounts to a morphism of Lie* algebras $L^\flat \to A^{Lie}$ which maps $1^\flat$ to 1_A.

As in 3.7.13, the image of $L^\flat$ in $U(L)^\flat$ defines the PBW filtration on $U(L)^\flat$, and one has a canonical surjective map of coisson algebras

$$\mathrm{Sym}^{\cdot} L^\ell \twoheadrightarrow \mathrm{gr}.U(L)^\flat. \tag{3.7.20.1}$$

One has the twisted version of the Poincaré-Birkhoff-Witt theorem:

PROPOSITION. *If L is $\mathcal{O}_X$-flat then (3.7.20.1) is an isomorphism.*

Proof. Notice that $\varepsilon := 1 - 1^\flat$ is a central horizontal section of $U(L^\flat)^\ell$. It defines the endomorphism $\varepsilon\cdot$ of the $\mathcal{D}$-module $U(L^\flat)$, and $U(L)^\flat$ is its cokernel. This endomorphism shifts the PBW filtration of $U(L^\flat)$ by 1, and the corresponding endomorphism of $\operatorname{gr} U(L^\flat)^\ell = \operatorname{Sym} L^{\flat\ell}$ (see 3.7.14) is multiplication by $1^\flat \in L^\ell$. It is injective with the cokernel equal to $\operatorname{Sym} L^\ell$. This implies our PBW. □

The coproduct on $U(L^\flat)$ (see 3.7.12) yields a coaction of the Hopf chiral algebra $U(L)$ on the chiral algebra $U(L)^\flat$. If L is $\mathcal{O}_X$-flat, then $U(L)^\flat$ is a $U(L)$-cotorsor (see 3.4.16). The functor (we use the notation of 2.5.8 and 3.4.16)

$$\mathcal{P}(L) \to \mathcal{P}(U(L)), \quad L^\flat \mapsto U(L)^\flat, \tag{3.7.20.1}$$

is naturally a morphism of Picard groupoids.

EXERCISE. Show that (3.7.20.1) is an equivalence of Picard groupoids.[60]

3.7.21. The morphism $L^\flat \to U(L)^\flat$ yields a functor $\mathcal{M}(X, U(L)^\flat) \to \mathcal{M}(X, L)$ (since any $*$ action of ω_X on any module is trivial, $L^\flat$-modules are the same as L-modules). It admits a left adjoint functor $\operatorname{Ind}^\flat = \operatorname{Ind}^\flat_L \colon \mathcal{M}(X, L) \to \mathcal{M}(X, U(L)^\flat)$ (see 3.7.15). For an L-module M we have $\operatorname{Ind}^\flat_L M = \operatorname{Ind}_{L^\flat} M/(1-1^\flat)\operatorname{Ind}_{L^\flat} M$. One defines the canonical filtration on $\operatorname{Ind}^\flat M$ as the image of the canonical filtration on $\operatorname{Ind}_{L^\flat} M$, so we have a canonical morphism of graded $\operatorname{Sym} L^\ell$-modules

$$\operatorname{Sym}^{\cdot} L^\ell \otimes M \twoheadrightarrow \operatorname{gr}_{\cdot} \operatorname{Ind}^\flat M. \tag{3.7.21.1}$$

LEMMA. *If L is $\mathcal{O}_X$-flat, then (3.7.21.1) is an isomorphism.*

Proof. Same as that of the proposition in 3.7.20 (see Remark (i) in 3.7.15). □

3.7.22. Here is the twisted version of compatibility (3.7.19.1). Assume that we are in the situation of 3.7.19, so L is a Lie* algebra on U_x and let $L^\flat$ be an extension of L by ω_{U_x}. Let us compute the topological associative algbera $U(L)^{\flat as}_x = (U(L)^\flat)^{as}_x$.

The morphism of topological Lie algebras $\hat{h}^{Lie}_x(L^\flat) \twoheadrightarrow \hat{h}^{Lie}_x(L)$ is continuous and open. Its kernel is the image of $k = \hat{h}_x(j_{x*}\omega_{U_x}) \xrightarrow{1^\flat} \hat{h}^{Lie}_x(L^\flat)$. The twisted topological enveloping algebra $U(\hat{h}^{Lie}_x(L))^\flat$ is the quotient of $U(\hat{h}^{Lie}_x(L^\flat))$ modulo the closed ideal generated by the central element $1^\flat(1) - 1$.

PROPOSITION. *There is a canonical isomorphism of topological associative algebras*

$$U(\hat{h}^{Lie}_x(L))^\flat \xrightarrow{\sim} U(L)^{\flat as}_x. \tag{3.7.22.1}$$

Proof. The composition of the morphisms $U(\hat{h}^{Lie}_x(L^\flat)) \to U(L^\flat)^{as}_x \to U(L)^{\flat as}_x$ sends $1^\flat$ to 1, so it yields a morphism $U(\hat{h}^{Lie}_x(L))^\flat \to U(L)^{\flat as}_x$. As follows from 3.7.19, this morphism induces an equivalence between the categories of discrete modules; hence it is an isomorphism (use the fact that our algebras are complete and separated and their topologies are defined by systems of left ideals). □

[60]Hint: for $C \in \mathcal{P}(U(L))$ the corresponding ω-extension $L^\flat$ is the Lie* subalgebra of C defined as the preimage of $L \times C \subset U(L) \otimes C$ by the coaction morphism $\delta : C \to U(L) \otimes C$.

3.7.23. Let A be a chiral algebra and L a Lie* algebra that acts on A (see 3.3.3). For a chiral algebra B consider the set of pairs (i,ψ) where $i\colon A\to B$ is a morphism of chiral algebras and $\psi\colon L\to B$ a morphism of Lie* algebras, such that i is a morphism of L-modules (here L acts on B via ψ and the adjoint action of B). Such pairs depend on B functorially, and it is easy to see that there exists a universal pair (i,ψ); the corresponding chiral algebra B is denoted by $A\circledast U(L)$. Indeed, consider the L-action on A as an action of L on the Lie* algebra A^{Lie},[61] so we have the semi-direct product $A^{Lie}\rtimes L$ of the Lie* algebras A^{Lie} and L. Our $A\circledast U(L)$ is the quotient of $U(A^{Lie}\rtimes L)$ modulo the obvious relations (which say that the morphism $A\to A\circledast U(L)$ is a morphism of chiral algebras, not just of Lie* algebras).

Here is another construction of $A\circledast U(L)$. Recall that Ind^{ch}_L is a pseudo-tensor functor (see 3.7.18), so $\mathrm{Ind}^{ch}_L A$ is a Lie^{ch} algebra. The image of 1_A is the unit in $\mathrm{Ind}^{ch}_L A$, so $\mathrm{Ind}^{ch}_L A$ is a chiral algebra. The canonical morphism $i\colon A\to \mathrm{Ind}^{ch}_L A$ and the morphism $\psi\colon L\to \mathrm{Ind}^{ch}_L A$ that comes from the chiral action of L on 1_A satisfy the above properties. One checks immediately the universality property, so $A\circledast U(L)=\mathrm{Ind}_L A$.

We leave it to the reader to check that the PBW filtration on $\mathrm{Ind}_L A$ is a chiral algebra filtration, and the PBW morphism

$$A\otimes \mathrm{Sym}^{\cdot} L^{\ell}\to \mathrm{gr}_{\cdot}(A\circledast U(L)) \tag{3.7.23.1}$$

is a morphism of chiral algebras. According to Remark (i) in 3.7.15, if L is $\mathcal{O}_X$-flat, then (3.7.23.1) is an isomorphism.

3.7.24. The constructions of 3.7.23 and 3.7.20 easily combine. Namely, assume that we have A,L as in 3.7.23 and $L^\flat$ as in 3.7.20. Look at the pairs $i\colon A\to B$, $\psi^\flat\colon L^\flat\to B$ that satisfy the same conditions as i,ψ from 3.7.23 and assume that $\psi^\flat(1^\flat)=1_B$. As in 3.7.23, there exists a universal pair $(i,\psi^\flat)$, and we denote the corresponding B by $A\circledast U(L)^\flat$. One has $A\circledast U(L)^\flat=A\circledast U(L^\flat)/(1-1^\flat)A\circledast U(L^\flat)$. There is a canonical identification $A\circledast U(L)^\flat=\mathrm{Ind}^\flat_L A$. The canonical filtration on $\mathrm{Ind}^\flat_L A$ is a chiral algebra filtration, and the canonical morphism $A\otimes\mathrm{Sym}^{\cdot} L^\ell\to \mathrm{gr}_{\cdot}(A\circledast U(L)^\flat)$ is a morphism of chiral algebras. If L is $\mathcal{O}_X$-flat, then this is an isomorphism (see 3.7.21). This version of the PBW theorem will be of use in 3.8.

Remark. For $M\in\mathcal{M}(X)$ a structure of the $A\circledast U(L)^\flat$-module on M amounts to a chiral action on M of the semi-direct product $A^{Lie}\rtimes L^\flat$ of the Lie* algebras A^{Lie} and $L^\flat$ such that the chiral A^{Lie}-action is actually a chiral A-action and the chiral action of $\omega\subset L^\flat$ is the unit action. This follows immediately from 3.7.18 and the construction of $A\circledast U(L)^\flat$ as a quotient of $U(A^{Lie}\rtimes L^\flat)$.

3.7.25. Virasoro vectors. Consider the Virasoro extension $\Theta^c_{\mathcal{D}}$ of $\Theta_{\mathcal{D}}$ of central charge $c\in k$ (see (c) in 2.5.10). The twisted enveloping algebra $U(\Theta)^c:=U(\Theta_{\mathcal{D}})^c$ is called the *chiral Virasoro algebra of central charge c.*

For a chiral algebra A a *Virasoro vector*, a.k.a. *stress-energy tensor*, of central charge c is a morphism of chiral algebras $U(\Theta)^c\to A$ which is the same as a morphism of Lie* algebras $\iota^c:\Theta^c_{\mathcal{D}}\to A$ which sends $1^c:\omega_X\hookrightarrow\Theta^c_{\mathcal{D}}$ to 1_A.

[61] We use notation of 3.3.2.

A Virasoro vector yields two actions of the Lie algebra Θ on the sheaf A, called the *adjoint* and *Lie* action, defined, respectively, by the formulas

$$(3.7.25.1) \qquad \mathrm{ad}_\tau a := \mathrm{ad}_{\iota(\tau)} a, \quad \mathrm{Lie}_\tau a := \mathrm{ad}_\tau a - a\cdot\tau.$$

Here $\cdot$ is the action via the right $\mathcal{D}$-module structure and $\iota : \Theta_{\mathcal{D}} \to A/\omega 1_A$ is defined by ι^c. The adjoint action is $\mathcal{D}_X$-linear, while the Lie action is compatible with the usual Θ_X-action on $\mathcal{D}_X$.

EXAMPLE. *Sugawara's construction.* Let $\mathfrak{g}$ be a finite-dimensional Lie algebra, κ a symmetric ad-invariant bilinear form on $\mathfrak{g}$. So we have the Kac-Moody Lie* algebra $\mathfrak{g}_{\mathcal{D}}^\kappa$ (see 2.5.9); let A be the twisted enveloping algebra $U(\mathfrak{g}_{\mathcal{D}})^\kappa$.

Let $\gamma = \Sigma a_i \otimes b_i \in \mathfrak{g}\otimes\mathfrak{g}$ be a symmetric ad-invariant tensor. It yields endomorphisms $\mathrm{ad}_\gamma : x \mapsto \Sigma[a_i,[b_i,x]]$ and $\epsilon_\gamma : x \mapsto \Sigma\kappa(a_i,x)b_i$ of $\mathfrak{g}$ which commute with the adjoint action; one has $\mathrm{Tr}(\epsilon_\gamma) = \kappa(\gamma) := \Sigma\kappa(a_i,b_i)$. We say that γ is *κ-normalized* if $\mathrm{ad}_\gamma + \epsilon_\gamma = id_{\mathfrak{g}}$.

REMARK. A κ-normalized γ is unique; it exists if and only if the bilinear form $x, y \mapsto \kappa(x,y) + \frac{1}{2}\mathrm{Tr}(\mathrm{ad}_x\mathrm{ad}_y)$ on $\mathfrak{g}$ is non-degenerate.

Consider γ as a symmetric section of $\mathfrak{g}_{\mathcal{D}} \boxtimes \mathfrak{g}_{\mathcal{D}} \subset A\boxtimes A$. The image of γ by the Lie* bracket morphism $A\boxtimes A \to \Delta_* A$ equals $\kappa(\gamma)\phi 1_A$ where $\phi \in \Delta_*\omega_X$ is a canonical skew-symmetric section which can be written in terms of local coordinates as $(dx\cdot\delta(x-y))\partial_x$. Thus the skew-symmetric morphism of $\mathcal{O}_{X\times X}$-modules

$$(3.7.25.2) \qquad \mathcal{O}_{X\times X}(\Delta) \xrightarrow{\gamma} j_*j^*A\boxtimes A \xrightarrow{\mu} \Delta_* A$$

kills $\mathcal{O}_{X\times X}(-2\Delta)$; i.e., it yields a skew-symmetric morphism $\nu \to \Delta_* A$ where ν was defined in (c) in 2.5.10. The restriction of this morphism to $\omega_X \subset \nu$ equals $\kappa(\gamma)1_A$. Replacing ν by the induced $\mathcal{D}$-module and using (2.5.10.1), we get a morphism of $\mathcal{D}_X$-modules

$$(3.7.25.3) \qquad s_\gamma : \Theta_{\mathcal{D}}^{(-2\kappa(\gamma))} \to A$$

such that $s_\gamma 1^{-2\kappa(\gamma)} = 1_A$; this is the *Sugawara tensor.*

LEMMA. *If γ is κ-normalized, then s_γ is a Virasoro vector of central charge $-2\kappa(\gamma)$, and the corresponding adjoint action of $\Theta_{\mathcal{D}}$ on A coincides with the one defined by the obvious action*[62] *of $\Theta_{\mathcal{D}}$ on $\mathfrak{g}_{\mathcal{D}}^\kappa$.*

Proof. Straightforward computation. □

3.8. BRST, alias semi-infinite, homology

We considered the BRST reduction in the classical setting in 1.4.21–1.4.26; now we turn to the quantum version.

The Becchi-Rouet-Stora-Tyupin construction arose in mathematical physics. It came to representation theory in a pioneering work of B. Feigin [**F**] (his term "semi-infinite" reflects the view of a Tate vector space as a sum of its compact and discrete "halves"[63]) and was further developed in [**FGZ**]. These articles deal with a linear algebra setting; the BRST differential is given by an explicit formula written in terms of a redundant $\mathbb{Z}$-grading. Its conceptual characterization (the BRST

[62] See the first Remark in 2.5.9.

[63] Or, perchance, the view of our momentary existence suspended between the two eternities.

property of 3.8.10(iii) and 3.8.21(iii) below) is due to F. Akman [**A**]. An important example of the BRST construction is the quantum Drinfeld-Sokolov reduction which produces from (the chiral envelope of) an affine Kac-Moody algebra the *chiral* W-*algebra* (see Chapter 14 of [**FBZ**] and references therein). Some closely related subjects we do not touch: the BRST homology of finite quantum groups [**Ar**] and a general homological algebra approach to BRST-like constructions [**Vo**].

We discuss BRST homology first in the setting of chiral algebras (3.8.1–3.8.15) and then in the more traditional setting of Tate's linear algebra (3.8.17–3.8.22). The two pictures are compared in 3.8.23–3.8.25. W-algebras are briefly mentioned in 3.8.16; we refer the reader to [**FBZ**] for a thorough treatment.

We work in the DG super setting (see 1.1.16) skipping the adjective "DG super".

3.8.1. Chiral Weyl algebras. Let T be a $\mathcal{D}_X$-module; we assume that it has no $\mathcal{O}$-torsion. Let $\langle\ \rangle_T \in P_2^*(\{T,T\},\omega_X)$ be a skew-symmetric[64] $*$ pairing. Set $T^\flat := T \oplus \omega_X$. This is a Lie* algebra with commutator equal to $\langle\ \rangle_T$ (plus zero components). So $T^\flat$ is an extension of a commutative Lie* algebra T by ω_X. Let $\mathcal{W} = \mathcal{W}(T,\langle\ \rangle_T)$ be the corresponding twisted enveloping chiral algebra $U(T)^\flat$ (see 3.7.20). This is the *Weyl algebra* of $(T,\langle\ \rangle_T)$. It carries the PBW filtration so that $T^\flat = \mathcal{W}_1$, and $\mathrm{gr}.\mathcal{W} = \mathrm{Sym}^{\cdot} T$ (see 3.7.20).

The *classical*, or *coisson*, Weyl algebra $\mathcal{W}^c$ is $\mathrm{Sym}\, T$ equipped with a coisson bracket that equals $\langle\ \rangle_T$ on $T = \mathrm{Sym}^1 T$. It differs from $\mathrm{gr}.\mathcal{W}$ as a coisson algebra: the coisson bracket on $\mathrm{gr}.\mathcal{W}$ is trivial.

3.8.2. Assume our T is a vector $\mathcal{D}_X$-bundle and $\langle\ \rangle_T$ is non-degenerate; i.e., the corresponding morphism $T \to T^\circ$ (see 2.2.16) is an isomorphism. Then the center of $\mathcal{W}(T)$ equals ω_X. Let A be any chiral algebra. The tensor product $A \otimes \mathcal{W}(T)$ (see 3.4.15) contains A and $\mathcal{W}(T)$ as subalgebras, and A coincides with the centralizer of $T^\flat \subset \mathcal{W}(T) \subset A \otimes \mathcal{W}(T)$. We have a pair of adjoint functors

$$(3.8.2.1)\qquad \mathcal{M}(X,A) \to \mathcal{M}(X, A\otimes\mathcal{W}(T)),\quad \mathcal{M}(X,A\otimes\mathcal{W}(T)) \to \mathcal{M}(X,A),$$

where the left functor is $M \mapsto M \otimes \mathcal{W}(T)$ and the right one sends $N \in \mathcal{M}(X, A\otimes \mathcal{W}(T))$ to the centralizer of T.

3.8.3. Lemma. *The functor* $\mathcal{M}(X,A) \to \mathcal{M}(X,A\otimes\mathcal{W}(T))$ *is fully faithful. Its image consists of those* $A\otimes\mathcal{W}(T)$*-modules on which the Lie* algebra* $T^\flat$ *acts in a locally nilpotent way. In particular, if* T *is purely odd, then the functors (3.8.2.1) are mutually inverse equivalences of categories.* □

3.8.4. Let us return to situation 3.8.1 (so we drop the non-degeneracy condition). Assume that we have a direct sum decomposition $T = V \oplus V'$ such that $\langle\ \rangle_T$ vanishes on both V and V'. Let $\langle\ \rangle \in P_2^*(\{V',V\},\omega_X)$ be the restriction of $\langle\ \rangle_T$ (notice that the sign of $\langle\ \rangle$ depends on the order of V, V'). One recovers $(T,\langle\ \rangle_T)$ from $(V,V',\langle\ \rangle)$ in the obvious way, so we will write $\mathcal{W} = \mathcal{W}(V,V',\langle\ \rangle)$. There are obvious embeddings of chiral algebras $\mathrm{Sym}\, V, \mathrm{Sym}\, V' \hookrightarrow \mathcal{W}(V,V')$.

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[64] As always, "skew-symmetric" refers to the "super" tensor structure: if T is purely odd, this amounts to "naive symmetric."

Define an extra $\mathbb{Z}$-grading on $T^\flat$ setting $T^{\flat(-1)} := V$, $T^{\flat(1)} := V'$, $T^{\flat(0)} := \omega_X$. Therefore $\mathcal{W}$ acquires an extra $\mathbb{Z}$-grading $\mathcal{W}^{(\cdot)}$.[65] One has $\mathcal{W}_1^{(-1)} = V$, $\mathcal{W}_1^{(1)} = V'$, and $\mathrm{gr}_a \mathcal{W}^{(n)} = \mathrm{Sym}^{\frac{a-n}{2}} V \otimes \mathrm{Sym}^{\frac{a+n}{2}} V'$.

Remark. $\mathcal{W}$ carries another filtration – the one generated by $(\mathrm{Sym}\, V', V + \mathrm{Sym}\, V')$ (see 3.7.13). This filtration is commutative, and the associated graded algebra $\mathrm{gr}'_\cdot \mathcal{W}$ equals $\mathrm{Sym}\, V' \otimes \mathrm{Sym}^\cdot V$. So we have a canonical identification of graded algebras[66] $\mathrm{gr}' \mathcal{W} = \mathrm{Sym}(V' \oplus V) = \mathcal{W}^c$ which is compatible with coisson brackets. So if V, V' are mutually dual vector $\mathcal{D}_X$-bundles, then $\mathcal{W}$ is a $\mathrm{Sym}V'$-cdo (see 3.9.5 for terminology).

3.8.5. The Tate extension revisited. Consider the $\mathcal{D}_X$-module $\mathcal{W}_2^{(0)}$. It contains $\mathcal{W}_0^{(0)} = \mathcal{W}_0 = \omega$ and $\mathcal{W}_2^{(0)}/\omega = \mathrm{gr}_2 \mathcal{W}^{(0)} = V \otimes V'$. The adjoint action of $\mathcal{W}_2^{(0)}$ preserves $\mathcal{W}_1^{(\pm 1)}$. Since $\omega \subset \mathcal{W}_2^{(0)}$ acts trivially, we get $*$ actions of $V \otimes V'$ on V, V'. Since $\mathcal{W}_2^{(0)} = \mu(V, V') + \omega$, we see that $\mathcal{W}_2^{(0)}$ is a Lie* subalgebra of $\mathcal{W}$.

Recall that in 1.4.2 we defined the associative* algebra structure on $V \otimes V'$ together with its left action on V and right action on V'. We leave it to the reader to check that the above action of $V \otimes V'$ on V coincides with that of 1.4.2, the one on V' is opposite to that of 1.4.2, and the Lie* algebra structure on $V \otimes V' = \mathcal{W}_2^{(0)}/\omega$ coincides with the Lie* algebra structure associated with the associative* algebra structure from 1.4.2. So the quotient $V \otimes V' = \mathcal{W}_2^{(0)}/\omega$ is the Lie* algebra $\mathcal{G}^{Lie}$ from 2.7.2, and we have an extension of Lie* algebras

$$0 \to \omega \to \mathcal{W}_2^{(0)} \to \mathcal{G} \to 0. \tag{3.8.5.1}$$

This extension canonically identifies with the extension $\mathcal{G}^{-\flat}$ opposite to the Tate extension $\mathcal{G}^\flat$ from 2.7.2. Namely, consider the chiral product $j_* j^* V \boxtimes V' \to \Delta_* \mathcal{W}_2^{(0)}$. Its restriction to $V \boxtimes V'$ equals $-\langle\ \rangle : V \boxtimes V' \to \Delta_* \omega_X$, and its composition with the projection $\mathcal{W}_2^{(0)} \to V \otimes V'$ is the standard projection $j_* j^* V \boxtimes V' \to \Delta_*(V \otimes V')$. Therefore it yields a canonical identification of extensions (see the construction of $\mathcal{G}^\flat$ in 2.7.2)

$$\mathcal{W}_2^{(0)} \xrightarrow{\sim} \mathcal{G}^{-\flat}. \tag{3.8.5.2}$$

This identification is obviously compatible with the Lie* brackets.

3.8.6. Chiral Clifford algebras. Let L be a (DG super) $\mathcal{D}_X$-module; suppose that it is a finite complex of vector $\mathcal{D}_X$-bundles. Let L° be its dual; denote by $(\ ) \in P_2^*(\{L^\circ, L\}, \omega_X)$ the canonical non-degenerate pairing (see 2.2.16). Denote the corresponding pairing $\in P_2^*(\{L^\circ[-1], L[1]\}, \omega_X)$ by $\langle\ \rangle$; set $\mathcal{C}l = \mathcal{C}l(L, L^\circ, (\ ))$ $:= \mathcal{W}(L[1], L^\circ[-1], \langle\ \rangle)$. This is our *Clifford* chiral algebra.

As in 3.8.4, $\mathcal{C}l$ contains subalgebras $\mathrm{Sym}(L^\circ[-1])$ and $\mathrm{Sym}(L[1])$. It carries an extra $\mathbb{Z}$-grading $\mathcal{C}l^{(\cdot)}$ and the PBW filtration $\mathcal{C}l_\cdot$ with $\mathcal{C}l_1 = L[-1] \oplus L[1] \oplus \omega$ and $\mathrm{gr}\, \mathcal{C}l = \mathrm{Sym}(L^\circ[-1] \oplus L[1])$.

Consider the Lie* algebra $\mathcal{C}l_2^{(0)}$. The chiral product $j_* j^*(L^\circ \boxtimes L) = j_* j^*(L^\circ[-1] \boxtimes L[1]) = j_* j^*(\mathcal{C}l_1^{(1)} \boxtimes \mathcal{C}l_1^{(-1)}) \to \mathcal{C}l_2^{(0)}$ yields the $*$ pairing equal to $(\ )$, so it defines,

[65] Notice that, unless V, V' are purely odd, the $\mathbb{Z}$-grading mod 2 does *not* coincide with the structure "super" $\mathbb{Z}/2$-grading.

[66] The grading we consider comes from the grading on $\mathcal{W}$; we forget about the grading coming from the filtration.

as in 3.8.5, an isomorphism of ω-extensions of $\mathcal{G} = L \otimes L^\circ$ (cf. (3.8.5.2))

$$\mathcal{C}l_2^{(0)} \xrightarrow{\sim} \mathcal{G}^\flat \tag{3.8.6.1}$$

where $\mathcal{G}^\flat$ is the Tate extension defined by $L, L^\circ, (\ )$. This is an isomorphism of Lie* algebras, and it identifies the adjoint action of $\mathcal{C}l_2^{(0)}/\omega$ on $\mathcal{C}l_1^{(\pm 1)}$ with the usual action of $\mathcal{G}$ on $L^\circ[-1]$, $L[1]$. The proof coincides with that of the similar statement for Weyl algebras plus a sign exercise.

3.8.7. Let now L be a Lie* algebra on X which satisfies the conditions of 3.8.6 as a mere $\mathcal{D}_X$-complex. So we have the Clifford algebra $\mathcal{C}l$ and the Tate extension $\mathfrak{g}l(L)^\flat$ of the Lie* algebra $\mathfrak{g}l(L) = L \otimes L^\circ$ identified with $\mathcal{C}l(T)_2^{(0)}$. The adjoint action of L yields a morphism of Lie* algebras $L \to \mathfrak{g}l(L)$. Denote by $L^\flat$ the pull-back to L of the extension $\mathfrak{g}l(L)^\flat$; this is the *Tate extension* of L. So we have a canonical morphism of Lie* algebras $\beta :\ L^\flat \to \mathfrak{g}l(L)^\flat \subset \mathcal{C}l$ which sends $1^\flat \in L^{\flat\ell}$ to $1 \in \mathcal{C}l^\ell$.

3.8.8. Let A be a chiral algebra and $\alpha : L^\flat \to A$ a morphism of Lie* algebras such that $\alpha(1^\flat) = -1_A$; we refer to $(A, \alpha) = (A, L, \alpha)$ as *BRST datum.* Since α and β take opposite values on $1^\flat \in L^\flat$, we have a morphism of Lie* algebras

$$\ell^{(0)} := \alpha + \beta :\ L \to A \otimes \mathcal{C}l^{(0)}. \tag{3.8.8.1}$$

Consider the contractible Lie* algebra $L_\dagger$ (see 1.1.16); recall that as a mere complex it equals $\mathcal{C}one(id_L)$. Then $\ell^{(0)}$ extends to a morphism of mere graded Lie* algebras (we forget about the differential for a moment)

$$\ell : L_\dagger \to A \otimes \mathcal{C}l, \tag{3.8.8.2}$$

whose component $\ell^{(-1)} : L[1] \to A \otimes \mathcal{C}l^{(-1)}$ is the composition $L[1] \hookrightarrow \mathcal{C}l^{(-1)} \hookrightarrow A \otimes \mathcal{C}l$.

3.8.9. To define the BRST charge, we need a simple lemma. Consider the Chevalley differential δ on the commutative$^!$ algebra $\mathrm{Sym}(L^\circ[-1])$ (see 1.4.10). Let us identify $\mathrm{Sym}(L^\circ[-1])$ with its image by the embedding $\mathrm{Sym}(L^\circ[-1]) \subset \mathcal{C}l \subset A \otimes \mathcal{C}l$. The adjoint action of $L_\dagger$ on $A \otimes \mathcal{C}l$ via ℓ preserves $\mathrm{Sym}(L^\circ[-1])$. The corresponding action of $L_\dagger$ on the Chevalley DG algebra is compatible with the differentials: one has $ad_{\ell^{(0)}} = [\delta, ad_{\ell^{(-1)}(a)}] \in P_2^*(\{L, \mathrm{Sym}(L^\circ[-1])\}, \mathrm{Sym}(L^\circ[-1]))$. Restricting this identity to $L^\circ[-1] \subset \mathrm{Sym}(L^\circ[-1])$, we get (here $[\ ]$ is the bracket on $A \otimes \mathcal{C}l$):

LEMMA. *The $*$ operations $[\ell^{(0)}, id_{L^\circ[-1]}]$ and $[\ell^{(-1)}, \delta|_{L^\circ[-1]}]$ in $P_2^*(\{L, L^\circ[-1]\}, A \otimes \mathcal{C}l^{(1)})$ coincide.* □

Consider a chiral operation

$$\tilde{\chi} := \mu(\ell^{(0)}, id_{L^\circ[-1]}) - \mu(\ell^{(-1)}, \delta|_{L^\circ[-1]}) \in P_2^{ch}(\{L, L^\circ\}, A \otimes \mathcal{C}l^{(1)}[1]) \tag{3.8.9.1}$$

where μ is the chiral product on $A \otimes \mathcal{C}l$. The corresponding $*$ operation vanishes by the lemma, so it amounts to a morphism of complexes

$$\chi :\ L \otimes L^\circ \to A \otimes \mathcal{C}l^{(1)}[1]. \tag{3.8.9.2}$$

We have $h(L \otimes L^\circ) \xrightarrow{\sim} \mathrm{End}(L)$. Set

$$\mathfrak{d} = \mathfrak{d}_{A,\alpha} := \chi(id_L) \in h(A \otimes \mathcal{C}l^{(1)}[1]). \tag{3.8.9.3}$$

This is the *BRST charge* for (A, α). Its adjoint action is an odd derivation $d_{A,\alpha}$ of $A \otimes \mathcal{C}l$ called the *BRST differential*. It has degree 1 with respect to both the structure degree and the ${}^{(\cdot)}$-grading.

3.8.10. THEOREM. (i) *One has* $[\mathfrak{d}, \mathfrak{d}] = 0$*; hence* $d^2_{A,\alpha} = 0$.
The DG chiral algbebra $(A \otimes \mathcal{C}l, d_{A\otimes\mathcal{C}l} + d_{A,\alpha})$ *is denoted by* $\mathcal{C}_{\mathrm{BRST}}(L, A)$.
(ii) ℓ *from (3.8.8.2) is a morphism of DG Lie* algebras*

(3.8.10.1) $$\ell : L_\dagger \to \mathcal{C}_{\mathrm{BRST}}(L, A).$$

(iii) $\mathfrak{d}$ *is a unique element of* $h(A \otimes \mathcal{C}l^{(1)}[1])$ *which is even of structure degree 0 and such that* $ad_{\mathfrak{d}}\ell^{(-1)} = \ell^{(0)}$*. Similarly,* $d_{A,\alpha}$ *is a unique odd derivation of* $A \otimes \mathcal{C}l$ *of structure and* ${}^{(\cdot)}$ *degrees 1 such that* $d_{A,\alpha}\ell^{(-1)} = \ell^{(0)}$.

REMARKS. (i) The statement (ii) of the theorem says that $\ell^{(-1)}$ is a $d_{A,\alpha}$ (or $d_{A\otimes\mathcal{C}l} + d_{A,\alpha}$) homotopy between $\ell^{(0)} = \alpha + \beta$ and 0; we refer to it as the *BRST property* of $\mathfrak{d}$ or $d_{A,\alpha}$. According to (iii) of the theorem, the BRST property determines $\mathfrak{d}$ and $d_{A,\alpha}$ uniquely.

(ii) The construction of the BRST element and the above theorem remain literally valid in the setting of chiral algebras on an algebraic $\mathcal{D}_X$-space $\mathcal{Y}$ (see 3.3.10). We assume that L is a Lie* algebra on $\mathcal{Y}$ which is a vector $\mathcal{D}_X$-bundle on $\mathcal{Y}$, etc.

(iii) Let $\varphi : A \to A'$ be a morphism of chiral algebras. Then $(A', L, \varphi\alpha)$ is again a BRST datum. It follows from the statement (iii) of the theorem that the morphism $\varphi \otimes id_{\mathcal{C}l} : A \otimes \mathcal{C}l \to A' \otimes \mathcal{C}l$ sends $\mathfrak{d}_{A,\alpha}$ to $\mathfrak{d}_{A',\varphi\alpha}$; hence we have a morphism of DG chiral algebras $\mathcal{C}_{\mathrm{BRST}}(L, A) \to \mathcal{C}_{\mathrm{BRST}}(L, A')$.

(iv) For the "classical" version of the theorem see 1.4.24.

Proof. It is found in 3.8.11–3.8.13.

3.8.11. First we show that $d_{A,\alpha}\ell^{(-1)} = \ell^{(0)} \in \mathrm{Hom}(L, A \otimes \mathcal{C}l)$; i.e., $\mathfrak{d}$ and $d_{A,\alpha}$ satisfy (iii).

Consider the operation $\mu(\tilde{\chi}, \ell^{(-1)}) \in P_3^{ch}(\{L, L^\circ, L\}, A \otimes \mathcal{C}l)$. We label the variables by the lower indices 1, 2, 3. One has $\mu(\tilde{\chi}, \ell^{(-1)}) = \mu(\ell_1^{(0)}, \mu(id_{L^\circ[-1]\,2}, \ell_3^{(-1)})) + \mu(\mu(\ell_1^{(0)}, \ell_3^{(-1)}), id_{L^\circ[-1]\,2}) - \mu(\ell_1^{(-1)}, \mu(\delta|_{L^\circ[-1]\,2}, \ell_3^{(-1)})) - \mu(\mu(\ell_1^{(-1)}, \ell_3^{(-1)}), \delta|_{L^\circ[-1]\,2})$. Restricting it to $j_*(L \boxtimes L^\circ) \boxtimes L$, we get $\mu(\ell_1^{(0)}, \langle\ \rangle_{23}) + \mu(\ell^{(-1)}[\ ]_{13}, id_{L^\circ[-1]\,2}) + \mu(\ell_1^{(-1)}, ad^\circ_{L\,32}) = \mu(\ell_1^{(0)}, \langle\ \rangle_{23}) + [\mu(\ell_1^{(-1)}, id_{L^\circ[-1]\,2}), \ell_3^{(0)}]$. Both summands vanish on $L \boxtimes L^\circ \boxtimes L$, so we can consider them as operations in $P_2^*(\{L \otimes L^\circ, L\}, A \otimes \mathcal{C}l)$. Insert $id_L \in h(L \otimes L^\circ)$ in the first variable; the first summand yields $\ell^{(0)}$ and the second one 0. We are done.

3.8.12. Now let us prove the uniqueness statement (iii) in the theorem 3.8.10.

(a) Let $F \subset h(A \otimes \mathcal{C}l)$ be the centralizer of $\ell^{(-1)}$, so F contains $A \otimes \mathrm{Sym}(L[1])$. Let us show that actually $F = A \otimes \mathrm{Sym}(L[1])$. This statement follows from its "classical" version 1.4.22. Indeed, consider the ring filtration on $A \otimes \mathcal{C}l$ generated by $(A \otimes \mathrm{Sym}(L[1]), A \otimes (\mathrm{Sym}(L[1]) + L^\circ[-1]))$ (see 3.7.13). The associated graded algebra equals $A \otimes \mathcal{C}l_c$ (see 1.4.21). Consider the corresponding filtration on $h(A \otimes \mathcal{C}l)$; we have $\mathrm{gr}\, h(A \otimes \mathcal{C}l) = h(A \otimes \mathcal{C}l_c)$. Take $v \in h(A \otimes \mathcal{C}l^a)$; let n be the smallest number such that it lies in the nth term of the filtration, so the corresponding $\bar{v} \in \mathrm{gr}_n h(A \otimes \mathcal{C}l_c^a) = h(A \otimes \mathrm{Sym}^n(L^\circ[-1]) \otimes \mathrm{Sym}^{n-a}(L[1]))$ is non-zero (cf. the proof in 1.4.22). Then the composition $ad_v\ell^{(-1)}$ takes values in the previous term, and the corresponding morphism from $L[1]$ to $\mathrm{gr}_{n-1}(A\otimes \mathcal{C}l^{(a)}) \subset A\otimes \mathcal{C}l_c^{(a-1)}$ is the morphism $\psi(\bar v)$ from 1.4.22. So 1.4.22 implies that for $v\in F$ one has $n=0$; q.e.d.

(b) Consider the maps $h(A\otimes \mathcal{C}l^{(a)}) \to \mathrm{Der}^a(A\otimes \mathcal{C}l) \to \mathrm{Hom}(L[1], A\otimes \mathcal{C}l^{a-1})$, where the first arrow is the adjoint action and the second one is $\nu \mapsto \nu\ell^{(-1)}$.[67] Now both of them are injective for $a\geq 1$; this obviously implies 3.8.10(iii). The proof is an obvious modification of that of its "classical" counterpart 1.4.24(iii) (replace 1.4.22 by (a) above and $\mathcal{C}l_c$ by $\mathcal{C}l$).

3.8.13. Let us show that $[\mathfrak{d},\mathfrak{d}] = d_{A,\alpha}(\mathfrak{d})$ vanishes. According to (a) in 3.8.12, it suffices to prove that $[\mathfrak{d},\mathfrak{d}]\in F$, i.e., $d^2\ell^{(-1)}=0$. One has $d\ell^{(-1)}=\ell^{(0)}$ (see 3.8.11), so we want to show that $d\ell^{(0)}=0$. Again by (a) in 3.8.12 it suffices to show that $d\ell^{(0)}$ takes values in F, i.e., $[d\ell^{(0)},\ell^{(-1)}]=0$. One has[68] $[d\ell^{(0)},\ell^{(-1)}] = d[\ell^{(0)},\ell^{(-1)}] - [\ell^{(0)}, d\ell^{(-1)}] = d\ell^{(-1)}[\ ] - [\ell^{(0)}, d\ell^{(-1)}] = \ell^{(0)}[\ ] - [\ell^{(0)},\ell^{(0)}] = 0$, and we are done.

Finally 3.8.10(ii) follows from 3.8.11 and 3.8.13. □

3.8.14. The DG chiral algebra $\mathcal{C}_{\mathrm{BRST}}(L,A)$ considered as an object of the homotopy category $\mathcal{H}o\mathcal{CA}(X)$ (see 3.3.13) is called the *BRST reduction* of A with respect to α. The reduction is *regular* if $H^a_{\mathrm{BRST}}(L,A)=0$ for $a\neq 0$; then the BRST reduction is the plain chiral algebra $H^0_{\mathrm{BRST}}(L,A)$.

For any graded $A\otimes\mathcal{C}l$-module N the action $\mathfrak{d}_N$ of $\mathfrak{d}$ on N is a $d_{A,\alpha}$-derivation of square 0 commuting with the structure differential d_N. So N equipped with a differential $d_N+\mathfrak{d}_N$ is a DG $\mathcal{C}_{\mathrm{BRST}}(L,A)$-module. In particular, for an A-module M the module $M\otimes\mathcal{C}l$ (see 3.8.2) is a DG $\mathcal{C}_{\mathrm{BRST}}(L,A)$-module. We denote it by $\mathcal{C}_{\mathrm{BRST}}(L,M)$ and call the BRST complex of L with coefficients in M. Its cohomology $H^{\cdot}_{\mathrm{BRST}}(L,M)$ is an $H^{\cdot}_{\mathrm{BRST}}(L,A)$-module; this is the *BRST*, or *semi-infinite*, homology of L with coefficients in M. As follows from Remark (iii) in 3.8.10 (applied to the morphism $\varphi: U(L)^{-\flat}\to A$ defined by α) our $C_{\mathrm{BRST}}(L,M)$ depends, as a complex of plain $\mathcal{D}_X$-modules, only on the chiral $L^{-\flat}$-module structure on M.

VARIANT. Let $x\in X$ be a point, $j_x: U_x \hookrightarrow X$ its complement. Assume that our BRST datum is given on U_x. According to 3.6.6 and 3.6.13 the category of $j_{x*}A\otimes\mathcal{C}l$-modules supported at x coincides with the category of discrete $A_x^{as}\hat{\otimes}\mathcal{C}l_x^{as}$-modules. We can consider $\mathfrak{d}$ as an element of $h_x j_{x*}A\otimes\mathcal{C}l$, so its action yields a canonical differential on any discrete $A_x^{as}\hat{\otimes}\mathcal{C}l_x^{as}$-module. We will see in 3.8.25 that it can be described in terms of the Tate linear algebra.

REMARK. In this setting we have no canonical $j_{x*}\mathcal{C}l$-module supported at x, so to define the BRST homology of an A_x^{as}-module M, we should choose some $\mathcal{C}l_x^{as}$-module C and then define the BRST homology of M as the homology of $M\otimes C$. If L is extended to X, i.e., if we have a Lie* subalgebra $L_X\subset j_{x*}L$, then we can take for C the module corresponding to the fiber at x of the Clifford module for L_X.

3.8.15. The classical limit. Let us show that the BRST construction for coisson algebras from 1.4.21–1.4.25 can be considered as a "classical limit" of the chiral BRST construction.

Consider an increasing filtration on $\mathcal{C}l$ with terms $\mathrm{Sym}(L^\circ[-1])\cdot \mathrm{Sym}^{\leq n}(L[1])$ (it differs from the PBW filtration from 3.8.6). This is a commutative filtration and

[67] Here the Der^a are derivations of degree a with respect to the grading $A\otimes\mathcal{C}l^{(\cdot)}$.

[68] The first equality comes since d is a derivation, the second and the fourth ones follow since ℓ commutes with brackets, and the third one is 3.8.11.

$\mathrm{gr}.\mathcal{C}l = \mathrm{Sym}(L^\circ[-1]) \otimes \mathrm{Sym}^{\cdot}(L[1])$; this identification is, in fact, an isomorphism of coisson algebras (see 1.4.21, 1.4.23)

$$\mathrm{gr}\,\mathcal{C}l \xrightarrow{\sim} \mathcal{C}l_c. \tag{3.8.15.1}$$

Let now A, $\alpha : L^\flat \to A$ be a BRST datum. Suppose that A carries a commutative filtration $A.$ (see 3.3.12) such that $\alpha(L^\flat) \subset A_1$. Then $\mathrm{gr}\,A$ is a coisson algebra, and $\alpha_c := \alpha \bmod A_0 : L \to \mathrm{gr}_1 A$ is a morphism of Lie* algebras, so we have the corresponding classical BRST complex $\mathcal{C}_{\mathrm{BRST}}(L, \mathrm{gr}\,A)_c$ (see 1.4.24).

Equip $A \otimes \mathcal{C}l$ with the tensor product of our filtrations. This is a commutative filtration, and one has an isomorphism of coisson algebras $\mathrm{gr}(A \otimes \mathcal{C}l) = \mathrm{gr}\,A \otimes \mathrm{gr}\,\mathcal{C}l = (\mathrm{gr}\,A) \otimes \mathcal{C}l_c$. The image of the morphism ℓ from (3.8.8.2) lies in the first term of the filtration, and $\ell \bmod A_0$ equals its classical counterpart (1.4.23.1). The BRST differential preserves our filtration, and $\mathrm{gr}\,d$ satisfies the classical BRST property, hence equals the classical BRST differential. We have proved:

LEMMA. *There is a canonical isomorphism of DG coisson algebras*

$$\mathrm{gr}\mathcal{C}_{\mathrm{BRST}}(L, A) \xrightarrow{\sim} \mathcal{C}_{\mathrm{BRST}}(L, \mathrm{gr}A)_c. \tag{3.8.15.2}$$

In particular, if the "classical" BRST reduction of $\mathrm{gr}\,A$ *is regular, then the "quantum" BRST reduction of* A *is also regular.* □

3.8.16. W-algebras. An important example of the BRST construction is the quantum Drinfeld-Sokolov reduction. See Chapter 14 of [**FBZ**] for all details and proofs.

We use the notation of (a) in 2.6.8. As in loc. cit., we have a canonical embedding $\alpha : \mathfrak{n}(\mathfrak{F})_{\mathcal{D}} \hookrightarrow \mathfrak{g}(\mathfrak{F})^{\kappa}_{\mathcal{D}}$. Consider the twisted enveloping algebra $U^\kappa := U(\mathfrak{g}(\mathfrak{F})_{\mathcal{D}})^\kappa$ (see 3.7.20) and the embedding $\alpha_\psi := i\alpha + \psi \cdot 1_{U^\kappa} : \mathfrak{n}(\mathfrak{F})_{\mathcal{D}} \hookrightarrow U^\kappa$. Since the adjoint action of $\mathfrak{n}$ is nilpotent, the extension $\mathfrak{n}(\mathfrak{F})^\flat_{\mathcal{D}}$ of $\mathfrak{n}(\mathfrak{F})_{\mathcal{D}}$ from 3.8.7 is trivialized, so α_ψ extends to a BRST datum $\mathfrak{n}(\mathfrak{F})^\flat_{\mathcal{D}} \to U^\kappa$ denoted again by α_ψ.

THEOREM. *The BRST reduction of* U^κ *with respect to* α_ψ *is regular. Thus it reduces to a plain chiral algebra* $W^\kappa := H^0_{\mathrm{BRST}}(\mathfrak{n}(\mathfrak{F})_{\mathcal{D}}, U^\kappa)$ *called the* W*-algebra.* □

REMARK. The theorem for generic κ follows, by 3.8.15, from the classical counterpart (which is the first proposition in 2.6.8).

3.8.17. BRST in the Tate linear algebra setting. In the rest of this section we will show that the associative topological DG algebras $\mathcal{C}_{\mathrm{BRST}}(L, A)^{as}_x$ can be computed purely in terms of Tate's linear algebra. We assume that the reader is acquainted with the content of 2.7.7–2.7.9 and 3.6.1.

Notice that if T is a Tate vector space and R any topological vector space, then $R\hat{\otimes}T^*$ is equal to the space $\mathrm{Hom}(T, R)$ of continuous linear maps $T \to R$. More generally, $R\hat{\otimes}T^{*\hat{\otimes}n}$ equals the space of continuous polylinear maps $T \times \cdots \times T \to R$.

For a topological Lie algebra L its *topological enveloping algebra* $U(L)$ is a topological associative algebra equipped with a continuous morphism of Lie algebras $L \to U(L)$ which is universal in the obvious sense. Thus $U(L)$ is a completion of the plain enveloping algebra with respect to the topology made of left ideals generated by open subspaces $V \subset L$. If the topology on L is generated by open Lie subalgebras, then $U(L)$ satisfies the Poincaré-Birkhoff-Witt theorem: the associated graded algebra $\operatorname{gr} U(L)$ for the PBW filtration[69] equals the completed symmetric algebra $\operatorname{Sym} L$. This condition holds automatically if L is a Tate vector space.[70]

REMARK. If $L^\flat$ is a central extension of L by k, then we have the corresponding twisted topological enveloping algebra $U(L)^\flat$; this is a topological associative algebra equipped with a morphism of topological Lie algebras $L^\flat \to U(L)^\flat$ which sends $1^\flat \in k \subset L^\flat$ to $1_{U(L)^\flat}$ and is a universal morphism with these properties. The PBW theorem for $U(L)^\flat$ holds if the topology on $L^\flat$ is generated by open Lie subalgebras (and we assume that the topology induced on $k \subset L^\flat$ is discrete).

Let T be a Tate (super) vector space, $< \ > = < \ >_T$ a skew-symmetric (continuous) bilinear form on T. Consider $T^\flat := T \oplus k$ as a Lie algebra with the non-zero component of the bracket equal to $< \ >$. This is a central extension of T (considered as a commutative Lie algebra) by k. The twisted topological enveloping algebra $W = U(T)^\flat$ is called the *topological Weyl algebra* of $(T, < \ >)$.

Assume that $< \ >$ is non-degenerate. Then the category Wmod can be described as follows. Let $V \subset T$ be a c-lattice; then its orthogonal complement $V^\perp$ is also a c-lattice. Assume that the restriction of $< \ >$ to V is trivial; i.e., $V^\perp \supset V$. The restriction of $< \ >$ to $V^\perp$ comes from the finite-dimensional vector space $V^\perp/V$. The Weyl algebra $W(V^\perp)$ is equal to the subalgebra of W generated by $V^\perp$ (and coincides with the centralizer of V in W). The projection $W(V^\perp) \to W(V^\perp/V)$ yields the functor $W(V^\perp/V)\mathrm{mod} \to W\mathrm{mod}$, $M \mapsto W \underset{W(V^\perp)}{\otimes} M$. This functor is fully faithful. According to a variant of Kashiwara's lemma it identifies $W(V^\perp/V)$mod with the subcategory of those W-modules on which every $v \in V \subset W$ acts in a locally nilpotent way. Every discrete W-module is a union of submodules which satisfy this property with respect to smaller and smaller V's.

As in 3.8.6 we can replace skew-symmetric pairing by a symmetric pairing to get a *topological Clifford algebra.* As above, we are interested in the graded version of this construction, so we start with a Tate vector space F and consider the topological graded Clifford algebra $\mathcal{C}l = \mathcal{C}l^{(\cdot)}$ generated by the graded (super) Tate vector space $F^*[-1] \oplus F[1]$ such that the commutator pairing between $F^*[-1]$ and $F[1]$ equals the canonical pairing $F^*[-1] \otimes F[1] = F^* \otimes F \to k \subset \mathcal{C}l$. Our $\mathcal{C}l$ contains commutative graded subalgebras $\operatorname{Sym}(F^*[-1])$, $\operatorname{Sym}(F[1])$; here Sym denotes the completed symmetric algebra.[71] It carries the PBW filtration $\mathcal{C}l_\cdot$ compatible with the grading; the algebra $\operatorname{gr} \mathcal{C}l$ is equal to $\operatorname{Sym}(F^*[-1] \oplus F[1])$.

We have $\mathcal{C}l_0^{(0)} = k$, $\mathcal{C}l_1^{(1)} = F^*[-1]$ and $\mathcal{C}l_1^{(-1)} = F[1]$. Notice that $\mathcal{C}l_2(0)$ is equal to the simultaneous normalizer of $F^*[-1]$, $F[1]$ in $\mathcal{C}l$. The adjoint action of this Lie subalgebra preserves both $F^*[-1]$ and $F[1]$ and identifies $\mathcal{C}l_2^{(0)}/k$ with $\mathfrak{gl}(F)$ (as a topological Lie algebra).[72] Thus $\mathcal{C}l_2^{(0)}$ is a central extension of $\mathfrak{gl}(F)$ by k. It equals the Tate extension $\mathfrak{gl}(F)^\flat$ from 2.7.8:

3.8.18. PROPOSITION. *There is a canonical isomorphism of topological central extensions*

$$\mathcal{C}l_2^{(0)} \xrightarrow{\sim} \mathfrak{gl}(F)^\flat.$$

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[69] Its terms are closures of the terms of the PBW filtration on the plain enveloping algebra.

[70] Proof: for every c-lattice $V \subset L$ its normalizer in L intersected with V is an open Lie subalgebra of L contained in V.

[71] I.e., the symmetric algebra in the category of topological vector spaces for the product $\hat{\otimes}$.

[72] Use Exercise in 2.7.7.

Proof. According to (ii) in 2.7.1 and 2.7.8 such an isomorphism amounts to a pair of continuous sections $s_c : \mathfrak{gl}_c(F) \to \mathcal{C}l_2^{(0)}$, $s_d : \mathfrak{gl}_d(F) \to \mathcal{C}l_2^{(0)}$ which commute with the adjoint action of $\mathfrak{gl}(F)$, $s_c + s_d : \mathfrak{gl}_c(F) \oplus \mathfrak{gl}_d(F) \to \mathcal{C}l_2^{(0)}$ is an open map, and $s_c - s_d : \mathfrak{gl}_f(F) \to k \subset \mathcal{C}l_2^{(0)}$ equals tr. Recall (see Exercise in 2.7.7) that $\mathfrak{gl}_c(F)$, $\mathfrak{gl}_d(F)$ are completions of $F \otimes F^*$ with respect to appropriate topologies. We define s_c, s_d restricted to $F \otimes F^*$ as, respectively, compositions $F \otimes F^* = F^*[-1] \otimes F[1] \xrightarrow{\cdot} \mathcal{C}l_2^{(0)}$, $F \otimes F^* = F[1] \otimes F^*[-1] \xrightarrow{\cdot} \mathcal{C}l_2^{(0)}$ where $\cdot$ is the product map. It follows immediately from Exercise in 2.7.7 that s_c and s_d are continuous, so they extend to continuous maps $s_c : \mathfrak{gl}_c(F) \to \mathcal{C}l_2^{(0)}$, $s_d : \mathfrak{gl}_d(F) \to \mathcal{C}l_2^{(0)}$. We leave it to the reader to check that s_c, s_d satisfy the properties listed above. □

3.8.19. Let L be a topological Lie algebra which is a Tate vector space. The adjoint action yields a continuous morphism of Lie algebras $ad : L \to \mathfrak{gl}(L)$. The *Tate extension* of L is the pull-back $L^\flat$ of the Tate extension $\mathfrak{gl}(L)^\flat$ by ad.

Remark. For every Lie subalgebra $P \subset L$ which is a c-lattice the adjoint action maps P to $\mathfrak{gl}(L)_P$, so, according to Remark in 2.7.9, we have a canonical section $s_P : P \to L^\flat$. If $P' \subset P \subset L$ is another such subalgebra, then the restriction of s_P to P' equals $s_{P'} + \mathrm{tr}_{P/P'}$ (the trace of the adjoint action of P' on P/P').

According to 3.8.18 we have a canonical morphism of topological Lie algebras $\beta : L^\flat \to \mathfrak{gl}(L)^\flat \subset \mathcal{C}l^{(0)}$ where $\mathcal{C}l$ is the Clifford algebra as in 3.8.17 (for $F = L$). One has $\beta(1^\flat) = 1_{\mathcal{C}l}$.

Let A be a topological associative algebra. A *BRST datum* is a continuous morphism of Lie algebras $\alpha : L^\flat \to A$ such that $\alpha(1^\flat) = -1_A$. Since α and β take opposite values on $1^\flat$, we have a morphism of topological Lie algebras

$$\ell^{(0)} := \alpha + \beta : L \to A\hat{\otimes}\mathcal{C}l^{(0)}. \tag{3.8.19.1}$$

Consider the contractible topological Lie DG algebra $L_\dagger$ (see 1.1.16). Then $\ell^{(0)}$ extends to a morphism of topological graded Lie algebras (we forget about the differential for a moment)

$$\ell : L_\dagger \to A\hat{\otimes}\mathcal{C}l, \tag{3.8.19.2}$$

where $\ell^{(-1)}$ is the composition $L[1] \to \mathcal{C}l^{(-1)} \subset A\hat{\otimes}\mathcal{C}l^{(-1)}$.

Denote by δ the Chevalley differential on the topological commutative DG algebra $\mathrm{Sym}(L^*[-1])$ defined by the topological Lie algebra structure on L. Let us identify $\mathrm{Sym}(L^*[-1])$ with its image by the embedding $\mathrm{Sym}(L^*[-1]) \subset \mathcal{C}l \subset A\hat{\otimes}\mathcal{C}l$. The action of $L_\dagger$ on $A\hat{\otimes}\mathcal{C}l$ via ℓ and the adjoint action preserve $\mathrm{Sym}(L^*[-1])$. Considered as an action of $L_\dagger$ on the Chevalley DG algebra, it is compatible with the differentials: for $a \in L$ the action of $ad_{\ell^{(0)}(a)}$ on $\mathrm{Sym}(L^*[-1])$ coincides with $[\delta, ad_{\ell^{(-1)}(a)}]$.

3.8.20. Lemma. *The map $L \otimes L^* \to A\hat{\otimes}\mathcal{C}l[1]$, $a \otimes b \mapsto \ell^{(0)}(a) \cdot b - \ell^{(-1)}(a) \cdot \delta(b)$, where $\cdot$ is the product in $A\hat{\otimes}\mathcal{C}l$, extends by continuity to a morphism*

$$\chi : L\hat{\otimes}L^* \to A\hat{\otimes}\mathcal{C}l^{(1)}[1]. \tag{3.8.20.1}$$

Proof (cf. 3.8.9). We want to find for each open $U \subset A\hat{\otimes}\mathcal{C}l$ open subspaces $P \subset L$, $Q \subset L^*$ such that $\ell^{(0)}(a) \cdot b + \ell^{(-1)}(a) \cdot \delta(b) \in U$ if either $a \in P$ or $b \in Q$.

We can assume that U is a left ideal. Take P, Q such that $\ell^{(0)}(P)$, $\ell^{(-1)}(P)$, Q, $\delta(Q) \subset U$. Then our condition is satisfied since $[\ell^{(0)}(a), b] - [\ell^{(-1)}(a), \delta(b)] = 0$. □

Recall that $L\hat{\otimes}L^* \xrightarrow{\sim} \mathrm{End}(L)$. Set

$$\mathfrak{d} = \mathfrak{d}_{A,\alpha} := \chi(id_L) \in A\hat{\otimes}\mathcal{C}l^{(1)}[1]. \tag{3.8.20.2}$$

This is the *BRST charge*. Its adjoint action $d_{A,\alpha}$ is the *BRST differential*; this is a continuous odd derivation of $A\hat{\otimes}\mathcal{C}l$ of degree 1.

3.8.21. THEOREM. (i) *One has* $[\mathfrak{d}, \mathfrak{d}] = 0$*; hence* $d^2_{A,\alpha} = 0$.

The topological associative DG algbebra $(A\hat{\otimes}\mathcal{C}l, d_{A\hat{\otimes}\mathcal{C}l} + d_{A,\alpha})$ *is denoted by* $\mathcal{C}_{\mathrm{BRST}}(L, A)$.

(ii) ℓ *from (3.8.19.2) is a morphism of topological Lie DG algebras*

$$\ell : L_\dagger \to \mathcal{C}_{\mathrm{BRST}}(L, A). \tag{3.8.21.1}$$

(iii) $\mathfrak{d}$ *is a unique odd element in* $A\hat{\otimes}\mathcal{C}l^{(1)}$ *such that* $ad_{\mathfrak{d}}\ell^{(-1)} = \ell^{(0)}$. *Similarly,* $d_{A,\alpha}$ *is a unique continuous odd derivation of degree 1 of* $A\hat{\otimes}\mathcal{C}l$ *such that* $d_{A,\alpha}\ell^{(-1)} = \ell^{(0)}$.

We refer to (ii) as the *BRST property* of $\mathfrak{d}$ or $d_{A,\alpha}$.

Proof (cf. the proof of Theorem 3.8.10). (a) One has $d_{A,\alpha}\ell^{(-1)} = \ell^{(0)}$.

(b) The centralizer F of $L[1] \subset \mathcal{C}l \subset A\hat{\otimes}\mathcal{C}l$ equals $A\hat{\otimes}\mathrm{Sym}(L[1]) \subset A\hat{\otimes}\mathcal{C}l$. In particular, F has degrees ≤ 0.

(c) Let us check (iii). Property $[\mathfrak{d}, \ell^{(-1)}] = \ell^{(0)}$ determines $\mathfrak{d}$ uniquely up to an element of F^1. So $\mathfrak{d}$ is unique. Similarly, property $d\ell^{(-1)} = \ell^{(0)}$ determines d up to a derivation ∂ which kills the image of $L[1]$. Such a ∂ sends $L^*[-1] \subset \mathcal{C}l^{(1)} \subset A\hat{\otimes}\mathcal{C}l^{(1)}$ and $A \subset A\hat{\otimes}\mathcal{C}l^{(0)}$ to F. Since ∂ has degree 1, it kills $L^*[-1]$ and A according to (a). Thus $\partial = 0$; i.e., d is determined uniquely.

(d) Statement (i) follows from (b); the argument repeats 3.8.13. Statement (ii) follows from (i) and (a). □

3.8.22. For every discrete $A\hat{\otimes}\mathcal{C}l$-module N the action of $\mathfrak{d}$ on N is an odd $d_{A,\alpha}$-derivation of degree 1 and square 0, so it defines on N a canonical structure of a DG $\mathcal{C}_{\mathrm{BRST}}(L, A)$-module. Fix a graded discrete $\mathcal{C}l$-module C; then for a discrete A-module M the tensor product $M \otimes C$ becomes a DG $\mathcal{C}_{\mathrm{BRST}}(L, A)$-module called the *BRST, or semi-infinite homology,* complex of M (for the Clifford module C).

REMARKS. (i) If L is purely even, then an irreducible $\mathcal{C}l$-module is unique up to a twist by a 1-dimensional (super) vector space. So the BRST complex with respect to an irreducible C is uniquely defined up to a twist by a 1-dimensional graded (super) vector space.

(ii) $\mathfrak{d}$, hence BRST complexes, behave in the obvious manner with respect to morphisms of A's. The "smallest" A for the BRST datum is $U(L)^{-\flat}$. So the BRST complex of M depends only on the $L^{-\flat}$-module structure on M (coming from α).

(iii) Suppose L is discrete. Then $L^\flat$ is split by means of s_d, which identifies L-modules and $L^{-\flat}$-modules. The Chevalley homology complex of an L-module coincides with the BRST complex of the corresponding $L^{-\flat}$-module for the Clifford module $\mathrm{Sym}(L[1])$. Similarly, if L is compact, then we use the splitting s_c to identify L- and $L^{-\flat}$-modules. The Chevalley cohomology complex of an L-module coincides then with the BRST complex of the corresponding $L^{-\flat}$-module for the Clifford module $\mathrm{Sym}(L^*[-1])$. If L is finite-dimensional, then the splittings differ by the character $\mathrm{tr}\, ad : L \to k$, and we get the usual identification of the Chevalley homology complex of an L-module with the Chevalley cohomology complex of the L-module twisted by $\det L$.

3.8.23. Now we can compare the chiral and the Tate linear algebra versions of BRST. Let $x \in X$ be a (closed) point, $j_x : U_x \hookrightarrow X$ its complement.

Let L be a vector $\mathcal{D}$-bundle on U_x, L° the dual vector $\mathcal{D}$-bundle. According to 2.7.10 the Tate vector space $L^\circ_{(x)} := \hat{h}_x(j_{x*}L^\circ)$ is dual to $L_{(x)} := \hat{h}_x(j_{x*}L)$; i.e., $L^\circ_{(x)} = (L_{(x)})^*$.

Let $\mathcal{C}l^\cdot$ be the graded chiral Clifford algebra generated by $L^\circ[-1] \oplus L[1]$ (see 3.8.6), and let $\mathcal{C}l^\cdot_{(x)}$ be the topological graded Clifford algebra generated by $L^\circ_{(x)}[-1]$ $\oplus L_{(x)}[1]$ (see 3.8.17). According to 3.7.22, there is a canonical isomorphism of topological graded algebras $\mathcal{C}l^{\cdot as}_x \xrightarrow{\sim} \mathcal{C}l^\cdot_{(x)}$.

Here is a simple alternative construction of the identification $r^\flat$ of (2.7.14.1).[73] Consider the Tate extension $\mathfrak{gl}(L)^\flat = \mathcal{C}l^0_2$ (see (3.8.6.1)). The previous isomorphism yields then a morphism of Lie algebras $h_x j_{x*}\mathfrak{gl}(L)^\flat \to \mathcal{C}l^0_{(x)2}$. It is obviously continuous with respect to the $\Xi^{sp\flat}_x$-topology (see 2.7.13), so we have a morphism of topological Lie algebras $\mathfrak{gl}(L)^\flat_{(x)} \to \mathcal{C}l^0_{(x)2}$. Using the identification $\mathcal{C}l^0_{(x)2} = \mathfrak{gl}(L_{(x)})^\flat$ of 3.8.18, we can rewrite this morphism as $r^\flat : \mathfrak{gl}(L)^\flat_{(x)} \to \mathfrak{gl}(L_{(x)})^\flat$. By construction, this $r^\flat$ is a lifting of the isomorphism $r : \mathfrak{gl}(L)_{(x)} \xrightarrow{\sim} \mathfrak{gl}(L_{(x)})$ of 2.7.12 to our central k-extensions.

Now we assume that our L is a Lie* algebra, so we have the Tate extension $L^\flat$ (see 3.8.7). Then $L_{(x)}$ is a topological Lie algebra (see 2.5.12 and 2.5.13(ii)) and $L^\flat_{(x)}$ is a central extension of $L_{(x)}$ by k. By definition of $L^\flat$ the adjoint action of L lifts canonically to a morphism of topological extensions $L^\flat_{(x)} \to \mathfrak{gl}(L)^\flat_{(x)}$. Thus $r^\flat$ provides a canonical identification of $L^\flat_{(x)}$ with the Tate linear algebra extension $(L_{(x)})^\flat$ from 3.8.19.

Assume we have a chiral BRST datum, i.e., a chiral algebra A on U_x and a morphism of Lie* algebras $\alpha : L^\flat \to A$ such that $\alpha(1^\flat) = -1_A$ (see 3.8.8). This yields the topological associative algebra $A^{as}_x = (j_{x*}A)^{as}_x$ (see 3.6.2–3.6.6) and the morphism $\alpha_{(x)} : L^\flat_{(x)} \to A^{as}_x$ which sends $1^\flat$ to -1 which is a Tate linear algebra BRST datum (see 3.8.19).

According to 3.6.8 the functor $A \mapsto A^{as}_x$ commutes with tensor products, so $(A \otimes \mathcal{C}l)^{as}_x = A^{as}_x \hat{\otimes} \mathcal{C}l_{(x)}$.

3.8.24. Lemma. *The canonical morphism (3.6.6.2) of graded Lie algebras*

$$h_x j_{x*} A \otimes \mathcal{C}l \to (A \otimes \mathcal{C}l)^{as}_x = A^{as}_x \hat{\otimes} \mathcal{C}l_{(x)}$$

sends the chiral BRST charge (3.8.9.3) to its Tate linear algebra cousin (3.8.20.2).

Proof. The image of chiral $\mathfrak{d}$ satisfies the BRST property, so our statement follows from 3.8.21(iii). □

3.8.25. According to 3.6.6 the categories of graded $j_{x*}A \otimes \mathcal{C}l$-modules supported at x and discrete $A^{as}_x \hat{\otimes} \mathcal{C}l_{(x)}$-modules are canonically identified. Each of

[73] Our L is V of 2.7.14.

these objects carries a canonical BRST differential (defined as the action of the BRST charges), and the above lemma identifies these differentials. This provides a canonical identification of the BRST homology from 3.8.14 (Variant) with those from 3.8.22.

3.9. Chiral differential operators

In this section we explain what enveloping chiral algebras of Lie* algebroids are. In particular, we consider chiral algebras of differential operators or cdo which correspond to the tangent algebroid $\Theta_{\mathcal{Y}}$. The key difference with the case of Lie* algebras treated in 3.7 is that the construction of the enveloping algebra requires an extra structure (that of chiral extension) on the Lie* algebroid.

In 3.9.1–3.9.3 we consider chiral $R_{\mathcal{D}if}$-algebras for a commutative $\mathcal{D}_X$-algebra R. The important fact is that chiral $R_{\mathcal{D}if}$-algebras are local objects with respect to $\operatorname{Spec} R$, so they can live on any algebraic $\mathcal{D}_X$-space. An example of such algebras are R-cdo defined in 3.9.5. The notion of a chiral Lie algebroid is discussed in 3.9.6–3.9.10; in particular, we define an affine structure on the groupoid of chiral R-extensions of a given Lie* algebroid $\mathcal{L}$ in 3.9.7, construct rigidified chiral extensions in 3.9.8, and relate chiral extensions with mod t^2 quantizations in 3.9.10. The enveloping chiral algebras of chiral Lie algebroids come in 3.9.11. The Poincaré-Birkhoff-Witt theorem (in a form suggested by D. Gaiitsgory), which is a main technical result of this section, is in 3.9.11–3.9.14. To make the exposition clear, we first present another proof of the PBW (actually, of a slightly more general statement) for the usual Lie algebroids,[74] and then explain how to modify it to the chiral setting. The PBW theorem permits us to identify chiral extensions of $\Theta_{\mathcal{Y}}$ with cdo (see 3.9.15). In the rest of the section we describe the groupoid of chiral R-extensions explicitly. In 3.9.16–3.9.17 we define a canonical DG chiral extension of a certain DG Lie* algebroid over the de Rham-Chevalley algebra of $\mathcal{L}$. In particular, for $\mathcal{L} = \Theta_R$ we get a canonical DG cdo $\mathcal{D}_{DR}$ over the de Rham $\mathcal{D}_X$-algebra $DR = DR_{R^\ell/X}$ (see 3.9.18). We use it in 3.9.20 to identify chiral extensions of $\mathcal{L}$ with variants of Tate structures from 2.8.1; for $\mathcal{L} = \Theta$ we get exactly Tate structures on $\mathcal{Y}$. Thus concrete Tate structures from 2.8.15 and 2.8.17 provide examples of cdo. The formalism of Tate structures from 2.8 provides an obstruction to the existence of the global chiral extension of $\mathcal{L}$ (see 3.9.21–3.9.22). In particular, 2.8.13 yields a version of the Gorbounov-Malikov-Schechtman formula of 3.9.23 for the obstruction to the existence of a global cdo on a jet $\mathcal{D}_X$-scheme. In 3.9.24–3.9.26 we identify modules over the enveloping algebra of a chiral algebroid with chiral modules over the algebroid, describe the corresponding topological associative algebras as enveloping algebras of topological Lie algebroids, and prove a version of the PBW theorem. In particular, we show that the topological associative algebras coming from a cdo are topological tdo (see 3.9.27).

Much of the material was independently developed by Gorbounov-Malikov-Schechtman [**GMS2**], see also [**Sch1**] and [**Sch2**]. They consider the setting of graded vertex algebras; in our languge, these are chiral algebras on $X = \mathbb{A}^1$ equivariant with respect to the action of the group Aff of affine linear transformations (see 0.15). Their cdo and chiral algebroids live on the jet scheme of a constant fibration $Z = X \times B$ over X equipped with an evident Aff-action; the "vertex $\mathcal{O}_B$-algebroid" from loc. cit. is the degree 1 component of (the translation invariant

[74]The proof from 2.9.2 seems not to admit a chiral version.

part of) a chiral Lie algebroid with appropriate induced structure which permits us to recover the whole chiral algebroid. The notion of a chiral $R_{\mathcal{D}if}$-algebra is a chiral analog of that of the D-algebra from [**BB**] 1.1. The subject of 3.9.1–3.9.3 was also treated in [**KV**]. The canonical chiral DG tdo $\mathcal{D}_{DR}$ from 3.9.18 first appeared (in the graded vertex algebra setting) in the work of Malikov-Schechtman-Vaintrob [**MSV**] under the name of chiral de Rham complex. The original construction of [**MSV**] was coordinate-style (the standard "linear" chiral DG algebras, attached to coordinate neighbourhoods of B, are patched together by means of certain explicit liftings of coordinate transformations); it was elucidated in [**KV**] and [**Bre1**]. The formula in 3.9.23 (in the graded vertex algebra setting)[75] is due to Gorbounov-Malikov-Schechtman [**GMS1**], [**GMS2**] 8.7, [**Sch1**], see also [**Bre2**].[76] The first, to our knowledge, "non-linear" example of cdo was found by Feigin-Frenkel [**FF1**], [**FF2**]. They constructed a cdo (as a topological associative algebra) on the jet scheme of an open Schubert cell by certain mysterious explicit formulas. In fact, as was shown in [**GMS3**], for $X = \mathbb{A}^1$ the jet scheme of a flag space admits a unique Aff-equivariant cdo; its restriction to the Schubert cell is the Feigin-Frenkel cdo. From the point of view of this article, this follows from (3.9.20.2) and the exercise in 2.8.17. Similarly, the cdo on the jet scheme of a group constructed in [**AG**] and [**GMS3**] arise from the Tate structures from 2.8.15. The family of cdo on the jet scheme of a flag space parametrized by Miura opers was suggested by E. Frenkel and D. Gaitsgory (see 3.9.20 and 2.8.17). The subject of 3.9.27 is related to the Appendix in [**AG**]. For an application of chiral Lie algebroids to representation theory of affine Kac-Moody Lie algebras at the critical level, see [**FrG**].

3.9.1. Let R be a commutative chiral algebra; i.e., by (3.3.3.1), R^ℓ is a commutative $\mathcal{D}_X$-algebra.

We say that a chiral R-module M is an *R-differential module* or simply an *$R_{\mathcal{D}if}$-module* if M is a union of submodules which are successive extensions of central R-modules.[77] Equivalently, this means that there is a filtration $\{0\} = M_{-1} \subset M_0 \subset M_1 \subset \cdots$, $\bigcup M_i = M$, such that the $\mathrm{gr}_i M$ are central.[78] The subcategory $\mathcal{M}(X,R)_{\mathcal{D}if} \subset \mathcal{M}(X,R)$ of $R_{\mathcal{D}if}$-modules is closed under direct sums and subquotients. Any chiral R-module N has the maximal differential submodule $N^{\mathcal{D}if}$. Every chiral operation preserves maximal differential submodules. We denote by $\mathcal{M}(X,R)^{ch}_{\mathcal{D}if} \subset \mathcal{M}(X,R)^{ch}$ the full pseudo-tensor subcategory of $R_{\mathcal{D}if}$-modules.

Consider the sheaf of commutative Lie algebras $h(R)$ on X. It acts canonically on every chiral R-module M, so M is a $\mathrm{Sym}\, h(R)$-module. If $M \in \mathcal{M}(X,R)$ is an $R_{\mathcal{D}if}$-module, then the $h(R)$-action on M is locally nilpotent which means that every $m \in M$ is killed by some $\mathrm{Sym}^i h(R)$, $i \gg 0$. The converse is true if we know that for every $i > 0$ the subsheaf of M that consists of sections killed by $\mathrm{Sym}^i h(R)$ is $\mathcal{O}_X$-quasi-coherent. This always happens if M is either $\mathcal{O}_X$-flat or supported in finitely many points.

Let $f : R_1^\ell \to R_2^\ell$ be a morphism of $\mathcal{D}_X$-algebras. If M is an $R_{2\mathcal{D}if}$-module, then it is an $R_{1\mathcal{D}if}$-module via f, so we have faithful pseudo-tensor functors

$$f_\cdot : \mathcal{M}(X,R_2)^{ch}_{\mathcal{D}if} \to \mathcal{M}(X,R_1)^{ch}_{\mathcal{D}if}. \tag{3.9.1.1}$$

[75] Notice that in the translation equivariant setting ω_X is trivialized, so it disappears from the formula.

[76] Presumably, in the Aff-equivariant setting our construction reduces to that of [**GMS2**].

[77] See 3.3.7 and 3.3.10. In this definition R need not be commutative.

[78] One can define $M_\cdot$ by induction taking for M_i the preimage of $(M/M_{i-1})^{cent}$.

3.9.2. Consider the topological algebra with connection (R^{as}, ∇) defined in 3.6.14. Since R is commutative, R^{as} is also commutative. We have $R^\ell = R^{as}/I_R$ for an open ideal $I_R \subset R^{as}$. Let $R^{\mathcal{D}if}$ be the I_R-adic completion of R^{as}. Precisely, we consider on R^{as} the topology defined by those open ideals I for which $I_R^n \subset I$ for $n \gg 0$; our $R^{\mathcal{D}if}$ is the corresponding completion. This is a commutative topological $\mathcal{O}_X$-algebra equipped with a connection ∇.

REMARK. The functor $R \mapsto R^{\mathcal{D}if}$ commutes with coproducts.[79]

We define discrete $R^{\mathcal{D}if}[\mathcal{D}_X]$-modules in the same way as we defined discrete $R^{as}[\mathcal{D}_X]$-modules in 3.6.15; i.e., discrete $R^{\mathcal{D}if}[\mathcal{D}_X]$-modules are the same as discrete $R^{as}[\mathcal{D}_X]$-modules such that the action of R^{as} comes from the discrete action of $R^{\mathcal{D}if}$. They form an abelian category $\mathcal{M}^\ell(X, R^{\mathcal{D}if})$.

The equivalence from 3.6.16 identifies $R_{\mathcal{D}if}$-modules with discrete $R^{\mathcal{D}if}[\mathcal{D}_X]$-modules. Therefore we have an equivalence

$$\mathcal{M}(X,R)_{\mathcal{D}if} \xrightarrow{\sim} \mathcal{M}^\ell(X, R^{\mathcal{D}if}). \tag{3.9.2.1}$$

3.9.3. PROPOSITION. *Let $f : R_1^\ell \to R_2^\ell$ be an étale morphism of $\mathcal{D}_X$-algebras.*

(i) *The morphism of topological algebras $R_1^{\mathcal{D}if} \to R_2^{\mathcal{D}if}$ is also étale.*

(ii) *The pseudo-tensor functor $f_\cdot$ from (3.9.1.1) admits a left adjoint pseudo-tensor functor $f^\cdot : \mathcal{M}(X,R_1)^{ch}_{\mathcal{D}if} \to \mathcal{M}(X,R_2)^{ch}_{\mathcal{D}if}$. One has $f^\cdot M = R_2^{\mathcal{D}if} \underset{R_1^{\mathcal{D}if}}{\otimes} M$.*

(iii) *$R_{\mathcal{D}if}$-modules and chiral R-operations between them have the étale local nature.*

Proof. (i) Let $I_1 \subset R_1^{\mathcal{D}if}$ be an open ideal, $I_2 \subset R_2^{\mathcal{D}if}$ the closed ideal generated by $f(I_1)$. We want to show that I_2 is open and the morphism $R_1^{\mathcal{D}if}/I_1 \to R_2^{\mathcal{D}if}/I_2$ is étale. This follows from the fact that the morphism $R_1^{\mathcal{D}if} \to R_2^{\mathcal{D}if}$ is formally étale, which is an immediate corollary of 3.6.19.

(ii) It follows from (i) that $f^\cdot$ defined by the above formula sends discrete $R_1^{\mathcal{D}if}[\mathcal{D}_X]$-modules to discrete $R_2^{\mathcal{D}if}[\mathcal{D}_X]$-modules. The adjunction property on the level of the usual morphisms is evident. To check it for chiral R-operations one uses the lemma from 3.4.19 (the case of $J = I$); the details are left to the reader.

(iii) Follows from (i), (ii) and the remark in 3.9.2. □

According to 3.9.3(iii) one has the notion of an $\mathcal{O}_{\mathcal{D}if}$-module on any algebraic $\mathcal{D}_X$-space $\mathcal{Y}$; we also call these objects *$\mathcal{O}$-modules on $\mathcal{Y}_{\mathcal{D}if}$*. We denote the corresponding abelian pseudo-tensor category by $\mathcal{M}(\mathcal{Y})^{ch}_{\mathcal{D}if}$.

3.9.4. For a commutative R a *chiral $R_{\mathcal{D}if}$-algebra* is a chiral algebra A together with a morphism of chiral algebras $R \to A$ such that A is an $R_{\mathcal{D}if}$-module. According to 3.9.3(iii) these objects have the étale local nature, so we have the notion of a chiral $\mathcal{O}_{\mathcal{D}if}$-algebra on any algebraic $\mathcal{D}_X$-space $\mathcal{Y}$; we also call them *chiral algebras on $\mathcal{Y}_{\mathcal{D}if}$*. The corresponding category is denoted by $\mathcal{CA}(\mathcal{Y})_{\mathcal{D}if}$.

EXAMPLE. Let A be a chiral algebra equipped with a commutative filtration $R = A_0 \subset A_1 \subset \cdots$ (see 3.3.12). Then A is a chiral $R_{\mathcal{D}if}$-algebra. Notice that in this situation $\mathrm{gr}_1 A$ is a Lie* R-algebroid (see 1.4.11) in the obvious way.

Let A be a chiral $R_{\mathcal{D}if}$-algebra. We say that a chiral A-module M is *R-differential* if M considered as an R-module is. This notion is étale local, so for

[79] This follows immediately from a similar property of the functor $R \mapsto R^{as}$; see 3.6.8, 3.6.13.

any $\mathcal{Y}$ as above and $\mathcal{A} \in \mathcal{CA}(\mathcal{Y})_{\mathcal{D}if}$ we know what the $\mathcal{O}$-*differential* $\mathcal{A}$-*modules on* $\mathcal{Y}$, or simply the A-*modules on* $\mathcal{Y}_{\mathcal{D}if}$, are. These objects form an abelian pseudo-tensor category $\mathcal{M}(\mathcal{Y}, \mathcal{A})^{ch}_{\mathcal{D}if}$; we also have the corresponding sheaf of pseudo-tensor categories $\mathcal{M}(\mathcal{Y}_{\text{ét}}, \mathcal{A})^{ch}_{\mathcal{D}if}$ on $\mathcal{Y}_{\text{ét}}$.

3.9.5. Cdo. Assume for the moment that Ω_R is a finitely generated locally projective $R^\ell[\mathcal{D}_X]$-module; e.g., R is smooth (see 2.3.15). A *chiral algebra of twisted differential operators* on $\operatorname{Spec} R$ (we abbreviate it to *cdo* on $\operatorname{Spec} R$ or R-*cdo*) is a chiral $R_{\mathcal{D}if}$-algebra A which admits a commutative filtration $A_0 \subset A_1 \subset \cdots$ (see 3.3.12) such that $R = A_0$ and there is an isomorphism of coisson $\mathbb{Z}$-graded algebras $\operatorname{Sym}^{\cdot}\Theta_R \xrightarrow{\sim} \operatorname{gr}_{\cdot} A$ identical on R.[80]

Notice that the filtration $A_{\cdot}$ is uniquely defined: indeed, $A_0 = R$, A_1 is the normalizer of R in $\mathcal{D}$, and our filtration is 1-generated (see 3.7.13). We call it the *canonical filtration*. The above isomorphism of the coisson graded algebras is also uniquely defined.

It is clear that cdo are local objects for the étale topology. Thus we know what cdo on any smooth algebraic $\mathcal{D}_X$-space $\mathcal{Y}$ are. We denote the corresponding category by $\mathcal{CDO}(\mathcal{Y}) \subset \mathcal{CA}(\mathcal{Y})_{\mathcal{D}if}$. This is a groupoid. We have the corresponding sheaf of groupoids $\mathcal{CDO}(\mathcal{Y}_{\text{ét}})$ on $\mathcal{Y}_{\text{ét}}$.

The principal difference between cdo and the usual tdo (see [**BB**] or [**Kas1**]) is that there is no *canonical* ("non-twisted") cdo. We also do *not* know if cdo always exist locally in the étale topology. This is true (as follows from (3.9.20.2)) if $\Theta_{\mathcal{Y}}$ is a *locally trivial* vector $\mathcal{D}$-bundle (which happens when $\mathcal{Y}$ is a jet $\mathcal{D}_X$-scheme). If cdo exist locally, then they form a gerbe. We will see that this gerbe need not be trivial.

Similarly to the usual tdo, one handles cdo realizing them as twisted enveloping algebras of certain Lie* algebroids. We explain the chiral Lie algebroid basics in 3.9.6–3.9.14 returning to tdo in 3.9.15.

3.9.6. Chiral Lie algebroids. An important class of chiral $R_{\mathcal{D}if}$-algebras (which includes cdo) is formed by chiral enveloping algebras of Lie* R-algebroids. Contrary to the case of mere Lie* algebras, to define such an enveloping algebra, one needs an extra structure of the chiral extension of our algebroid. We first describe this structure and play with it a bit; the chiral envelopes enter in 3.9.11.

Let R be a commutative chiral algebra, B a chiral R-algebra, and $\mathcal{L}$ a Lie* R-algebroid. Suppose we have a $\mathcal{D}$-module extension

$$0 \to B \to \mathcal{L}^\flat \to \mathcal{L} \to 0 \tag{3.9.6.1}$$

together with a Lie* bracket and a chiral R-module structure $\mu_{R\mathcal{L}^\flat}$ on $\mathcal{L}^\flat$ such that the arrows in (3.9.12.1) are compatible with the Lie* algebra and chiral R-module structures. Then $\mathcal{L}^\flat$ acts as a Lie* algebra on itself (by adjoint action), on B (since $B \subset \mathcal{L}^\flat$), and on R (via the projection $\mathcal{L}^\flat \to \mathcal{L}$ and the $\mathcal{L}$-action on R). Assume that the following properties are satisfied:

(i) The chiral operations $\mu_B \in P_2^{ch}(\{B, B\}, B)$ and $\mu_{R\mathcal{L}^\flat} \in P_2^{ch}(\{R, \mathcal{L}^\flat\}, \mathcal{L}^\flat)$ are compatible with the Lie* actions of $\mathcal{L}^\flat$.

(ii) The structure morphism $R \to B$ is compatible with the Lie* $\mathcal{L}^\flat$-actions.

[80] Exercise: show that the existence of such an isomorphism amounts to the folowing conditions: the adjoint $*$ action of $\operatorname{gr}_1 \mathcal{D}$ on $\mathcal{D}_0 = R$ identifies it with Θ_R and $\operatorname{gr}_1 \mathcal{D}$ generates $\operatorname{gr}_{\cdot} \mathcal{D}$.

(iii) The $*$ operation that corresponds to $\mu_{R\mathcal{L}^\flat}$ is equal to $-\iota\tau^\sigma_{\mathcal{L}^\flat R}$ where $\tau_{\mathcal{L}^\flat R}$ is the $\mathcal{L}^\flat$-action on R, σ is the transposition of variables, and ι is the composition of the structure morphism $R \to B$ and the embedding $B \subset \mathcal{L}^\flat$.

We call such an $\mathcal{L}^\flat$ a *chiral B-extension* of $\mathcal{L}$. The particular case of chiral R-extensions (when $B = R$) is most interesting. We also call a pair $(\mathcal{L}, \mathcal{L}^\flat)$, where $\mathcal{L}^\flat$ is a chiral R-extension of $\mathcal{L}$, a *chiral Lie R-algebroid*. The image of 1_R in $\mathcal{L}^{\flat\ell}$ is denoted by $1^\flat$.

EXAMPLE. In the situation of the example in 3.9.4, A_1 is a chiral Lie R-algebroid, a chiral R-extension of $\mathrm{gr}_1 A$. More generally, suppose we have an embedding of chiral algebras $B \subset A$ and a $\mathcal{D}$-submodule $\mathcal{L}^\flat$ of A containing B. Assume that the structure morphism $R \to B$ is injective, $\mathcal{L}^\flat$ is a Lie*-subalgebra and a chiral R-submodule of A, and $\mathcal{L}^\flat$ normalizes $R \subset A$ (in the $*$ sense). Then $\mathcal{L} := \mathcal{L}^\flat/B$ is naturally a Lie* R-algebroid and $\mathcal{L}^\flat$ is its chiral B-extension.

We will see in 3.9.11 that every $\mathcal{L}^\flat$ that satisfies a certain flatness condition comes in this way.

The quadruples $(R, B, \mathcal{L}, \mathcal{L}^\flat)$ form a category in the obvious manner. For fixed R, B and $\mathcal{L}$ the chiral B-extensions of $\mathcal{L}$ form a groupoid $\mathcal{P}^{ch}(\mathcal{L}, B)$; if $B = R$, we write simply $\mathcal{P}^{ch}(\mathcal{L})$. We denote the category of chiral Lie R-algebroids by $\mathcal{L}ie\mathcal{A}lg^{ch}(R) = \mathcal{L}ie\mathcal{A}lg^{ch}(\mathrm{Spec}\, R^\ell)$. By 3.9.3(iii) chiral B-extensions have the étale local nature (just as Lie* algebroids), so we can replace $\mathrm{Spec}\, R^\ell$ by any algebraic $\mathcal{D}_X$-space $\mathcal{Y}$. The corresponding categories are denoted by $\mathcal{P}^{ch}(\mathcal{Y}, \mathcal{L}, B)$, etc., and the sheaves of categories on $\mathcal{Y}_{\text{ét}}$ by $\mathcal{P}^{ch}(\mathcal{Y}_{\text{ét}}, \mathcal{L}, B)$, etc.

3.9.7 The affine structure on $\mathcal{P}^{ch}(\mathcal{L})$. Let $\mathcal{L}$ be a Lie* R-algebroid. The important point is that $\mathcal{P}^{ch}(\mathcal{L})$ is *not* a Picard groupoid: the notion of a *trivial* chiral extension of $\mathcal{L}$ makes no sense. We will see in a moment that this groupoid carries an *affine* structure instead.

As in 2.8.2, we define a *classical R-extension* $\mathcal{L}^c$ of $\mathcal{L}$ as an extension of $\mathcal{L}$ by R in the category of Lie* R-algebroids. In other words, $\mathcal{L}^c$ is a Lie* R-algebroid equipped with a horizontal section $1^c \in \mathcal{L}^{c\ell}$ and an identification of Lie* algebroids[81] $\mathcal{L}^c/R1^c \xrightarrow{\sim} \mathcal{L}$; the morphism $\iota = \iota_{\mathcal{L}^c} : R \to \mathcal{L}^c$, $r \mapsto r1^c$ is assumed to be injective. These objects have the étale local nature, so, as above, we have the groupoid of classical extensions $\mathcal{P}^{cl}(\mathcal{L}) = \mathcal{P}^{cl}(\mathcal{Y}, \mathcal{L})$.

The Baer sum of classical extensions makes $\mathcal{P}^{cl}(\mathcal{L})$ a Picard groupoid. We can also form k-linear combinations of classical extensions, so $\mathcal{P}^{cl}(\mathcal{L})$ is actually a k-vector space category.

If $\mathcal{L}^\flat$ is a chiral R-extension and $\mathcal{L}^c$ is a classical R-extension of $\mathcal{L}$, then we have the Baer sum of extensions $\mathcal{L}^c + \mathcal{L}^\flat$. This is a Lie* algebra, a chiral R-module, and an R-extension of $\mathcal{L}$ in the obvious way. The compatibilities in the definition in 3.9.6 are obviously satisfied, so $\mathcal{L}^c + \mathcal{L}^\flat$ is a chiral R-extension of $\mathcal{L}$. Therefore we have defined an action of the Picard groupoid $\mathcal{P}^{cl}(\mathcal{L})$ on $\mathcal{P}^{ch}(\mathcal{L})$.

LEMMA. *If $\mathcal{P}^{ch}(\mathcal{L})$ is non-empty, then it is naturally a $\mathcal{P}^{cl}(\mathcal{L})$-torsor.*

Proof. The Baer difference of two chiral extensions is a classical extension. □

[81] Notice that $R \cdot 1^c \subset \mathcal{L}^c$ is automatically a Lie* ideal which acts trivially on R, so $\mathcal{L}^c/R1^c$ is a Lie* R-algebroid.

We will describe $\mathcal{P}^{ch}(\mathcal{L})$ in down-to-earth terms later in 3.9.20–3.9.21. Presently we see that the whole of $\mathcal{P}^{ch}(\mathcal{L})$ is at hand (modulo the understanding of $\mathcal{P}^{cl}(\mathcal{L})$) the moment we find *some* chiral R-extension. Here is a way to construct one:

3.9.8. Rigidified chiral extensions. We consider the general setting of 3.9.6, so we have a chiral R-algebra B. Let L be a Lie* algebra acting on R and B in a compatible way; denote by τ_{LR}, τ_{LB} the L-actions. Let $\mathcal{L}$ be a (L, τ_{LR})-rigidified Lie* R-algebroid (see 1.4.13), so we have a morphism of Lie* algebras $\psi : L \to \mathcal{L}$ which yields an isomorphism of R-modules $R^\ell \otimes L \xrightarrow{\sim} \mathcal{L}$.

PROPOSITION. *Suppose that L is $\mathcal{O}_X$-flat. Then there is a chiral B-extension $\mathcal{L}^\flat$ equipped with a lifting $\psi^\flat : L \to \mathcal{L}^\flat$ of ψ such that $\psi^\flat$ is a morphism of Lie* algebras and the adjoint action of L on B via $\psi^\flat$ equals τ_{LB}. Such $(\mathcal{L}^\flat, \psi^\flat)$ is unique.*

We call $(\mathcal{L}^\flat, \psi^\flat)$ the (L, τ_{LB})-*rigidified* B-extension of $\mathcal{L}$.

Proof. Existence: Consider the chiral algebra $B \circledast U(L)$ (see 3.7.23). It carries the PBW filtration and $\mathrm{gr}_\cdot B \circledast L = B \otimes \mathrm{Sym}^\cdot L^\ell$. Define $\mathcal{L}^\flat$ as the pull-back of the extension $0 \to B \to (B \circledast U(L))_1 \to \mathrm{gr}_1 B \circledast L \to 0$ by the morphism $\mathcal{L} = R^\ell \otimes L \to B^\ell \otimes L = \mathrm{gr}_1 B \circledast L$. It is an (L, τ_{LB})-rigidified B-extension of $\mathcal{L}$ in the evident way.

Uniqueness: Suppose we have some $(\mathcal{L}^\flat, \psi^\flat)$. Consider the chiral pairing $\eta := \mu_{R\mathcal{L}^\flat}(id_R, \psi^\flat) \in P_2^{ch}(\{R, L\}, \mathcal{L}^\flat)$. The corresponding $*$ pairing equals $-\iota\tau_{LR}^\sigma$ by property (iii) in 3.9.6, and the composition of η with the projection $\mathcal{L}^\flat \to \mathcal{L}$ is the chiral pairing that corresponds to the rigidification $R \otimes L \xrightarrow{\sim} \mathcal{L}$. Therefore η identifies $\Delta_*\mathcal{L}^\flat$ with the push-out of the extension $0 \to R \boxtimes L \to j_*j^*R \boxtimes L \to \Delta_* R^\ell \otimes L \to 0$ by $R \boxtimes L \xrightarrow{\tau_{LR}} \Delta_* R \to \Delta_* B$.

Therefore every two (L, τ_{LB})-rigidified B-extensions of $\mathcal{L}$ are canonically identified. This identification is compatible then with the chiral R-actions (use the trick from the remark in 3.3.6, or, equivalently, consider the chiral R-module structure as the R^{as}-module structure; see 3.6.16) and with the Lie* brackets (use property (i) in 3.9.6). We are done. □

3.9.9. For $B = R$ the above proposition can be rephrased as follows. For (L, τ) as above its R-*extension* is a $\mathcal{D}$-module extension $L^\flat$ of L by R together with a Lie* algebra structure on $L^\flat$ such that the projection $\pi : L^\flat \to L$ is a morphism of Lie* algebras and the adjoint action of $L^\flat$ on $R \subset L^\flat$ coincides with $\tau\pi$. Such extensions form a Picard groupoid (in fact, a k-vector space category) $\mathcal{P}(L, \tau)$. For an (L, τ)-rigidified Lie* R-algebroid $(\mathcal{L}, \psi)$ the pull-back by ψ of a classical or a chiral R-extension of $\mathcal{L}$ is an R-extension of L. So we have defined functors

$$\mathcal{P}^{cl}(\mathcal{L}) \to \mathcal{P}(L, \tau), \quad \mathcal{P}^{ch}(\mathcal{L}) \to \mathcal{P}(L, \tau). \tag{3.9.9.1}$$

The first of them is a morphism of Picard groupoids; the second one transforms the $\mathcal{P}^{cl}(\mathcal{L})$-action on $\mathcal{P}^{ch}(\mathcal{L})$ to the translation action of $\mathcal{P}(L, \tau)$ on itself.

COROLLARY. *If L is $\mathcal{O}_X$-flat, then these are equivalences of groupoids.*

Proof. Since $\mathcal{P}^{ch}(\mathcal{L})$ is non-empty, it suffices to treat the first functor. Its inverse assigns to an R-extension $L^\flat$ of L the push-forward of the $R \otimes R$-extension $(L^\flat)_R$ of the Lie* R-algebroid L_R by the product map $R \otimes R \to R$. □

3.9.10. Quantization mod t^2. Here is a different application of the notion of a chiral extension.

Suppose we are given a coisson structure on R. It yields a structure of Lie* algebroid on the $R[\mathcal{D}_X]$-module $\Omega = \Omega^1_{R/X}$ (see 1.4.17), so we have the corresponding groupoid $\mathcal{P}^{ch}(\Omega)$ of chiral R-extensions.

Consider the groupoid $\mathcal{Q}^{ch} = \mathcal{Q}^{ch}(R, \{\ \})$ of mod t^2 quantizations of the coisson structure (see 3.3.11); let $\mathcal{Q}^{cl} = \mathcal{Q}^{cl}(R, \{\ \})$ be the groupoid of $k[t]/t^2$-deformations of our coisson algebra. Just as in the conventional Poisson setting, the Baer sum operation makes $\mathcal{Q}^{cl}$ a Picard groupoid (actually, a k-vector space category) and $\mathcal{Q}^{ch}$ a $\mathcal{Q}^{cl}$-torsor.

There are canonical morphisms of Picard groupoids and their torsors[82]

$$\mathcal{P}^{cl}(\Omega) \to \mathcal{Q}^{cl}, \quad \mathcal{P}^{ch}(\Omega) \to \mathcal{Q}^{ch} \tag{3.9.10.1}$$

defined as follows. For a classical extension Ω^c the corresponding coisson deformation R^c, as a plain $\mathcal{D}$-module, is the pull-back of Ω^c by $d : R \to \Omega$. Let $p : R^c \to R$, $d^c : R^c \to \Omega^c$ be the projections. The coisson bracket is defined by the property that p and d^c are morphisms of Lie* algebras. The product on $R^{c\ell}$ is determined by the requirement that p is a morphism of algebras and d^c is a derivation (Ω^c is an R^ℓ-, hence $R^{c\ell}$-, module). The morphism of algebras $k[t]/t^2 \to R^{c\ell}$ is determined by the condition that $d^c(t) = 2 \cdot 1^c$ and $p(t) = 0$.

The chiral definitions are similar. The quantization $R^\flat$ corresponding to $\Omega^\flat$, as a plain $\mathcal{D}$-module, equals the pull-back of $\Omega^\flat$ by $d : R \to \Omega$. Let $p : R^\flat \to R$, $d^\flat : R^\flat \to \Omega^\flat$ be the projections. We define $\{\ \}^{(1)}$ demanding that p and $d^\flat$ be morphisms of Lie* algebras. The chiral product $\mu_{R^\flat} \in P_2^{ch}(R^\flat)$ is defined by the properties $p\mu_{R^\flat} := \mu_R(p,p)$, $d^\flat\mu_{R^\flat} := \mu_{\Omega^\flat}(p, d^\flat) - \mu^\sigma_{\Omega^\flat}(d^\flat, p)$ where σ is the transposition of arguments. The $k[t]/t^2$-algebra structure $k[t]/t^2 \hookrightarrow R^{\flat\ell}$ is $p(1_{R^\flat}) = 1_R$, $d^\flat(1_{R^\flat}) = 0$ and $p(t) = 0$, $d^\flat(t) = 2 \cdot 1^\flat$.

The compatibilities are straightforward; we leave them to the reader.

PROPOSITION. *Functors (3.9.10.1) are fully faithful. If R^ℓ is formally smooth, then the "classical" functor is an equivalence, and the "chiral" functor is an equivalence if $\mathcal{P}^{ch}(\Omega)$ is non-empty étale locally on* $\operatorname{Spec} R^\ell$.[83]

Proof. The "classical" statement: we recover Ω^c from R^c as $\Omega_{R^c}/t\Omega_{R^c}$, where $\Omega_{R^c} = \Omega^1_{R^c/X}$ is the Lie* algebroid defined by the coisson structure on R^c, and $1^c \in \Omega^{c\ell}$ is the image of $d(t)/2$. For an arbitrary R^c this could be an extension of Ω by some quotient of R. It is always R itself if[84] $\Omega^{(-1)}_{R/X} = 0$, as happens to be true for a formally smooth R.

The "chiral" statement follows from the "classical" one. □

3.9.11. Suppose we have $(R, B, \mathcal{L}, \mathcal{L}^\flat)$ as in 3.9.6, so B is a chiral R-algebra and $\mathcal{L}^\flat$ is a chiral B-extension of a Lie* R-algebroid $\mathcal{L}$. For a test chiral algebra A a *chiral morphism* $\varphi^\flat : \mathcal{L}^\flat \to A$ is a morphism of $\mathcal{D}_X$-modules $\mathcal{L}^\flat \to A$ compatible with all the structures. Precisely, this means that $\varphi^\flat$ is a morphism of Lie* algebras, $\phi := \varphi^\flat|_B : B \to A$ is a morphism of chiral algebras, and $\varphi^\flat$ is a morphism of chiral R-modules (where R acts on A via the morphism $R \to B \xrightarrow{\phi} A$). It is easy to see

[82] I.e., "torsors" unless $\mathcal{P}^{ch}(\Omega)$ is empty.
[83] We hope that the latter condition is redundant.
[84] See 2.3.13 for the notation.

that there is a universal $\varphi^\flat$; we denote the corresponding A by $U(\mathcal{L})^\flat = U(B,\mathcal{L})^\flat$. This is the *chiral envelope* of $\mathcal{L}^\flat$.

It is clear that $U(\mathcal{L})^\flat$ is a chiral $R_{\mathcal{D}if}$-algebra and the enveloping algebra functor is compatible with the étale localization of R. Therefore the above constructions make sense over any algebraic $\mathcal{D}_X$-space $\mathcal{Y}$. Notation: $U(\mathcal{L})^\flat_{\mathcal{Y}}$.

We define the *Poincaré-Birkhoff-Witt filtration* $U(\mathcal{L})^\flat_\cdot$ on $U(\mathcal{L})^\flat$ as the filtration generated by $U(\mathcal{L})^\flat_0 := \phi(B)$ and $U(\mathcal{L})^\flat_1$ which is the sum of $U(\mathcal{L})^\flat_0$ and the image of $\mu_{U(\mathcal{L})^\flat}(\varphi^\flat,\phi) \in P_2^{ch}(\{\mathcal{L}^\flat, B\}, U(\mathcal{L})^\flat)$ (see 3.7.13). Consider the chiral algebra $\mathrm{gr}.U(\mathcal{L})^\flat$. This is a chiral R-algebra.[85] Our $\varphi^\flat$ yields a morphism $\varphi : \mathcal{L} \to \mathrm{gr}_1 U(\mathcal{L})^\flat$ whose image is central in $\mathrm{gr}\, U(\mathcal{L})^\flat$. By 3.4.15, φ and ϕ yield the surjective *PBW morphism* of graded chiral algebras

$$B \underset{R^\ell}{\otimes} \mathrm{Sym}^\cdot_{R^\ell}\mathcal{L}^\ell \twoheadrightarrow \mathrm{gr}.U(\mathcal{L})^\flat. \tag{3.9.11.1}$$

THEOREM. *If R and B are $\mathcal{O}_X$-flat and $\mathcal{L}$ is a flat R^ℓ-module, then the PBW morphism is an isomorphism.*

Notice that if $A = R$, then the PBW filtration is commutative and the PBW morphism $\mathrm{Sym}^\cdot_{R^\ell}\mathcal{L}^\ell \twoheadrightarrow \mathrm{gr}.U(\mathcal{L})^\flat$ is a morphism of coisson algebras.

Proof. We deduce the statement from the ordinary PBW theorem for Lie* algebras (see 3.7.14). The reduction is of a quite general nature. A similar argument proves the PBW theorem (or its more general version parallel the above theorem) for the usual Lie algebroids. To make the exposition more transparent, we first discuss the setting of the usual Lie algebroids (see 3.9.12 and 3.9.13) and then we explain the slight modifications needed to cover the chiral situation (see 3.9.14).

3.9.12. We have already discussed the PBW theorem for the usual Lie algebroids in 2.9.2. The argument from loc. cit. does not generalize to the chiral setting though. Below we give a different proof.

Here is a version of the PBW theorem parallel to the statement from 3.9.11.

Let R be a commutative k-algebra, B an associative R-algebra, and $\mathcal{L}$ a Lie R-algebroid. Suppose we have an extension $0 \to B \to \mathcal{L}^\flat \to \mathcal{L} \to 0$ together with a Lie algebra and an R-module structure on $\mathcal{L}^\flat$ such that the arrows are compatible with the Lie algebra and R-module structures. Then $\mathcal{L}^\flat$ acts on itself by adjoint action, on B (since $B \subset \mathcal{L}^\flat$), and on R (via the projection $\mathcal{L}^\flat \to \mathcal{L}$ and the $\mathcal{L}$-action on R). Assume that the following properties are satisfied:

(i) $\mathcal{L}^\flat$ acts on B by derivations; its action on $\mathcal{L}^\flat$ is compatible with the R-action (via the $\mathcal{L}^\flat$-action on R), so $\mathcal{L}^\flat$ is a Lie R-algebroid.

(ii) The structure morphism $R \to B$ is compatible with the $\mathcal{L}^\flat$-actions.

We call such an $\mathcal{L}^\flat$ a *B-extension* of $\mathcal{L}$.

For such a B-extension $\mathcal{L}^\flat$ we can consider morphisms $\mathcal{L}^\flat \to A$, where A is a test associative algebra, compatible in the obvious manner with the above structures. There is a universal morphism $\varphi^\flat : \mathcal{L}^\flat \to U(\mathcal{L})^\flat$. Then $U(\mathcal{L})^\flat$ carries the PBW filtration $U(\mathcal{L})^\flat_\cdot$ where $U(\mathcal{L})^\flat_0$ is the image of B, $U(\mathcal{L})^\flat_1 = \varphi^\flat(\mathcal{L}^\flat)\cdot U(\mathcal{L})^\flat_0$, and $U(\mathcal{L})^\flat_n = (U(\mathcal{L})^\flat_1)^n$ for $n \geq 1$. One gets a canonical surjective morphism of graded algebras $B \underset{R}{\otimes} \mathrm{Sym}^\cdot_R\mathcal{L} \twoheadrightarrow \mathrm{gr}.U(\mathcal{L})^\flat$.

[85] Therefore $U(\mathcal{L})^\flat$ is an $R_{\mathcal{D}if}$-algebra.

THEOREM. *If $\mathcal{L}$ is a flat R-module, then this is an isomorphism.*

The proof consists of three steps (a)–(c). First we consider the case when $\mathcal{L}^\flat$ admits a rigidification. Then our statement follows from the usual PBW theorem. Next, in (b), we construct for any $\mathcal{L}$ an associative filtered DG algebra $\mathcal{V}$ together with an identification $H^0\mathcal{V} = U(\mathcal{L})^\flat$ such that $\mathrm{gr}\,\mathcal{V}$ is controllable (here step (a) is used). Finally, we show that if $\mathcal{L}$ is R-flat, then $\mathcal{V}$ is quasi-isomorphic to $U(\mathcal{L})^\flat$ as a filtered algebra, which yields the theorem.

3.9.13. (a) Suppose we have a Lie algebra L acting on R and B by derivations. Consider the rigidified Lie algebroid $\mathcal{L} := R \otimes \mathcal{L}$. Then the trivial R-module extension $\mathcal{L}^\flat := B \oplus R \otimes L$ carries a unique A-extension structure in the sense of 3.9.12 such that the morphism $L \to \mathcal{L}^\flat$, $l \mapsto 1_R \otimes l$ is a morphism of Lie algebras and the adjoint action of $L \subset \mathcal{L}^\flat$ on $A \subset \mathcal{L}^\flat$ coincides with the structure action. We call $\mathcal{L}^\flat$ the *rigidified B-extensions* of $\mathcal{L}$.

Denote by $U(B, L)$ the universal associative k-algebra equipped with an associative algebra morphism $\phi : B \to U(B, L)$ and a Lie algebra morphism $\varphi_L : L \to U(B, L)$ which are compatible with respect to the L-action on B. Consider the morphism $\varphi^\flat : \mathcal{L}^\flat \to U(B, L)$, $rl + b \mapsto \phi(r)\varphi_L(l) + \phi(b)$. It satisfies all the properties from 3.9.12 and is universal, so $U(B, L) = U(\mathcal{L})^\flat$.

The algebra $U(B, L)$ can be described explicitly as follows. The L action on B amounts to an action of the Hopf algebra $U(L)$ on the algebra B. Denote by $B \circledast U(L)$ the corresponding twisted tensor product of B and $U(L)$: this is an associative algebra which equals $B \otimes U(L)$ as a k-module with the product $(a \otimes u) \cdot (b \otimes v) = \Sigma a u_i(b) \otimes u'_i v$ where $\Sigma u_i \otimes u'_i$ is the coproduct of u. One checks in a moment that the obvious morphisms $B \to B \circledast U(L)$ and $L \to B \circledast U(L)$ satisfy the universality property, so we have a canonical isomorphism $U(\mathcal{L})^\flat = U(B, L) \xrightarrow{\sim} B \circledast U(L)$. It identifies the PBW filtration on $U(\mathcal{L})^\flat$ with the filtration $B \circledast U(L)_i := B \otimes U(L)_i$. Therefore the PBW theorem for $\mathcal{L}^\flat$ follows from the usual PBW for L.

(b) Let us return to the general situation of 3.9.12, so $\mathcal{L}$ is an arbitrary Lie R-algebroid and $\mathcal{L}^\flat$ is its arbitrary B-extension. Let us define the DG associative filtered algebra $\mathcal{V}$ that was promised in loc. cit.

Denote by L our $\mathcal{L}^\flat$ considered as a mere Lie k-algebra acting on R and B. Let K be the kernel of the morphism of R-modules $R \otimes L \to \mathcal{L}$, $r \otimes l^\flat \mapsto rl$. Our L acts on $R \otimes L$ (by the tensor product of the L-action on R and the adjoint action), so it acts on K.

Set $\tilde{R} := \mathrm{Sym}_R(K[1])$ and $\tilde{B} := B \underset{R}{\otimes} \tilde{R}$. So $\tilde{B}$ is an associative $\tilde{R}$-algebra, and L acts on these algebras. Let $U(\tilde{B}, L)$ be the enveloping algebra (see (a)). This is a $\mathbb{Z}$-graded associative super algebra whose 0 component equals $U(B, L)$.

The algebra $U(\tilde{B}, L)$ carries the PBW filtration; by (a), the corresponding associated graded algebra equals $\tilde{B} \otimes \mathrm{Sym}_k L$. We want to consider a different filtration, namely, the filtration which on a graded component of degree a equals the PBW filtration shifted by a; this is a ring filtration compatible with the grading. We denote $U(\tilde{B}, L)$ equipped with this filtration by $\mathcal{V}$. Shifting the PBW isomorphism appropriately, we get the identification of graded modules $\mathrm{gr}_{\cdot}\mathcal{V} = \tilde{B}^{\cdot} \otimes \mathrm{Sym}^{\cdot} L =$

$B \underset{R}{\otimes} \mathrm{Sym}^{\cdot}_R(K[1]) \underset{R}{\otimes} \mathrm{Sym}_R(R\otimes L) = B \underset{R}{\otimes} \mathrm{Sym}^{\cdot}_R(K[1]\oplus R\otimes L)$. Therefore

$$\mathrm{gr}_{\cdot}\mathcal{V} \xrightarrow{\sim} B \underset{R}{\otimes} \mathrm{Sym}^{\cdot}_R(K[1]\oplus R\otimes L). \tag{3.9.13.1}$$

Let us define a natural derivation d on $\mathcal{V}$. We have the maps $\alpha, \beta : K \to U(B, L)$ where α is the composition $K \hookrightarrow R\otimes L \to U(B, L)$, the second arrow is the product map, and β comes from the projection $R\otimes L \to \mathcal{L}^\flat$, $r\otimes l^\flat \mapsto rl^\flat$, which sends K to $B \subset \mathcal{L}^\flat$, and the morphism $B \to U(B, L)$.

LEMMA. *There is a unique odd derivation d of degree 1 on $\mathcal{V}$ which kills the images of B and L and equals $\alpha - \beta$ on $\tilde{R}^{-1} = K[1]$.*

Proof of Lemma. The uniqueness of d is clear since our conditions define d on the generators of $\mathcal{V}$. It remains to show that d is correctly defined.

Set $\Lambda := \mathrm{Sym}(k[1]) = k \oplus k[1]$; this is a $\mathbb{Z}$-graded commutative algebra. Consider the $\mathbb{Z}$-graded associative Λ-algebra $\mathcal{V}_\Lambda = \mathcal{V}\otimes\Lambda$. We have the morphism of graded associative algebras $\tilde{B} \to \mathcal{V}_\Lambda$ and that of Lie algebras $L \to \mathcal{V}_\Lambda$ which are compatible via the L-action on $\tilde{B}$. They satisfy the obvious universality property.

An odd derivation of degree 1 of $\mathcal{V}$ is the same as an automorphism of the $\mathbb{Z}$-graded Λ-algebra $\mathcal{V}_\Lambda$ which induces the identity on its quotient $\mathcal{V}$. We construct d using the universality property of $\mathcal{V}_\Lambda$.

The conditions on d mean that the corresponding automorphism $a(d)$ of $\mathcal{V}_\Lambda$ fixes the images of B and L, and on $\tilde{R}^{-1} = K[1]$ it is the morphism $\nu : K[1] \to \mathcal{V}_\Lambda = \mathcal{V}\oplus\mathcal{V}[1]$, $k \mapsto k + \alpha(k) - \beta(k)$. Now ν takes values in the centralizer $Z_\Lambda \subset \mathcal{V}_\Lambda$ of B; it is a morphism of R-modules, and it is compatible with the L-actions. Thus we get $a(d)$ from the universality property of $\mathcal{V}_\Lambda$ the moment we extend ν to a morphism of R-algebras $\tilde{R} = \mathrm{Sym}_R(K[1]) \to Z_\Lambda$. Such an extension is unique, and its existence amounts to the fact that on the image of ν the product of $\mathcal{V}_\Lambda$ is commutative. The latter property is equivalent to the commutativity of the image of the map $K[1] \to \mathcal{V}_\Lambda$, $k \mapsto k + \alpha(k)$, which is evident. □

It is clear that $d^2 = 0$ (check on the generators) and d preserves the filtration, so $\mathcal{V}$ is a filtered DG algebra which sits in degrees ≤ 0. The differential acts on $\mathrm{gr}_1 U = K[1]\oplus\mathcal{L}_R$ as the canonical embedding $K \to \mathcal{L}_R$, so we can rewrite (3.9.13.1) as a canonical isomorphism of graded DG algebras

$$\mathrm{gr}_{\cdot}\mathcal{V} = B \underset{R}{\otimes} \mathrm{Sym}^{\cdot}_R \mathcal{C}one(K \to R\otimes L). \tag{3.9.13.2}$$

The projection $\mathcal{V}^0 = U(B, L) \to U(B, \mathcal{L})^\flat$ has kernel equal to the image of d, so we have an isomorphism of filtered algebras

$$H^0\mathcal{V} \xrightarrow{\sim} U(\mathcal{L})^\flat. \tag{3.9.13.3}$$

(c) Now suppose that $\mathcal{L}$ is R-flat. Then $0 \to K \to R\otimes L \to \mathcal{L} \to 0$ is a short exact sequence of flat R-modules. So the projection $\mathrm{Sym}^{\cdot}_R\mathcal{C}one(K \to \mathcal{L}) \twoheadrightarrow \mathrm{Sym}^{\cdot}_R\mathcal{L}$ is a quasi-isomorphism. By (3.9.13.2) one has $H^a\mathrm{gr}_{\cdot}\mathcal{V} = 0$ for $a \neq 0$ (hence $H^0\mathrm{gr}_{\cdot}\mathcal{V} = \mathrm{gr}_{\cdot}H^0\mathcal{V}$) and $H^0\mathrm{gr}_{\cdot}\mathcal{V} = B \underset{R}{\otimes} \mathrm{Sym}^{\cdot}_R\mathcal{L}$. Therefore, by (3.9.13.3), one has $\mathrm{gr}_{\cdot}U(\mathcal{L})^\flat = B \underset{R}{\otimes} \mathrm{Sym}^{\cdot}_R\mathcal{L}$, and we are done. □

3.9.14. *Proof of the theorem in 3.9.11.* Let us return to the chiral setting. The above arguments should be modified as follows. Again, we proceed in three steps.

(a) Suppose that L is an $\mathcal{O}_X$-flat Lie* algebra acting on R and B, $\mathcal{L}^\flat$ is an (L, τ_{LR})-rigidified Lie* algebroid, and $\mathcal{L}^\flat$ its (L, τ_{LB})-rigidified B-extension (see 3.9.8). One check immediately that $U(B, \mathcal{L})^\flat$ equals the chiral algebra $B \circledast L$ from 3.7.23 (see the proof in 3.9.8), so the PBW theorem for $\mathcal{L}^\flat$ follows from the PBW result from 3.7.23.

Steps (b) and (c) literally repeat those of 3.9.13. □

3.9.15. Using the language of 3.9.4, one can rephrase the PBW theorem for chiral Lie algebroids as follows.

Let $\mathcal{Y}$ be an $\mathcal{O}_X$-flat algebraic $\mathcal{D}_X$-space, $\mathcal{L}$ an $\mathcal{O}_\mathcal{Y}$-flat Lie* algebroid on $\mathcal{Y}$. Then the functors $\mathcal{L}^\flat \mapsto U(\mathcal{L})^\flat_\mathcal{Y}$, $(A., \alpha) \mapsto A_1$ are mutually inverse equivalences between the groupoid $\mathcal{P}^{ch}(\mathcal{Y}, \mathcal{L})$ of chiral $\mathcal{O}_\mathcal{Y}$-extensions of $\mathcal{L}$ and that of pairs $(A., \alpha)$ where $A.$ is a chiral algebra on $\mathcal{Y}_{\mathcal{D}if}$ equipped with a commutative filtration $A_0 \subset A_1 \subset \cdots$ such that $\mathrm{gr}.A^\ell = \mathrm{Sym}(\mathrm{gr}_1 A^\ell)$ and $\alpha : \mathrm{gr}_1 \mathcal{A} \xrightarrow{\sim} \mathcal{L}$ is an isomorphism of Lie* algebroids.

In particular, for smooth $\mathcal{Y}$ there is a canonical equivalence of groupoids (see 3.9.5)

$$\mathcal{CDO}^{ch}(\mathcal{Y}) \xrightarrow{\sim} \mathcal{P}^{ch}(\mathcal{Y}, \Theta_\mathcal{Y}). \tag{3.9.15.1}$$

3.9.16. We are going to describe in "classical" terms the groupoid $\mathcal{P}^{ch}(\mathcal{L})$ under the assumption that $\mathcal{Y}$ is $\mathcal{O}_X$-flat and $\mathcal{L}$ is a vector $\mathcal{D}_X$-bundle. This will be done in 3.9.20 below. The first step is to define certain *canonical* DG chiral Lie algebroid $\mathcal{L}^\flat_\mathcal{C}$ related to $\mathcal{L}$. The construction, inspired by [**MSV**], is of independent interest.

At the moment we work locally, so we have an $\mathcal{O}_X$-flat commutative $\mathcal{D}_X$-algebra R^ℓ and a Lie* R-algebroid $\mathcal{L}$ which is a locally projective $R^\ell[\mathcal{D}_X]$-module of finite rank. Let $\mathcal{C} = \mathcal{C}_R(\mathcal{L})$ be the de Rham-Chevalley complex of $\mathcal{L}$ (see 1.4.14). This is a commutative DG $\mathcal{D}_X$-algebra which equals $\mathrm{Sym}(\mathcal{L}^\circ[-1])$ as a mere graded module; here $\mathcal{L}^\circ$ is the dual $R^\ell[\mathcal{D}_X]$-module.

Let us define a DG Lie* $\mathcal{C}$-algebroid $\mathcal{L}_\mathcal{C}$. Our $\mathcal{C}$ carries an action of the Lie* DG algebra $\mathcal{L}_\dagger$ (see 1.4.14), so we have the corresponding induced DG Lie* $\mathcal{C}$-algebroid $(\mathcal{L}_\dagger)_\mathcal{C}$ (see 1.4.13). Denote by $\mathcal{K}^{-1} \subset (\mathcal{L}_\dagger)^{-1}_\mathcal{C}$ the kernel of the product map $(\mathcal{L}_\dagger)^{-1}_\mathcal{C} = R^\ell \otimes \mathcal{L} \xrightarrow{\cdot} \mathcal{L}$; let $\mathcal{K} \subset (\mathcal{L}_\dagger)_\mathcal{C}$ be the DG $\mathcal{C}$-ideal generated by $\mathcal{K}^{-1}$. Then $\mathcal{K}$ is also a Lie* ideal which acts trivially on $\mathcal{C}$ (since $\mathcal{K}^{-1}$ acts trivially on $\mathcal{C}$ and is normalized by $\mathcal{L}_\dagger$), and we set $\mathcal{L}_\mathcal{C} := (\mathcal{L}_\dagger)_\mathcal{C}/\mathcal{K}$.

Our $\mathcal{L}_\mathcal{C}$ looks as follows. By construction, it sits in degrees ≥ -1, and we have a morphism of DG Lie* algebras $\iota : \mathcal{L}_\dagger \to \mathcal{L}_\mathcal{C}$ such that $\iota^{-1} : \mathcal{L} \xrightarrow{\sim} \mathcal{L}^{-1}_\mathcal{C}$. The $\mathcal{C}$-submodule $\mathcal{L}_{\mathcal{C}+}$ generated by $\mathcal{L}^{-1}_\mathcal{C} = \mathcal{L}$ is not closed under the action of d. In fact, as one checks in a moment, $\mathcal{L}_{\mathcal{C}+}$ is freely $\mathcal{C}$-generated by $\mathcal{L}^{-1}_\mathcal{C}$, and the "Kodaira-Spencer" map $\mathcal{L}_{\mathcal{C}+} \to \mathcal{L}_\mathcal{C}/\mathcal{L}_{\mathcal{C}+}$, $\ell \mapsto d(\ell) \mathrm{mod} \mathcal{L}_{\mathcal{C}+}$, is an isomorphism. So we have an exact sequence of graded $\mathcal{C}$-modules

$$0 \to \mathcal{C} \underset{R}{\otimes} \mathcal{L}[1] \to \mathcal{L}_\mathcal{C} \to \mathcal{C} \underset{R}{\otimes} \mathcal{L} \to 0. \tag{3.9.16.1}$$

3.9.17. The following proposition can be considered as a version of 3.9.8 for L replaced by a single super symmetry:

PROPOSITION. *A DG chiral $\mathcal{C}$-extension $\mathcal{L}^\flat_{\mathcal{C}}$ of $\mathcal{L}_{\mathcal{C}}$ exists and is unique up to a unique isomorphism.*

Proof. Existence. Let us construct $\mathcal{L}^\flat_{\mathcal{C}}$. Consider the rigidified DG chiral $\mathcal{C}$-extension $(\mathcal{L}_\dagger)^\flat_{\mathcal{C}}$ that corresponds to the action of $\mathcal{L}_\dagger$ on $\mathcal{C}$ (see 3.9.8). In degree -1 it coincides with $(\mathcal{L}_\dagger)_{\mathcal{C}}$, and we define $\mathcal{K}^\flat \subset (\mathcal{L}_\dagger)^\flat_{\mathcal{C}}$ as the DG $\mathcal{C}$-submodule generated by $\mathcal{K}^{-1}$. It is automatically a Lie* ideal, and its image in $(\mathcal{L}_\dagger)_{\mathcal{C}}$ equals $\mathcal{K}$. Set $\mathcal{L}^\flat_{\mathcal{C}} := (\mathcal{L}_\dagger)^\flat_{\mathcal{C}}/\mathcal{K}^\flat$.

This is a chiral DG extension of $\mathcal{L}_{\mathcal{C}}$ by $\mathcal{C}/J_{\mathcal{C}}$, $J_{\mathcal{C}} := \mathcal{C} \cap \mathcal{K}^\flat$; it remains to show that $J_{\mathcal{C}} = 0$. Since $J_{\mathcal{C}}$ is closed under the action of $\mathcal{L}_\dagger$, hence $\mathcal{L}[1] \subset \mathcal{L}_\dagger$, it suffices to check that $J := J^0_{\mathcal{C}} = R \cap \mathcal{K}^\flat$ vanishes.

Let $\mathcal{K}^\flat_+ \subset \mathcal{K}^\flat$ be the $\mathcal{C}$-submodule generated by $\mathcal{K}^{-1}$. The composition $\mathcal{K}^{-1} \xrightarrow{d} \mathcal{K}^{\flat 0} \to \mathcal{K}^{\flat 0}/\mathcal{K}^{\flat 0}_+$ is R-linear, so $\mathcal{K}^{\flat 0}_+ + d(\mathcal{K}^{-1})$ is an R-submodule; hence $\mathcal{K}^{\flat 0} = \mathcal{K}^{\flat 0}_+ + d(\mathcal{K}^{-1})$. Since the composition $(\mathcal{L}_\dagger)^{-1}_{\mathcal{C}} \xrightarrow{d} (\mathcal{L}_\dagger)^0_{\mathcal{C}} \twoheadrightarrow (\mathcal{L}_\dagger)^0_{\mathcal{C}}/\mathcal{C}^1((\mathcal{L}_\dagger)^{-1}_{\mathcal{C}})$ is an isomorphism, we see that $J = R \cap \mathcal{K}^{\flat 0}_+$.

Let $(\mathcal{L}_\dagger)^\flat_{\mathcal{C}+} \subset (\mathcal{L}_\dagger)^\flat_{\mathcal{C}}$ be the preimage of $\mathcal{C} \otimes \mathcal{L}[1] \subset (\mathcal{L}_\dagger)_{\mathcal{C}}$. This is a graded chiral $\mathcal{C}$-module which is an extension of $\mathcal{C} \otimes \mathcal{L}[1]$ by $\mathcal{C}$. As we saw in the proof in 3.9.8, $\Delta_*(\mathcal{L}_\dagger)^\flat_{\mathcal{C}+}$ is the push-forward of the extension $0 \to \mathcal{C}\boxtimes\mathcal{L}[1] \to j_*j^*\mathcal{C}\boxtimes\mathcal{L}[1] \to \Delta_*\mathcal{C}\otimes\mathcal{L}[1] \to 0$ by the morphism $\mathcal{C}\boxtimes\mathcal{L}[1] \to \mathcal{C}$ defined by the $*$ action of $\mathcal{L}[1] \subset \mathcal{L}_\dagger$ on $\mathcal{C}$. The above $\mathcal{D}_{X\times X}$-modules are $R^\ell \boxtimes R^\ell$-modules in the obvious way and the $*$ action is R^ℓ-bilinear, so $(\mathcal{L}_\dagger)^\flat_{\mathcal{C}+}$ is an $R^\ell\otimes R^\ell$-module. Set $I^\ell := \mathrm{Ker}(R^\ell\otimes R^\ell \xrightarrow{\cdot} R^\ell)$. Then $\mathcal{K}^\flat_+ = I^\ell \cdot (\mathcal{L}_\dagger)^\flat_{\mathcal{C}+}$ since the $R^\ell \otimes R^\ell$-action commutes with the chiral $\mathcal{C}$-action. Since $\mathcal{L}$ and $\mathcal{C}$ are flat R^ℓ-modules, we see that $\mathcal{C} \cap \mathcal{K}^\flat_+ = 0$, hence $J = 0$; q.e.d.

Uniqueness. Let $\mathcal{L}^{\flat\prime}_{\mathcal{C}}$ be any DG chiral $\mathcal{C}$-extension of $\mathcal{L}_{\mathcal{C}}$. The projection $\mathcal{L}^{\flat\prime}_{\mathcal{C}} \to \mathcal{L}_{\mathcal{C}}$ is an isomorphism in degree -1, so ι lifts in a unique way to a morphism of complexes $\iota^{\flat\prime} : \mathcal{L}_\dagger \to \mathcal{L}^{\flat\prime}_{\mathcal{C}}$. This is automatically a morphism of Lie* algebras.[86]

According to 3.9.8, $\iota^{\flat\prime}$ extends in a unique way to a morphism of DG chiral Lie $\mathcal{C}$-algebroids $\iota^\flat_{\mathcal{C}} : (\mathcal{L}_\dagger)^\flat_{\mathcal{C}} \to \mathcal{L}^\flat_{\mathcal{C}}$. It vanishes on $(\mathcal{K}^\flat)^{-1}$, hence factors through a morphism $\mathcal{L}^\flat_{\mathcal{C}} \to \mathcal{L}^{\flat\prime}_{\mathcal{C}}$.

It remains to note that $\mathcal{L}^\flat_{\mathcal{C}}$ has no non-trivial automorphisms since it is generated (as a DG chiral Lie $\mathcal{C}$-algebroid) by the degree -1 part which coincides with that of $\mathcal{L}_{\mathcal{C}}$. □

The next lemma (cf. 3.8.10) will not be used in the sequel.

LEMMA. *There is a unique element $\mathfrak{d} \in h(\mathcal{L}^{\flat 1})$, called the de Rham-Chevalley charge, whose adjoint action is the differential $d = d_{\mathcal{L}^\flat_{\mathcal{C}}}$ of $\mathcal{L}^\flat_{\mathcal{C}}$. It acts on $\mathcal{C}$ as $d_{\mathcal{C}}$, and one has $[\mathfrak{d}, \mathfrak{d}] = 0$.*

Proof. Uniqueness. It suffices to show that elements of $h(\mathcal{L}^{\flat 1}_{\mathcal{C}})$ are uniquely determined by its adjoint action on $\mathcal{L}^{\flat -1}_{\mathcal{C}}$. In fact, the adjoint action morphism $h(\mathcal{L}^{\flat i}_{\mathcal{C}}) \to \mathrm{Hom}(\mathcal{L}^{\flat -1}_{\mathcal{C}}, \mathcal{L}^{\flat i-1}_{\mathcal{C}})$ is injective for any $i \geq 1$.

To see this notice that $\mathcal{L}^\flat_{\mathcal{C}}$, considered as a mere graded $\mathcal{C}$-module, carries a canonical 3-step filtration $\mathcal{C} = \mathcal{L}^\flat_{\mathcal{C}0} \subset \mathcal{L}^\flat_{\mathcal{C}1} \subset \mathcal{L}^\flat_{\mathcal{C}2} = \mathcal{L}^\flat_{\mathcal{C}}$ with $\mathrm{gr}\,\mathcal{L}^\flat_{\mathcal{C}} = \mathcal{C} \oplus \mathcal{C} \otimes L[1] \oplus \mathcal{C} \otimes \mathcal{L}$ (see (3.9.16.1)). The bracket with $\mathcal{L}^{\flat -1}_{\mathcal{C}} = \mathcal{L}[1]$ preserves the filtration; on $\mathrm{gr}\,\mathcal{L}^\flat_{\mathcal{C}}$ this is the obvious "convolution" coming from the duality. Thus for every

[86]Check first that $\iota^{\flat\prime}$ is compatible with $[\,,\,]_{0,-1}$ and then with $[\,,\,]_{0,0}$.

$i \geq 1$ the corresponding morphism $\operatorname{gr} h(\mathcal{L}_{\mathcal{C}}^{\flat i}) = h(\operatorname{gr}\mathcal{L}_{\mathcal{C}}^{i}) \to \operatorname{Hom}(\mathcal{L}_{\mathcal{C}}^{\flat -1}, \operatorname{gr}\mathcal{L}_{\mathcal{C}}^{\flat i-1})$ is injective, and we are done.

Existence. Let us find $\mathfrak{d} \in h(\mathcal{L}_{\mathcal{C}}^{\flat 1})$ whose adjoint action on $\mathcal{L}_{\mathcal{C}}^{\flat -1}$ equals d.

Notice that the quotient layer of the map $h(\mathcal{L}_{\mathcal{C}}^{\flat 1}) \hookrightarrow \operatorname{Hom}(\mathcal{L}_{\mathcal{C}}^{\flat -1}, \mathcal{L}_{\mathcal{C}}^{\flat 0})$ is the usual identification of $h(\mathcal{L}^\circ \underset{R}{\otimes} \mathcal{L})$ with $R^\ell[\mathcal{D}_X]$-linear endomorphisms of $\mathcal{L}$. Choose any $\mathfrak{d}' \in h(\mathcal{L}_{\mathcal{C}}^{\flat 1})$ which lifts the identity endomorphism $\mathrm{id}_{\mathcal{L}}$. The derivation $d_{\mathcal{C}} - \tau(\mathfrak{d}')$ of $\mathcal{C}$ is R-linear; hence such is the derivation $d' := d - \mathrm{ad}_{\mathfrak{d}'}$ of $\mathcal{L}_{\mathcal{C}}^{\flat}$. One has $d'(\mathcal{L}_{\mathcal{C}}^{\flat -1}) \subset \mathcal{L}_{\mathcal{C}1}^{\flat 0}$ and the R-bilinear pairing $\in P_2^*(\mathcal{L})$, $\ell, \ell' \mapsto [\ell . d'(\ell')]$, is skew-symmetric (here $\ell, \ell' \in \mathcal{L} = \mathcal{L}_{\mathcal{C}}^{\flat -1}[-1]$). Now the image of the adjoint action map $h(\mathcal{L}_{\mathcal{C}1}^{\flat 1}) \hookrightarrow \operatorname{Hom}(\mathcal{L}_{\mathcal{C}}^{\flat -1}, \mathcal{L}_{\mathcal{C}}^{\flat 0})$ consists exactly of all $R^\ell[\mathcal{D}_X]$-linear morphisms $\phi : \mathcal{L} \to \mathcal{L}_{\mathcal{C}1}^{\flat 0}$ such that the pairing $\ell, \ell' \mapsto [\ell, \phi(\ell')]$ is skew-symmetric. Therefore, adding to $\mathfrak{d}'$ an element of $h(\mathcal{L}_{\mathcal{C}1}^{\flat 1})$ that acts on $\mathcal{L}_{\mathcal{C}}^{\flat -1}$ as d', we get the promised $\mathfrak{d}$.

Let us prove that the adjoint action of $\mathfrak{d}$ coincides with d on the whole of $\mathcal{L}_{\mathcal{C}}^{\flat}$. We check it on every $\mathcal{L}_{\mathcal{C}}^{\flat i}$ by induction by i. The case $i = -1$ is known. Induction step: Suppose we know that $d - \mathrm{ad}_{\mathfrak{d}}$ kills $\mathcal{L}_{\mathcal{C}}^{\flat i}$. As we have seen above, to check that it kills $\mathcal{L}_{\mathcal{C}}^{\flat i+1}$, it suffices to verify that $[\ell, (d - \mathrm{ad}_{\mathfrak{d}})(f)] = 0$ for every $\ell \in \mathcal{L}_{\mathcal{C}}^{\flat -1}$, $f \in \mathcal{L}_{\mathcal{C}}^{\flat i+1}$. One has $[\ell, (d - \mathrm{ad}_{\mathfrak{d}})(f)] = [(d - \mathrm{ad}_{\mathfrak{d}})(\ell), f] - (d - \mathrm{ad}_{\mathfrak{d}})([\ell, f]) = 0$, and we are done.

Looking at $\mathcal{C} \subset \mathcal{L}_{\mathcal{C}}^{\flat}$, we see that $\tau(\mathfrak{d})$ acts on $\mathcal{C}$ as $d_{\mathcal{C}}$. Finally, $[\mathfrak{d}, \mathfrak{d}] \in h(\mathcal{L}_{\mathcal{C}}^{\flat 2})$ equals 0 since its adjoint action on $\mathcal{L}_{\mathcal{C}}^{-1}$ vanishes (being equal to $2d^2$). □

Question. Can one find an explicit formula for the de Rham-Chevalley charge similar to the formula for the BRST charge from 3.8.9?

3.9.18. Important example. Suppose that R^ℓ is smooth and $\mathcal{L}$ is the tangent algebroid Θ_R. Then $\mathcal{C}$ is equal to the relative de Rham complex $DR = DR_{R^\ell/X}$, which is a smooth DG commutative $\mathcal{D}_X$-algebra, and $\mathcal{L}_{\mathcal{C}}$ is its tangent algebroid. So, combining 3.9.17 with (3.9.15.1), we see that DR admits a unique DG cdo $\mathcal{D}_{DR}$.

3.9.19. Passing to the components of degree 0 in 3.9.17 and 3.9.18, we get a canonical Lie* R-algebroid $\mathcal{T} = \mathcal{T}(\mathcal{L}) := \mathcal{L}_{\mathcal{C}}^{0}$ and its chiral R-extension $\mathcal{T}^\flat = \mathcal{T}(\mathcal{L})^\flat := \mathcal{L}_{\mathcal{C}}^{\flat 0}$. Let us describe them explicitly.

By (3.9.16.1) $\mathcal{T}$ is an extension of $\mathcal{L}$ by the ideal $\mathcal{L}^\circ \underset{R}{\otimes} \mathcal{L}$. Our $\mathcal{T}$ acts on $\mathcal{L}$ (as on a mere vector $\mathcal{D}_X$-bundle) by the adjoint action on $\mathcal{L}_{\mathcal{C}}^{-1} = \mathcal{L}$. This action is R-linear with respect to the $\mathcal{T}$-variable,[87] so $\mathcal{L}$ is a $\mathcal{T}$-module (see 1.4.12). It identifies $\mathcal{L}^\circ \underset{R}{\otimes} \mathcal{L} \subset \mathcal{T}$ with $\mathfrak{gl}(\mathcal{L})$ in the usual way. The canonical section $\iota = \iota^0 : \mathcal{L} \to \mathcal{T}$ (see 3.9.16) is a morphism of Lie* algebras but *not* of R^ℓ-modules: for $f \in R^\ell$, $\ell \in \mathcal{L}$ the endomorphism $\iota^0(f\ell) - f\iota^0(\ell) \in \mathfrak{gl}(\mathcal{L})$ is $\ell' \mapsto \tau_{\mathcal{L}}(\ell')(f)\ell$.

Remark. If Θ_R, hence the Lie* algebroid $\mathcal{E}(\mathcal{L})$, is well defined, then the above action identifies $\mathcal{T}$ with the pull-back of the extension $0 \to \mathfrak{gl}(\mathcal{L}) \to \mathcal{E}(\mathcal{L}) \to \Theta_R \to 0$ by the action morphism $\tau_{\mathcal{L}} : \mathcal{L} \to \Theta_R$.

Now let us consider $\mathcal{T}^\flat$. Its restriction to $\mathfrak{gl}(\mathcal{L}) \subset \mathcal{T}$ identifies canonically with the Tate extension $\mathfrak{gl}(\mathcal{L})^\flat$ as defined in 2.7.6. Indeed, our extension does not depend on the Lie* algebra structure on $\mathcal{L}$, and in case $R^\ell = \mathcal{O}_X$ our identification

[87] Since $\mathcal{L}_{\mathcal{C}}^{-1}$ acts trivially on R.

is (3.8.6.1). The case of general R^ℓ is similar; we actually defined the identification at the end of the proof of the existence statement in 3.9.17.

The canonical Lie* algebra splitting[88] $\iota^\flat = \iota^{\flat 0} : \mathcal{L} \to \mathcal{T}^\flat$ is *not* R-linear: precisely, the chiral operation $\mu_{\mathcal{T}^\flat}(id_R, \iota^\flat) - \iota^\flat \mu_{\mathcal{L}} \in P_2^{ch}(\{R, \mathcal{L}\}, \mathfrak{gl}(\mathcal{L})^\flat)$ is equal to[89] $\mu_{\mathcal{C}l}(d, id_{\mathcal{L}})$ where $\mu_{\mathcal{C}l} \in P_2^{ch}(\{\mathcal{L}^\circ, \mathcal{L}\}, \mathfrak{gl}(\mathcal{L})^\flat)$ is the canonical chiral pairing and $d : R \to \mathcal{L}^\circ = \mathcal{C}^1$ is the differential in $\mathcal{C}$.[90]

Summing up: as a Lie* algebra, $\mathcal{T}^\flat$ is the semi-direct product of $\mathcal{L}$ and $\mathfrak{gl}(\mathcal{L})^\flat$ with respect to the canonical action of $\mathcal{L}$ on $\mathfrak{gl}(\mathcal{L})^\flat$ coming from the adjoint action of $\mathcal{L}$. The chiral action of R on $\mathcal{T}^\flat$ is the usual R^ℓ-action on $\mathfrak{gl}(\mathcal{L})^\flat$, and on $\mathcal{L} \subset \mathcal{T}^\flat$ it is given by the above formula.

3.9.20. Consider a pair $(\mathcal{T}^c, \rho)$ where $\mathcal{T}^c \in \mathcal{P}^{cl}(\mathcal{T}(\mathcal{L}))$ and $\rho : \mathcal{T}^c|_{\mathfrak{gl}(\mathcal{L})} \xrightarrow{\sim} \mathfrak{gl}(\mathcal{L})^\flat$ is an isomorphism of extensions which identifies the adjoint action of $\mathcal{L} \subset \mathcal{T}$ with the canonical action of $\mathcal{L}$ on $\mathfrak{gl}(\mathcal{L})^\flat$ coming from the adjoint action. Such pairs form a groupoid $\mathcal{P}^{\mathcal{T}}(\mathcal{L})$.

The Picard groupoid $\mathcal{P}^{cl}(\mathcal{L})$ acts naturally on $\mathcal{P}^{\mathcal{T}}(\mathcal{L})$: the translation by $\mathcal{L}^c \in \mathcal{P}^{cl}(\mathcal{L})$ sends $(\mathcal{T}^c, \rho)$ to the Baer sum of $\mathcal{T}^c$ and the pull-back of $\mathcal{L}^c$ by the projection $\mathcal{T} \twoheadrightarrow \mathcal{L}$. This action makes $\mathcal{P}^{\mathcal{T}}(\mathcal{L})$ a $\mathcal{P}^{cl}(\mathcal{L})$-torsor.

PROPOSITION. *There is a canonical* anti-*equivalence of* $\mathcal{P}^{cl}(\mathcal{L})$*-torsors*

$$\mathcal{P}^{ch}(\mathcal{L}) \xrightarrow{\sim} \mathcal{P}^{\mathcal{T}}(\mathcal{L}). \tag{3.9.20.1}$$

Proof. For $\mathcal{L}^\flat \in \mathcal{P}^{ch}(\mathcal{L})$ the corresponding $(\mathcal{T}^c, \rho)$ is defined as follows. The pull-back of $\mathcal{L}$ by $\mathcal{T} \twoheadrightarrow \mathcal{L}$ is a chiral R-extension of $\mathcal{T}$ trivialized over $\mathfrak{gl}(\mathcal{L}) \subset \mathcal{T}$; this trivialization is invariant under the adjoint action of $\mathcal{T}$. Our $\mathcal{T}^c$ is the Baer difference of the canonical chiral extension $\mathcal{T}^\flat$ and this chiral extension. The above trivialization provides[91] ρ. □

Of course, the above equivalence has the local nature, so it works in the setting of an $\mathcal{O}_X$-flat algebraic $\mathcal{D}_X$-space $\mathcal{Y}$ and a Lie* algebroid $\mathcal{L}$ on $\mathcal{Y}$ which is a vector $\mathcal{D}_X$-bundle.

For example, if $\mathcal{Y}$ is smooth, then $\mathcal{T}(\Theta_{\mathcal{Y}}) = \mathcal{E}(\Theta_{\mathcal{Y}})$ and the groupoid $\mathcal{P}^{\mathcal{T}}(\Theta_{\mathcal{Y}})$ coincides with the groupoid $\mathcal{T}ate(\mathcal{Y})$ of Tate structures on $\mathcal{Y}$ as defined in 2.8.1. Combining (3.9.20.1) with (3.9.15.1), we get a canonical anti-equivalence of $\mathcal{P}^{cl}(\Theta_{\mathcal{Y}})$-torsors

$$\mathcal{CDO}(\mathcal{Y}) \xrightarrow{\sim} \mathcal{T}ate(\mathcal{Y}). \tag{3.9.20.2}$$

We discussed some concrete constructions of Tate structures in 2.8.14–2.8.17. Now they can be seen as examples of cdo. If $\mathcal{Y}$ is equipped with an action of a group $\mathcal{D}_X$-scheme G affine over X, then weakly G-equivariant Tate structures correspond to cdo equipped with a G-action (see 3.4.17) which extends the G-action on $\mathcal{O}_{\mathcal{Y}}$.

EXAMPLE. According to 2.8.17, we have a canonical family of cdo on the jet scheme of a flag space parametrized by Miura opers.

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[88] $\iota^\flat : \mathcal{L}_\dagger \to \mathcal{L}^\flat_{\mathcal{C}}$ was defined in the beginning of the proof of the uniqueness statement of 3.9.17.

[89] Use the fact that $\iota^{\flat -1} : \mathcal{L} \xrightarrow{\sim} \mathcal{L}^{\flat -1}_{\mathcal{C}}$ is R-linear and the Leibnitz formula.

[90] I.e., the map dual to the action $\tau_{\mathcal{L}}$.

[91] Recall that the restriction of $\mathcal{T}^\flat$ to $\mathfrak{gl}(\mathcal{L})$ equals $\mathfrak{gl}(\mathcal{L})^\flat$; see 3.9.19.

Exercise. Suppose we are in the situation of 2.8.16 and G is connected. Let A be the cdo on $\mathcal{Y}$ that corresponds to the Tate structure $\mathcal{E}(\Theta_{\mathcal{Y}})^\flat$. Show that the morphism $\alpha : L^\flat \to \mathcal{E}(\Theta)^\flat$ from loc. cit. yields a morphism of Lie* algebras $\alpha : L^\flat \to A$ which sends $1^\flat$ to -1_A. The corresponding BRST reduction satisfies $H^{<0}_{\mathrm{BRST}}(L, A) = 0$ and $H^0_{\mathrm{BRST}}(L.A)$ is the cdo on $\mathcal{Z}$ that corresponds to the Tate structure $\mathcal{E}(\Theta_{\mathcal{Z}})^\flat_\alpha$.

3.9.21. Back in 2.8.3–2.8.9 we described Tate structures on a vector $\mathcal{D}_X$-bundles in terms of connections. This material can be easily rendered to the situation when Θ_R is replaced by an arbitrary $\mathcal{L}$ as above. One has only to replace mere connections by $\mathcal{L}$-connections (see 1.4.17). Let us formulate the principal statements; for the proofs and details we refer to 2.8.

Consider $(\mathcal{Y}, \mathcal{L})$ as above. Let $\mathcal{C} = \mathcal{C}_{\mathcal{O}_{\mathcal{Y}}}(\mathcal{L})$ be the de Rham-Chevalley DG commutative $\mathcal{O}_{\mathcal{Y}}[\mathcal{D}_X]$-algebra. We have the 2-term complex $h(\mathcal{C})_{[1,2)} := \tau_{\leq 2}\sigma_{\geq 1} h(\mathcal{C})$ of sheaves and the Picard groupoid $h(\mathcal{C})_{[1,2)}$-*tors* of $h(\mathcal{C})_{[1,2)}$-torsors. There is a canonical equivalence of Picard torsors

$$\mathcal{P}^{cl}(\mathcal{L}) \xrightarrow{\sim} h(\mathcal{C})_{[1,2)}\text{-}tors \tag{3.9.21.1}$$

which assigns to $\mathcal{L}^c$ the $h(\mathcal{C}^1)$-torsor of $\mathcal{L}$-connections on $\mathcal{L}^c$ and the curvature map $c : C \to h(\mathcal{C}^2)^{closed}$ (see 2.8.3).

Now we consider the set $\mathfrak{T}$ of pairs $(\mathcal{T}^c, \nabla^c)$ where $\mathcal{T}^c = (\mathcal{T}^c, \rho) \in \mathcal{P}^{\mathcal{T}}(\mathcal{L})$ and ∇^c is an $\mathcal{L}$-connection on $\mathcal{T}^c$. Let $\mathfrak{C}$ be the set of pairs $(\nabla, c(\nabla)^\flat)$ where ∇ is an $\mathcal{L}$-connection on $\mathcal{T}$ and $c(\nabla)^\flat \in h(\mathcal{C}^2 \otimes \mathfrak{gl}(\mathcal{L})^\flat)$ is a lifting of $c(\nabla) \in h(\mathcal{C}^2 \otimes \mathfrak{gl}(\mathcal{L}))$ such that $d^\flat_\nabla(c(\nabla)^\flat) = 0$. Here $d^\flat_\nabla$ is a $d_{\mathcal{C}}$-derivation of $\mathcal{C} \otimes \mathfrak{gl}(\mathcal{L})^\flat$ that comes from the canonical $\mathcal{T}$-action on $\mathcal{L}$ and ∇. There is a canonical bijection (see 2.8.5)

$$\mathfrak{T} \xrightarrow{\sim} \mathfrak{C}, \quad (\mathcal{T}^c, \nabla^c) \mapsto (\nabla, c(\nabla^c)). \tag{3.9.21.2}$$

The above $\mathfrak{T}$ and $\mathfrak{C}$ are actually groupoids (the morphisms in $\mathfrak{T}$ are those between $\mathcal{T}^c$; for $\mathfrak{C}$ see the formulas in 2.8.6), and (3.9.21.2) lifts to an isomorphism of groupoids. They have the local nature, so one can consider the corresponding sheafified groupoids. The one for $\mathfrak{T}$ is equivalent to $\mathcal{P}^{\mathcal{T}}(\mathcal{L})$, so we get an equivalence between $\mathcal{P}^{\mathcal{T}}(\mathcal{L})$ and the sheafification of $\mathfrak{C}$.

As in 2.8.8 this description can be made concrete by choosing a hypercovering $\mathcal{Y}.$ of $\mathcal{Y}$, an $\mathcal{L}$-connection for $\mathcal{T}$ on $\mathcal{Y}_0$, a lifting $\chi \in h(\mathcal{C}^2 \otimes \mathfrak{gl}(\mathcal{L})^\flat)$ of $C(\nabla)$ on $\mathcal{Y}_0$, and a section μ of $h(\mathcal{C}^1 \otimes \mathfrak{gl}(\mathcal{L})^\flat)$ on $\mathcal{Y}_1$ that lifts $\partial_0^*\nabla - \partial_1^*\nabla$. The same formulas as in 2.8.8 produce from this datum a Čech cocycle $\zeta \in C^3(\mathcal{Y}., h(\mathcal{C}^{\geq 1}))$. Its cohomology class $ch_1^{\mathcal{D}\mathcal{L}}(\mathcal{L}) \in H^3(\mathcal{Y}, h(\mathcal{C}^{\geq 1}))$ does not depend on the auxiliary choices. It vanishes if and only if $\mathcal{P}^{\mathcal{T}}(\mathcal{L})$ is non-empty. If this happens, then $\mathcal{P}^{\mathcal{T}}(\mathcal{L})$ identifies canonically with the groupoid of elements in $C^2(\mathcal{Y}., h(\mathcal{C}^{\geq 1}))$ whose differential equals ζ; the set of morphisms $a \to a'$ is $d_C^{-1}(a - a') \subset C^1(\mathcal{Y}., h(\mathcal{C}^{\geq 1}))$.

Suppose now that $\mathcal{Y}$ is smooth, so the Lie* algebroids $\Theta_{\mathcal{Y}}$ and $\mathcal{E}(\mathcal{L})$ are well defined. There is a canonical morphism $\mathcal{T} \to \mathcal{E}(\mathcal{L})$ (the action of $\mathcal{T}$ on $\mathcal{L}$) which provides the pull-back morphism of groupoids

$$\mathcal{T}ate(\mathcal{L}) \to \mathcal{P}^{\mathcal{T}}(\mathcal{L}). \tag{3.9.21.3}$$

Here $\mathcal{L}$ in the left-hand side is considered as a mere vector $\mathcal{D}_X$-bundle on $\mathcal{Y}$. A connection for a Tate structure on $\mathcal{L}$ provides a connection on the corresponding extension $\mathcal{T}^c$, which yields a morphism of the corresponding $\mathfrak{C}$-groupoids. In particular, we get:

3.9.22. COROLLARY. *The obstruction $ch_1^{\mathcal{D}\mathcal{L}}(\mathcal{L}) \in H^3(\mathcal{Y}, h(\mathcal{C}^{\geq 1}))$ to the existence of the global chiral extension of $\mathcal{L}$ is the image of $ch_1^{\mathcal{D}}(\mathcal{L})$ by the canonical morphism of DG algebras*[92] $DR_{\mathcal{Y}/X} \to \mathcal{C}$. □

Of course, the case of $\mathcal{L} = \Theta_{\mathcal{Y}}$ is all in 2.8. Looking at (3.9.20.2), we see, in particular, that the obstruction to the existence of a global cdo is $ch_1^{\mathcal{D}}(\Theta_{\mathcal{Y}}) \in H^4(\mathcal{Y}, DR_X(DR^{\geq 1}_{\mathcal{Y}/X}))$.

Consider, for example, the situation when our $\mathcal{Y}$ is a jet space, $\mathcal{Y} = \mathcal{J}Z$ for an algebraic space Z/X smooth over X (see 2.3.2). Let $p : \mathcal{Y} \to Z$ be the canonical projection and $\Theta_{Z/X}$ the relative tangent bundle. We will see in a moment that $\Theta_{\mathcal{Y}}$ is locally trivial, so cdo exist locally on $\mathcal{Y}$ (see Remark in 2.8.1). Consider the map $H^4(Z, DR_Z^{\geq 2}) \xrightarrow{p^*} H^4(\mathcal{Y}, DR_{\mathcal{Y}}^{\geq 2}) \to H^4(\mathcal{Y}, DR_X(DR^{\geq 1}_{\mathcal{Y}/X}))$ (see 2.8.12).

3.9.23. COROLLARY. *The obstruction $ch_1^{\mathcal{D}}(\Theta_{\mathcal{Y}})$ to the existence of global cdo is equal to the image of $ch_2(\Theta_{Z/X}\omega_X^{1/2})$ in $H^4(\mathcal{Y}, DR_X(DR^{\geq 1}_{\mathcal{Y}/X}))$.*

Proof. The morphism of $\mathcal{O}$-modules $dp : p^*\Omega^1_{Z/X} \to \Omega^1_{\mathcal{Y}/X}$ yields an isomorphism of $\mathcal{O}_{\mathcal{Y}}[\mathcal{D}_X]$-modules $(p^*\Omega^1_{Z/X})_{\mathcal{D}} \xrightarrow{\sim} \Omega^1_{\mathcal{Y}/X}$. Passing to dual modules (and using an $\mathcal{O}_{\mathcal{Y}}[\mathcal{D}_X]$-version of (2.2.16.1)), we see that $\Theta_{\mathcal{Y}} = (p^*\Theta_{Z/X}\omega_X)_{\mathcal{D}}$. Now use 2.8.13. □

Since p has contractible fibers, the map $p^* : H^{\cdot}_{DR}(Z) \to H^{\cdot}_{DR}(\mathcal{Y})$ is an isomorphism. The image of $ch_1^{\mathcal{D}}(\Theta_{\mathcal{Y}})$ by $H^4(\mathcal{Y}, DR_X(DR^{\geq 1}_{\mathcal{Y}/X})) \to H^4_{DR}(\mathcal{Y}) \xrightarrow{p^{*-1}} H^4_{DR}(Z)$ is the conventional de Rham Chern class $ch_2(\Theta_{Z/X}\omega_X^{1/2})$ (see Remark in 2.8.12). Thus if $ch_2(\Theta_{Z/X}\omega_X^{1/2}) \in H^4_{DR}(Z)$ does not vanish, then $\mathcal{Y}$ admits no global cdo.

3.9.24. The rest of the section comprises a few comments on modules over the chiral enveloping algebras parallel to 3.7.16–3.7.19. In particular, we show that for a chiral R-tdo A the topological algebras A_x^{as} are topological chiral R_x^{as}-tdo.

Let $(R, B, \mathcal{L}, \mathcal{L}^\flat)$ be as in 3.9.6. For a $\mathcal{D}_X$-module M a *chiral* $(B, \mathcal{L}^\flat)$*-module structure*, or *chiral* $(B, \mathcal{L}^\flat)$*-action*, on M is a chiral action $\mu_{\mathcal{L}^\flat M} \in P_2^{ch}(\{\mathcal{L}^\flat, M\}, M)$ of the Lie* algebra $\mathcal{L}^\flat$ on M (see 3.7.16) such that:

(i) The restriction μ_{BM} of $\mu_{\mathcal{L}^\flat M}$ to $B \subset \mathcal{L}^\flat$ is a unital chiral B-module structure on M.

(ii) The operations $\mu_{\mathcal{L}^\flat M}(\mu_{R\mathcal{L}^\flat}, id_M)$ and $\mu_{RM}(id_R, \mu_{\mathcal{L}^\flat M}) - \mu_{\mathcal{L}^\flat M}(id_{\mathcal{L}^\flat}, \mu_{RM})$ $\in P_3^{ch}(\{R, \mathcal{L}^\flat, M\}, M)$ coincide. Here μ_{RM} is the restriction of μ_{BM} to R.

Denote the category of chiral $(B, \mathcal{L}^\flat)$-modules by $\mathcal{M}(X, (B, \mathcal{L}^\flat)^{ch})$.

LEMMA. *There is a canonical equivalence of categories*

$$\mathcal{M}(X, U(B, \mathcal{L})^\flat) \xrightarrow{\sim} \mathcal{M}(X, (B, \mathcal{L}^\flat)^{ch}). \tag{3.9.24.1}$$

Proof. For any $U(B, \mathcal{L})^\flat$-module M the restriction of the $U(B, \mathcal{L})^\flat$-action to $\mathcal{L}^\flat$ is an $(B, \mathcal{L}^\flat)$-module structure on M, so we have a functor $\mathcal{M}(X, U(B, \mathcal{L})^\flat) \to$

[92] Which comes from the action morphism $\mathcal{L} \to \Theta_{\mathcal{Y}}$.

$\mathcal{M}(X,(B,\mathcal{L}^\flat))$. Its invertibility follows easily from the remark in 3.7.24 since $U(B,\mathcal{L})^\flat$ is a quotient of $B \circledast U(L)^\flat$ modulo the evident relations. □

One can consider a weaker structure of the $(B,\mathcal{L}^\flat)$-*module* on a $\mathcal{D}_X$-module M which is a pair consisting of a unital B-module structure μ_{BM} and a Lie* action $\tau_{\mathcal{L}^\flat M}$ of $\mathcal{L}^\flat$ on M such that:

(i) The Lie* action of B^{Lie} on M that corresponds to μ_{BM} equals the restriction of the $\mathcal{L}^\flat$-action on B.

(ii) The restriction $\mu_{RM} \in P^{ch}(\{R,M\},M)$ of μ_{BM} to R is compatible with the Lie* actions of $\mathcal{L}^\flat$ on R and M.

Denote the category of $(B,\mathcal{L}^\flat)$-modules by $\mathcal{M}(X,(B,\mathcal{L}^\flat))$. The evident forgetful functor $\mathcal{M}(X,(B,\mathcal{L}^\flat)^{ch}) \to \mathcal{M}(X,(B,\mathcal{L}^\flat))$ admits a left adjoint functor

$$\text{(3.9.24.2)} \qquad \operatorname{Ind}: \mathcal{M}(X,(B,\mathcal{L}^\flat)) \to \mathcal{M}(X,(B,\mathcal{L}^\flat)^{ch}).$$

The proof of this fact is similar to the proof of the proposition in 3.7.15; it also provides a construction of Ind in terms of the enveloping algebra functor. As in the corollary in 3.7.18, Ind extends naturally to a pseudo-tensor functor.

For a $(B,\mathcal{L}^\flat)$-module M we define the PBW filtration on $\operatorname{Ind} M$ as the image by the chiral action of $j_*j^*U(B,\mathcal{L}^\flat)_\cdot \boxtimes M$. If M is a central R-module, then $\operatorname{gr.Ind}M$ is also a central R-module, and we have a surjective morphism $M \underset{R^\ell}{\otimes} \operatorname{Sym}_{R^\ell}\mathcal{L} \twoheadrightarrow \operatorname{gr.Ind}M$.

Exercise. If the assumptions of the PBW theorem (see 3.9.11) are satisfied, then this is an isomorphism (cf. Remark (i) in 3.7.15).

3.9.25. Let $x \in X$ be a point, $j_x : U_x \hookrightarrow X$ its complement, and suppose that on U_x we have R, a Lie* algebroid $\mathcal{L}$, and its chiral R-extension. Let us describe the topological associative algebra $U(R,\mathcal{L})^{\flat as}_x$ (see 3.6.2–3.6.7).

Consider the set $\Xi^{R\mathcal{L}^\flat}_x$ of pairs $(R_\xi,\mathcal{L}^\flat_\xi) \in \Xi^{as}_x(j_{x*}R)\times\Xi_x(j_{x*}\mathcal{L}^\flat)$ such that $\mathcal{L}^\flat_\xi$ is a chiral Lie R_ξ-subalgebroid of $j_{x*}\mathcal{L}^\flat$; i.e., $\mathcal{L}^\flat_\xi$ is a Lie* subalgebra and R_ξ-submodule of $j_{x*}\mathcal{L}^\flat$, and its action on $j_{x*}R$ preserves R_ξ (cf. 2.5.19). The corresponding R_ξ and $\mathcal{L}^\flat_\xi$ form topologies at x on, respectively, $j_{x*}R$ and $j_{x*}\mathcal{L}^\flat$ which we denote by $\Xi^{as\mathcal{L}^\flat}_x$ and Ξ^{LieR}_x. Let $R^{as\mathcal{L}^\flat}_x$ and $\hat{h}^{LieR}_x(\mathcal{L}^\flat)$ be the corresponding completions. Then $R^{as\mathcal{L}^\flat}_x$ is a commutative topological algebra and $\hat{h}^{LieR}_x(\mathcal{L}^\flat)$ is a topological Lie $R^{as\mathcal{L}^\flat}_x$-algebroid.[93] The section $1^\flat$ provides a canonical central element $1^\flat \in \hat{h}^{LieR}_x(\mathcal{L}^\flat)$ (the $1^\flat$-image of $1 \in k = \hat{h}_x(j_{x*}\omega)$) which acts trivially on $R^{as\mathcal{L}^\flat}_x$.

Let Q be any commutative topological algebra, $P^\flat$ a topological Q-algebroid (see 2.5.18), and $1^\flat \in P^\flat$ a central element which acts trivially on Q. Then $P := P^\flat/Q1^\flat$ is a topological Lie Q-algebroid. The *topological enveloping algebra* $U(Q,P)^\flat$ of $(Q,P^\flat)$ is the quotient of the topological enveloping algebra of the topological Lie algebra $P^\flat$ (see 3.8.17) modulo the following relations:

(i) The composition $Q \xrightarrow{\cdot 1^\flat} P^\flat \to U(Q,P)^\flat$ is a morphism of unital associative algebras.

(ii) The map $P^\flat \to U(Q,P)^\flat$ is a morphism of Q-modules; here Q acts on $U(Q,P)^\flat$ by the left product via (i).

[93] See 2.5.18 for the definition.

The topological algebra $U(Q,P)^\flat$ has a filtration $U(Q,P)^\flat_\cdot$ with $U(Q,P)^\flat_0 :=$ the closure of the image of Q, and $U(Q,P)^\flat_i :=$ the closure of the image of the map $(Q^\flat)^{\otimes i} \to U(Q,P)^\flat$, $\ell_1^\flat \otimes \cdots \otimes \ell_i^\flat \mapsto \ell_1^\flat \cdots \ell_i^\flat$; we refer to it as the PBW filtration. It is clear that $\bigcup U(Q,P)^\flat_i$ is dense in $U(Q,P)^\flat$ and $\operatorname{gr} U(Q,P)^\flat$ is a topological commutative Q-algebra topologically generated by $U(Q,P)^\flat_1$.

Taking $Q = R_x^{as\mathcal{L}^\flat}$ and $P^\flat = \hat{h}_x^{LieR}(\mathcal{L}^\flat)$, we get a filtered topological associative algebra $U^\flat_{\mathcal{L}} := U(R_x^{as\mathcal{L}^\flat}, \hat{h}_x^{LieR}(\mathcal{L}))^\flat$.

LEMMA. *There is a canonical isomorphism of filtered topological algebras*

$$U^\flat_{\mathcal{L}} \xrightarrow{\sim} U(R,\mathcal{L})_x^{\flat as}. \tag{3.9.25.1}$$

Proof. The morphism $U^\flat_{\mathcal{L}} \to U(R,\mathcal{L})_x^{\flat as}$ comes from the universality property of $U^\flat_{\mathcal{L}}$ since the morphism $\mathcal{L}^\flat \to U(R,\mathcal{L})^\flat$ yields a morphism of topological Lie algebras $\hat{h}_x^{LieR}(\mathcal{L}^\flat) \to U(R,\mathcal{L})_x^{\flat as}$ that satisfies properties (i)–(ii) above. The fact that it is an isomorphism of topological algebras follows from (3.9.24.1) just as in the proof of the proposition in 3.7.22. The strict compatibility with filtrations is evident. □

3.9.26. Consider the topological R_x^{as}-module $\hat{h}_x^R(\mathcal{L})$ (see 2.4.11). The evident continuous morphisms $R_x^{as} \to U^\flat_{\mathcal{L}0}$, $\hat{h}_x^R(\mathcal{L}) \to \operatorname{gr}_1 U^\flat_{\mathcal{L}1}$ yield a continuous morphism

$$\operatorname{Sym}_{R_x^{as}} \hat{h}_x^R(\mathcal{L}) \to \operatorname{gr}_\cdot U^\flat_{\mathcal{L}} \tag{3.9.26.1}$$

of graded topological algebras with dense image.[94] One has the following version of the PBW theorem:

PROPOSITION. *Suppose that R^ℓ is a finitely generated $\mathcal{D}_{U_x}$-algebra, $\mathcal{L}$ is a finitely generated projective $R^\ell[\mathcal{D}_{U_x}]$-module, and $\mathcal{L}^\flat$ is a chiral R-extension of $\mathcal{L}$. Then (3.9.26.1) is an isomorphism.*

Proof. (i) We can assume that $\mathcal{L}$ is a free $R^\ell[\mathcal{D}_{U_x}]$-module.

If not, choose a finitely generated projective $R^\ell[\mathcal{D}_{U_x}]$-module T such that $\tilde{\mathcal{L}} := \mathcal{L} \oplus T$ is free. Then $\tilde{\mathcal{L}}$ is a Lie* algebra with respect to a bracket whose only non-zero component is the Lie* bracket on $\mathcal{L}$; it is a Lie* algebroid in the evident way. Similarly, $\tilde{\mathcal{L}}^\flat := \mathcal{L}^\flat \oplus T$ is a chiral R-extension of $\tilde{\mathcal{L}}$; the embedding and the projection $\mathcal{L}^\flat \leftrightarrows \tilde{\mathcal{L}}^\flat$ are morphisms of chiral Lie R-algebroids. Suppose that our PBW holds for $\tilde{\mathcal{L}}^\flat$; let us show that it is also holds for $\mathcal{L}^\flat$. Indeed, the morphism $\mathcal{L}^\flat \to \tilde{\mathcal{L}}^\flat$ yields a morphism $U^\flat_{\mathcal{L}} \to U^\flat_{\tilde{\mathcal{L}}}$ of filtered topological algebras and hence a map $\operatorname{gr} U^\flat_{\mathcal{L}} \to \operatorname{gr} U^\flat_{\tilde{\mathcal{L}}}$. Due to our assumption, we can rewrite it as $\operatorname{gr} U^\flat_{\mathcal{L}} \to \operatorname{Sym}_{R_x^{as}} \hat{h}_x^R(\tilde{\mathcal{L}})$. Its image lies in the topological R_x^{as}-subalgebra generated by $\hat{h}_x^R(\mathcal{L}) \subset \hat{h}_x^R(\tilde{\mathcal{L}})$ which equals $\operatorname{Sym}_{R_x^{as}} \hat{h}_x^R(\mathcal{L})$. We have defined a morphism $\operatorname{gr} U^\flat_{\mathcal{L}} \to \operatorname{Sym}_{R_x^{as}} \hat{h}_x^R(\tilde{\mathcal{L}})$; it is evidently inverse to (3.9.26.1).

(ii) Let $\{\ell_i^\flat\}$ be a finite set of sections of $\mathcal{L}^\flat$ such that the corresponding $\ell_i \in \mathcal{L}$ freely generate $\mathcal{L}$ as an $R^\ell[\mathcal{D}_{U_x}]$-module and let t be a parameter at x which we assume (by shrinking X) to be invertible on U_x. Take any $R_\xi \in \Xi_x^{as}(j_{x*}R)$ which is a finitely generated $\mathcal{D}_X$-algebra. For an integer n denote by $L_n^\flat = L_{\xi n}^\flat$ the chiral

[94] Here $\operatorname{Sym}_{R_x^{as}} \hat{h}_x^R(\mathcal{L})$ is the topological symmetric algebra, i.e., the topological commutative R_x^{as}-algebra freely generated by the topological R_x^{as}-module $\hat{h}_x^R(\mathcal{L})$.

R_ξ-submodule of $j_{x*}\mathcal{L}^\flat$ generated by sections $\{t^n \ell_i^\flat\}$ and $1^\flat$. Let $L_n = L_{\xi n}$ be its image in $j_{x*}\mathcal{L}$ which is a free $R_\xi^\ell[\mathcal{D}_X]$-module with base $\{\ell_i\}$.

Let us show that for n sufficiently big $L_n^\flat$ is a chiral Lie R_ξ-subalgebroid of $j_{x*}\mathcal{L}^\flat$ which is a chiral R_ξ-extension of L_n.

(a) For sufficiently big n the Lie* action of $L_n \subset j_{x*}\mathcal{L}$ preserves $R_\xi \subset j_{x*}R$ (since R_ξ is finitely generated).

(b) For such n, $L_n^\flat$ is an extension of L_n by R_ξ; i.e., $L_n^\smile := L_n^\flat / R_\xi \xrightarrow{\sim} L_n$. To see this, notice that $L_n^\smile$ is the chiral R_ξ-submodule of $j_{x*}\mathcal{L}^\flat/R_\xi$ generated by the images $t^n\ell_i^\smile$ of $t^n\ell_i^\flat$. Now (a) means that the $t^n\ell_i^\smile$ are R_ζ-central sections (see 3.3.7). Thus $L_n^\smile$ is actually an $R_\zeta^\ell[\mathcal{D}_X]$-module; hence it projects isomorphically to L_n.

(c) It remains to show that for n as in (a) and sufficiently big m our $L_{m+n}^\flat$ is a Lie* subalgebra of $j_{x*}\mathcal{L}^\flat$. Equivalently, this assertion means that all the sections $[t^{m+n}\ell_i^\flat, t^{m+n}\ell_j^\flat] \in \Delta_* j_{x*}\mathcal{L}^\flat$ lie in $\Delta_* L_{m+n}^\flat$, or that their images $[t^{m+n}\ell_i^\flat, t^{m+n}\ell_j^\flat]^\smile \in \Delta_*(j_{x*}\mathcal{L}^\flat/R_\xi)$ lie in $\Delta_* L_{m+n}^\smile$. Now, since $L_n^\smile \in \Xi_x(j_{x*}\mathcal{L}^\flat/R_\xi)$, one can find $a, b \geq 0$ such that $[t^n\ell_i^\flat, t^n\ell_j^\flat]^\smile \in \Delta_\cdot(t^{-a}\Sigma R_\xi^\ell t^n \ell_i^\smile)\cdot \mathcal{D}_{X\times X}^{\leq b} \subset \Delta_*(j_{x*}\mathcal{L}^\flat/R_\xi)$ where $\mathcal{D}_{X\times X}^{\leq b}$ is the sheaf of differential operators of degree $\leq b$ on $X \times X$. It is clear that any $m \geq a+b$ will do.

(iii) Algebras R_ξ as in (ii) form a base of the topology $\Xi_x^{as}(j_{x*}R)$, $L_{\xi n}$ a base of the topology $\Xi_x^{R_\xi}(j_{x*}\mathcal{L})$ and $(R_\xi, L_{\xi n}^\flat)$ a base of $\Xi_x^{R\mathcal{L}^\flat}$. The latter fact implies that the $U(R_\xi, L_{\xi n})^\flat$ form a base of the Ξ_x^{as}-topology of $j_{x*}U(R,\mathcal{L})^\flat$, so $U(R,\mathcal{L})_x^{\flat as} = \varprojlim U(R_\xi, L_{\xi n})_x^{\flat \ell}$. According to 3.4.11 applied to R_ξ and $L_{\xi n}^\flat$, we have $\operatorname{gr} U(R,\mathcal{L})_x^{\flat as} = \operatorname{Sym}_{R_x^{as}} \hat{h}_x^R(\mathcal{L})$. Now use (3.9.25.1) to finish the proof. □

Remark. We do not know if the $R^\ell[\mathcal{D}_{U_x}]$-projectivity assumption on $\mathcal{L}$ can be weakened to the $R^\ell[\mathcal{D}_{U_x}]$-flatness assumption.

3.9.27. Let Q be a reasonable commutative topological algebra, so Θ_Q is well defined (see 2.5.21), and let $Q \to F$ be a morphism of topological algebras. We say that F is a *topological Q-tdo* if there is a commutative ring filtration $F_0 \subset F_1 \subset \cdots$ by closed subspaces of F such that F_0 is the image of Q, there is an isomorphism of topological graded Poisson algebras $\operatorname{Sym}_Q \Theta_Q \xrightarrow{\sim} \operatorname{gr} F$, and F is the completion of $\bigcup F_i$ with respect to the topology formed by left ideals I such that $I \cap F_i$ is open in F_i for each i.

Corollary. *Suppose that R is smooth and we have $\Theta_R^\flat \in \mathcal{P}^{ch}(\Theta_R)$. Then $U(R,\Theta_R)_x^{\flat as}$ is a topological R_x^{as}-tdo.*

Proof. Our $(R, \Theta_R^\flat)$ satisfy the conditions of 3.9.26. By the proposition in 2.5.21, one has $\hat{h}_x^R(\Theta_R) = \Theta_{R_x^{as}}$. We are done by (3.9.25.1) and 3.9.26. □

Set $\Theta_{R_x^{as}}^\flat := \hat{h}_x^{LieR}(\Theta_R^\flat)$. This is an R_x^{as}-extension of the topological Lie R_x^{as}-algebroid $\Theta_{R_x^{as}}$, and $U(R,\Theta_R)_x^{\flat as} = U(R_x^{as}, \Theta_{R_x^{as}})^\flat$. Notice that for any *classical* R-extension $\Theta_R^c \in \mathcal{P}^{cl}(\Theta_R)$ the completion $\Theta_{R_x^{as}}^c := \hat{h}_x^R(\Theta_R^c)$ is also an R_x^{as}-extension of the topological Lie R_x^{as}-algebroid $\Theta_{R_x^{as}}$.

Consider our extensions as mere extensions of topological R_x^{as}-modules. Those that correspond to classical R-extensions of Θ_R split, so, by the lemma in 3.9.7, the class of $\Theta_{R_x^{as}}^\flat$ does not depend on the choice of a particular chiral extension $\Theta_R^\flat \in \mathcal{P}^{ch}(\Theta_R)$.

EXERCISES. Suppose that $R^\ell = \mathrm{Sym}(\mathcal{D}^n_{U_x})$, $n \geq 1$, and $\Theta^\nu_{R^{as}_x}$ is any topological R^{as}_x-extension of the Lie R^{as}_x-algebroid $\Theta_{R^{as}_x}$. Show that if $\Theta^\nu_{R^{as}_x}$ splits as an extension of topological R^{as}_x-modules, then the enveloping algebra $U(R^{as}_x, \Theta_{R^{as}_x})^\nu$ (see 3.9.25) vanishes. In particular, the extension $\Theta^\flat_{R^{as}_x}$ is *non-trivial*, and for any $\Theta^c_R \in \mathcal{P}^{cl}(\Theta_R)$ one has $U(R^{as}_x, \Theta_{R^{as}_x})^c = 0$.

3.10. Lattice chiral algebras and chiral monoids

The lattice chiral algebras are important examples of non-commutative chiral algebras that do *not* arise naturally as enveloping algebras of Lie* algebras. They appeared as vertex algebras in [**B1**] (and, to some extent, in an earlier article [**FK**]) and played a prominent role in [**FLM**]; see [**K**] 5.4, 5.5, for a nice presentation.

From the geometric perspective, the lattice chiral algebras arise naturally from the geometry of local Picard ind-schemes, namely, from their canonical chiral monoid structure. The structure of a lattice chiral algebra at a point $x \in X$ is controlled by an appropriate Heisenberg group whose commutator pairing is defined by the Contou-Carrère symbol [**CC**] of the geometric class field theory.

The exposition below divides naturally into three parts:

(i) *The description of lattice chiral algebras in terms of θ-data.* The lattice chiral algebras are defined in 3.10.1. The commutative lattice algebras are identified with the jet algebras of principal bundles for the dual torus in 3.10.2. The parallel non-commutative objects are θ-data defined in 3.10.3; the theorem identifying lattice algebras and θ-data is stated in 3.10.4. The construction of the θ-datum corresponding to a lattice algebra is given in 3.10.6. The converse construction, which will be of use in 4.9, is presented in 3.10.8. It is based on an interpretation of a θ-datum as a line bundle on the group of divisors equipped with a certain factorization structure (see 3.10.7), which is a local version of the self-duality of the Picard variety as described in [SGA 4] Exp. XVIII 1.3, 1.5. In 3.10.9 the Lie* Heisenberg subalgebra of a lattice algebra is considered. Subsection 3.10.10 treats a particular example (called the "boson-fermion correspondence"; see, e.g., [**K**] 5.2 or [**FBZ**] 4.3). In 3.10.11 we briefly mention another procedure of reconstruction of an arbitrary lattice chiral algebra from its θ-datum which is due to Roitman [**Ro**].

It is not difficult to check that in the translation equivariant setting, lattice chiral algebras are essentially the same as lattice vertex algebras from the references above.[95]

(ii) *The Heisenberg groups.* We begin with introductory material on group ind-schemes, their super extensions, and group algebras (see 3.10.12). After a brief reminder on the Contou-Carrère symbol (we refer to [**CC**], [**D2**] 2.9, and [**BBE**] 3.1–3.3 for all details), we define the Heisenberg groups and describe their representations (see 3.10.13). In 3.10.14 we assign to a lattice algebra A and a point $x \in X$ a Heisenberg group whose twisted group algebra coincides with A^{as}_x. Thus A-modules supported at x are identified with representations of the Heisenberg group, which implies the results of Dong [**Don**]. The Heisenberg group description of A^{as}_x has an immediate twisted version (see 3.10.15).

[95] In the vertex algebra literature the lattice vertex algebras are constructed rather than defined, and the cotorsor structure from 3.10.1 is implicit.

(iii) *The chiral monoids.* These are geometric objects that underlie the construction of lattice chiral algebras from 3.10.8. We give the first definitions and explain how ind-finite chiral monoids generate chiral algebras (see 3.10.16).

Let us mention that non-ind-finite chiral monoids can also be used to construct some important chiral algebras (such as the chiral Hecke algebra). Such constructions require some extra data. This subject will not be discussed in the book.

3.10.1. Let Γ be a lattice (= a free abelian group of finite rank), $\Gamma^\vee := \mathrm{Hom}(\Gamma,\mathbb{Z})$ the dual lattice and $\langle\, ,\rangle : \Gamma\times\Gamma^\vee \to \mathbb{Z}$ the pairing, $T := \mathbb{G}_m\otimes\Gamma = \operatorname{Spec} k[\Gamma^\vee]$ and $T^\vee := \mathbb{G}_m\otimes\Gamma^\vee = \operatorname{Spec} k[\Gamma]$ the corresponding tori, $\mathfrak{t} := k\otimes\Gamma$ and $\mathfrak{t}^\vee = k\otimes\Gamma^\vee = \mathfrak{t}^*$ their Lie algebras. For $\gamma\in\Gamma$ we write the corresponding homomorphism $\mathbb{G}_m\to T$ as $f\mapsto f^{\otimes\gamma}$. Let $\mathrm{Tors}(X,T)$, $\mathrm{Tors}(X,T^\vee)$ be the Picard groupoids of T- and $T^\vee$-torsors on X.

Let $\mathcal{T}_X^\vee = \operatorname{Spec} F_X^\ell := \mathcal{J}T_X^\vee$ be the jet $\mathcal{D}_X$-scheme of the group X-scheme $T_X^\vee := T^\vee\times X$. This is a commutative group $\mathcal{D}_X$-scheme, so $F = F_X$ is a commutative and cocommutative counital Hopf chiral algebra (see 3.4.16). We will consider chiral algebras equipped with a $\mathcal{T}_X^\vee$-action (which is the same as a counital F-coaction; see 3.4.17). As in 3.4.16, we have the Picard groupoid $\mathcal{P}^{Heis}(X,\Gamma) := \mathcal{P}(X,F)$ of F-cotorsors.

Definition. An object of $\mathcal{P}^{Heis}(X,\Gamma)$ is called a *lattice Heisenberg*, or simply *lattice*, chiral algebra on X (with respect to Γ).

Remark. The Picard groupoid $\mathcal{P}^{Heis}(X,\Gamma)$ carries an evident action of the group $\mathrm{Aut}\,(\Gamma)$. Therefore for any group acting on Γ, one can consider the corresponding equivariant lattice algebras. The lattice algebras equivariant with respect to the involution $\gamma\mapsto-\gamma$ are called *symmetric* lattice algebras.

3.10.2. The commutative case. Let $\mathcal{P}^{Heis}(X,\Gamma)^0\subset\mathcal{P}^{Heis}(X,\Gamma)$ be the Picard subgroupoid of *commutative* lattice chiral algebras.[96] The functor $A\mapsto \operatorname{Spec}A^\ell$ identifies $\mathcal{P}^{Heis}(X,\Gamma)^0$ with the Picard groupoid of $\mathcal{D}_X$-scheme $\mathcal{T}_X^\vee$-torsors. According to Remark (iv) in 3.4.17, the latter objects are the same as $T_X^\vee$-torsors, so we have defined a canonical equivalence of Picard groupoids (see (3.4.17.2))

$$\mathcal{P}^{Heis}(X,\Gamma)^0 \xrightarrow{\sim} \mathrm{Tors}(X,T^\vee). \tag{3.10.2.1}$$

3.10.3. θ-data. We want to extend (3.10.2.1) to the non-commutative situation. The $T_X^\vee$-torsors are replaced by the following objects:

Definition. A *θ-datum* for Γ on X is a triple $\theta=(\kappa,\lambda,c)$ where $\kappa:\Gamma\times\Gamma\to\mathbb{Z}$ is a symmetric bilinear form, λ is a rule that assigns to each $\gamma\in\Gamma$ a super line bundle λ^γ on X, and c is a rule that assigns to each pair γ_1,γ_2 of elements of Γ an isomorphism $c^{\gamma_1,\gamma_2}:\lambda^{\gamma_1}\otimes\lambda^{\gamma_2}\xrightarrow{\sim}\lambda^{\gamma_1+\gamma_2}\otimes\omega^{\otimes\kappa(\gamma_1,\gamma_2)}$. We demand c to be associative and to satisfy a twisted commutativity property:

(i) The isomorphisms $c^{\gamma_1,\gamma_2+\gamma_3}(id_{\lambda^{\gamma_1}}\otimes c^{\gamma_2,\gamma_3})$ and $c^{\gamma_1+\gamma_2,\gamma_3}(c^{\gamma_1,\gamma_2}\otimes id_{\lambda^{\gamma_3}})$: $\lambda^{\gamma_1}\otimes\lambda^{\gamma_2}\otimes\lambda^{\gamma_3}\xrightarrow{\sim}\lambda^{\gamma_1+\gamma_2+\gamma_3}\otimes\omega^{\otimes\kappa(\gamma_1,\gamma_2)+\kappa(\gamma_2,\gamma_3)+\kappa(\gamma_1,\gamma_3)}$ coincide.

(ii) $c^{\gamma_1,\gamma_2}=(-1)^{\kappa(\gamma_1,\gamma_2)}c^{\gamma_2,\gamma_1}\sigma$ where $\sigma:\lambda^{\gamma_1}\otimes\lambda^{\gamma_2}\xrightarrow{\sim}\lambda^{\gamma_2}\otimes\lambda^{\gamma_1}$ is the commutativity constraint.

[96] If A_1, A_2 are commutative lattice algebras, then $A_1\overset{F}{\otimes}A_2$ is also commutative (as a chiral subalgebra of $A_1\otimes A_2$).

Notice that by (ii) the parity of the super line λ^γ equals $\kappa(\gamma,\gamma)$ mod2. There is a canonical identification $\mathcal{O}_X \xrightarrow{\sim} \lambda^0$ which identifies $1 \in \mathcal{O}_X$ with the unique section $1 \in \lambda^0$ such that $c^{0,0}(1\otimes 1) = 1$.

Sometimes we call $\theta = (\kappa, \lambda, c)$ a κ-*twisted* θ-datum and write it as $\theta^\kappa = (\lambda, c)$. All θ-data for Γ on X form a groupoid $\mathcal{P}^\theta(X,\Gamma)$ which is the disjoint union of the groupoids $\mathcal{P}^\theta(X,\Gamma)^\kappa$ of κ-twisted θ-data. Our $\mathcal{P}^\theta(X,\Gamma)$ is a Picard groupoid with the product $(\kappa,\lambda,c)\otimes(\kappa',\lambda',c') = (\kappa+\kappa', \lambda\otimes\lambda', c\otimes c')$ where $(\lambda\otimes\lambda')^\gamma := \lambda^\gamma \otimes {\lambda'}^\gamma$, $(c\otimes c')^{\gamma_1,\gamma_2} := c^{\gamma_1,\gamma_2}\otimes {c'}^{\gamma_1,\gamma_2}$.

Notice that $\mathcal{P}(X,\Gamma) := \mathcal{P}^\theta(X,\Gamma)^0$ is a Picard subgroupoid of $\mathcal{P}^\theta(X,\Gamma)$ and each $\mathcal{P}^\theta(X,\Gamma)^\kappa$ is a $\mathcal{P}(X,\Gamma)$-torsor (it is non-empty by the lemma below).

Since θ-data have the étale local nature, we have a sheaf of Picard groupoids $\mathcal{P}^\theta(\Gamma)$ on $X_{\acute{e}t}$, etc.

LEMMA. (i) There is a canonical equivalence of Picard groupoids

$$\mathcal{P}(X,\Gamma) \xrightarrow{\sim} \mathrm{Tors}(X, T^\vee). \tag{3.10.3.1}$$

(ii) For any symmetric bilinear form κ on Γ one has $\mathcal{P}^\theta(X,\Gamma)^\kappa \neq \emptyset$. Therefore $\mathcal{P}^\theta(X,\Gamma)^\kappa$ is a $\mathcal{P}(X,\Gamma)$-torsor and $\mathcal{P}^\theta(\Gamma)^\kappa$ is a neutral $T^\vee$-gerbe on $X_{\acute{e}t}$.

Proof. (i) Take any $\theta = (\lambda, c) \in \mathcal{P}(X,\Gamma)$. Then $N(\theta) := \sum \lambda^\gamma$ is a Γ-graded commutative $\mathcal{O}_X$-algebra (with product c) and $\operatorname{Spec} N(\theta)$ is a $T^\vee$-torsor over X. Our equivalence is $\theta \mapsto \operatorname{Spec} N(\theta)$.

(ii) Let us construct a κ-twisted θ-datum. First we choose a central super $\mathbb{G}_m$-extension Γ^ϵ of Γ such that the commutator pairing $\Gamma\times\Gamma \to \mathbb{G}_m$ equals $\gamma_1,\gamma_2 \mapsto (-1)^{\kappa(\gamma_1,\gamma_2)}$. This means that for every $\gamma\in\Gamma$ we have a super line ϵ^γ and for every γ_1,γ_2 an isomorphism $c_\epsilon^{\gamma_1,\gamma_2} : \epsilon^{\gamma_1}\otimes\epsilon^{\gamma_2} \xrightarrow{\sim} \epsilon^{\gamma_1+\gamma_2}$ such that c_ϵ is associative, i.e., $c_\epsilon^{\gamma_1,\gamma_2+\gamma_3}(id_{\epsilon^{\gamma_1}}\otimes c_\epsilon^{\gamma_2,\gamma_3}) = c_\epsilon^{\gamma_1+\gamma_2,\gamma_3}(c_\epsilon^{\gamma_1,\gamma_2}\otimes id_{\epsilon^{\gamma_3}})$, and one has $c_\epsilon^{\gamma_1,\gamma_2} = (-1)^{\kappa(\gamma_1,\gamma_2)} c_\epsilon^{\gamma_2,\gamma_1}\sigma$ where $\sigma : \epsilon^{\gamma_1}\otimes\epsilon^{\gamma_2} \xrightarrow{\sim} \epsilon^{\gamma_2}\otimes\epsilon^{\gamma_1}$ is the commutativity constraint. Such Γ^ϵ always exists.[97] Next, choose a line bundle of half-forms $\omega^{\otimes 1/2}$ (this step is superfluous if κ is even). We get $\theta^\kappa = (\lambda, c) \in \mathcal{P}^\theta(X,\Gamma)^\kappa$ where $\lambda^\gamma := (\omega^{\otimes 1/2})^{\otimes -\kappa(\gamma,\gamma)} \otimes \epsilon_\gamma$ and c is the evident product multiplied by c_ϵ. □

REMARKS. (i) Pairs (Γ, θ) where Γ is a lattice, $\theta \in \mathcal{P}^\theta(X,\Gamma)$ form naturally a category $\mathcal{P}^\theta(X)$. Namely, a morphism $\phi : (\Gamma,\theta) \to (\Gamma',\theta')$ is a pair $(\phi_\Gamma, \phi_\lambda)$ where $\phi_\Gamma : \Gamma \to \Gamma'$ is a morphism of lattices and ϕ_λ is a collection of isomorphisms $\phi_\lambda^\gamma : \lambda^\gamma \xrightarrow{\sim} {\lambda'}^{\phi_\Gamma(\gamma)}$, $\gamma\in\Gamma$, such that ϕ_Γ is compatible with κ and κ' and ϕ_λ is compatible with c and c' in the obvious way. This is a symmetric monoidal category with the operation $(\Gamma,\theta)\oplus(\Gamma',\theta') = (\Gamma\oplus\Gamma', \theta\oplus\theta')$ where $\theta\oplus\theta' = (\kappa\oplus\kappa', \lambda\otimes\lambda', c\otimes c')$.

(ii) The group $\mathrm{Aut}\,(\Gamma)$ acts on $\mathcal{P}^\theta(X,\Gamma)$ by transport of structure. So, as in the remark in 3.10.1, we can consider equivariant θ-data. For example, symmetric θ-data form a sheaf of Picard groupoids on $X_{\acute{e}t}$; its κ-components are neutral $\mu_2\otimes\Gamma^\vee$-gerbes. In particular, symmetric θ-data are rigid.

3.10.4. Here is the promised generalization of 3.10.2:

THEOREM. *There is a canonical equivalence of Picard groupoids*

$$\mathcal{P}^{Heis}(X,\Gamma) \xrightarrow{\sim} \mathcal{P}^\theta(X,\Gamma). \tag{3.10.4.1}$$

[97] To construct one, choose a morphism $v : \Gamma \to \mathbb{Z}/2$ and a bilinear pairing $r : \Gamma\times\Gamma \to \mathbb{Z}/2$ so that $\kappa(\gamma_1,\gamma_2)$ mod2 $= v(\gamma_1)v(\gamma_2) + r(\gamma_1,\gamma_2) + r(\gamma_2,\gamma_1)$. Take for Γ^ϵ the $\mathbb{G}_m$-extension of Γ defined by the 2-cocycle $(-1)^r$, considered as a super extension with parity of the line corresponding to γ equal to $v(\gamma)$.

The equivalence is compatible with the $\mathrm{Aut}(\Gamma)$-actions on the Picard groupoids, so it yields the equivalences between the Picard groupoids of equivariant objects.

Proof. It can be found in 3.10.5–3.10.8. We define the functor (3.10.4.1) in 3.10.6 (after the necessary preliminaries of 3.10.5), essentially repeating the construction of 3.10.2. The inverse functor is constructed in 3.10.8 after we reinterpret the notion of θ-datum in 3.10.7; for a different construction see 3.10.11.

3.10.5. Preliminaries on $\mathcal{T}_X^\vee$-modules. (i) The canonical projection $\mathcal{T}_X^\vee \twoheadrightarrow T_X^\vee$ is a morphism of group X-schemes, so we have an embedding of the counital Hopf $\mathcal{O}_X$-algebras $\mathcal{O}_X[\Gamma] \hookrightarrow F_X^\ell$, $\gamma \mapsto \chi_\gamma$. The constant connection on $T_X^\vee$ yields a horizontal section of the above projection $T_X^\vee \hookrightarrow \mathcal{T}_X^\vee$ which is a morphism of group schemes, so we have a morphism of the counital Hopf $\mathcal{D}_X$-algebras $F_X^\ell \twoheadrightarrow \mathcal{O}_X[\Gamma]$ left inverse to the above embedding.

Consider the category of $\mathcal{O}_X$-flat $\mathcal{O}_X$-modules N equipped with a counital F_X^ℓ-action (we call such N simply a $\mathcal{T}_X^\vee$-module). This is a tensor category: for $\mathcal{T}_X^\vee$-modules M, N their tensor product as $\mathcal{T}_X^\vee$-modules is, by definition, the $\mathcal{T}_X^\vee$-submodule $M \overset{F}{\otimes} N \subset M \otimes N$ that consists of those sections n for which the images of n by the morphisms $\delta_M \otimes id_N, id_N \otimes \delta_M : M \otimes N \to F_X^\ell \otimes M \otimes N$ coincide. The unit object for $\overset{F}{\otimes}$ equals F_X^ℓ.

Every $\mathcal{T}_X^\vee$-module N is automatically a $T_X^\vee$-module = the counital $\mathcal{O}_X[\Gamma]$-comodule (via $T_X^\vee \hookrightarrow \mathcal{T}_X^\vee$). Therefore it carries a canonical Γ-grading $N = \oplus N^\gamma$ where $N^\gamma = \{n \in N : \delta_N(n) \in F_X^{\ell\gamma} \otimes N\}$. Let $N_0 \subset N$ be the maximal submodule on which the $\mathcal{T}_X^\vee$-action factors through the projection $\mathcal{T}_X^\vee \twoheadrightarrow T_X^\vee$. One has

$$N_0^\gamma = \{n \in N : \delta_N(n) = \chi_\gamma \otimes n\}. \tag{3.10.5.1}$$

The functor $N \mapsto N_0$ commutes with the tensor products: one has $(M \overset{F}{\otimes} N)_0^\gamma = M_0^\gamma \otimes N_0^\gamma$ (as $\mathcal{O}_X$-submodules of $M \otimes N$). In particular, if N is invertible with respect to $\overset{F}{\otimes}$, then all N_0^γ are line bundles on X.

(ii) It is convenient to keep in mind the dual $F^{\ell *}$-module picture:

Set $F_X^{\ell *} := \mathcal{H}om_{\mathcal{O}_X}(F_X^\ell, \mathcal{O}_X)$. The coalgebra structure on F_X^ℓ defines a topological $\mathcal{O}_X$-algebra structure on $F_X^{\ell *}$. The topology admits a base of open ideals $\mathcal{I}_\alpha$ such that $F_X^{\ell *}/\mathcal{I}_\alpha$ are locally free $\mathcal{O}_X$-modules of finite rank.[98] Therefore the ind-scheme $\mathrm{Spec}\, F_X^{\ell *} := \bigcup \mathrm{Spec}\, F_X^{\ell *}/\mathcal{I}_\alpha$ is the inductive limit of subschemes finite and flat over X. It is also a commutative group ind-X-scheme (the product comes from the algebra structure on F_X^ℓ) equipped with a connection along X (defined by the connection on F_X^ℓ). The above $T_X^\vee \rightleftarrows \mathcal{T}_X^\vee$ yield a projection $\mathrm{Spec}\, F_X^{\ell *} \twoheadrightarrow \Gamma_X$ and its section $\Gamma_X \xrightarrow{\sim} (\mathrm{Spec}\, F_X^{\ell *})_{red} \hookrightarrow \mathrm{Spec}\, F_X^{\ell *}$.

For an $\mathcal{O}_X$-module N, a $\mathcal{T}_X^\vee$-action $\delta_N : N \to F_X^\ell \otimes N$ amounts to a discrete $F_X^{\ell *}$-module structure on N. The tensor product $M \overset{F}{\otimes} N$ is formed by sections of the $F_X^{\ell *} \hat{\otimes} F_X^{\ell *}$-module $M \otimes N$ that are supported (scheme-theoretically) on the diagonal $\mathrm{Spec}\, F_X^{\ell *} \hookrightarrow \mathrm{Spec}\, F_X^{\ell *} \underset{X}{\times} \mathrm{Spec}\, F_X^{\ell *}$. The submodules N_0 and N^γ of N are formed by sections supported, respectively, on $(\mathrm{Spec}\, F_X^{\ell *})_{red}$ and the connected component of $\mathrm{Spec}\, F_X^{\ell *}$ labeled by γ.

For a geometric interpretation of $\mathrm{Spec}\, F_X^{\ell *}$ see 3.10.8.

[98] Our F_X^ℓ is locally free as an $\mathcal{O}_X$-module.

3.10.6. From latice algebras to θ-data. Let A be any lattice chiral algebra. According to 3.10.5, A carries a Γ-grading $A = \oplus A^\gamma$. One checks in a moment that it is compatible with the chiral product: μ_A sends $j_*j^*A^{\gamma_1} \boxtimes A^{\gamma_2}$ to $\Delta_* A^{\gamma_1+\gamma_2}$.

By 3.10.5, for each $\gamma \in \Gamma$ we have a (super) line bundle $\lambda_A^\gamma := A_0^{\ell\gamma} \subset A^{\ell\gamma}$. For $\gamma_1, \gamma_2 \in \Gamma$ let $\kappa_A(\gamma_1, \gamma_2)$ be the order of zero at the diagonal of the ope morphism $\circ_A$ (see 3.5.10, 3.5.11) restricted to the line bundle $\lambda_A^{\gamma_1} \boxtimes \lambda_A^{\gamma_2} \subset A^\ell \boxtimes A^\ell$.[99] In other words, $\kappa_A(\gamma_1, \gamma_2)$ is the largest integer n such that $\mu_A : j_*j^*A \boxtimes A \to \Delta_* A$ vanishes on $\lambda_A^{\gamma_1} \boxtimes \lambda_A^{\gamma_2}(n\Delta)$. The top Taylor coefficient of $\circ_A$ is then a non-zero morphism of $\mathcal{O}_X$-modules $c_A^{\gamma_1,\gamma_2} : \lambda_A^{\gamma_1} \otimes \lambda_A^{\gamma_2} \to A^\ell \otimes \omega^{\otimes \kappa_A(\gamma_1,\gamma_2)}$.

By (3.10.5.1) (and the fact that δ_A is a morphism of chiral algebras), $c_A^{\gamma_1,\gamma_2}$ takes values in $\lambda_A^{\gamma_1+\gamma_2} \otimes \omega^{\otimes \kappa_A(\gamma_1,\gamma_2)} \subset A^\ell \otimes \omega^{\otimes \kappa_A(\gamma_1,\gamma_2)}$.

By 3.10.5(i), our picture is compatible with the $\overset{F}{\otimes}$-product of lattice algebras. Namely, for two lattice algebras A, B the λ^γ line for $A \overset{F}{\otimes} B$ equals $\lambda_A^\gamma \otimes \lambda_B^\gamma$ as a submodule of $A^\ell \otimes B^\ell$. Due to compatibility of the ope with tensor products (see 3.5.11), κ and c for $A \overset{F}{\otimes} B$ are equal to, respectively, $\kappa_A + \kappa_B$ and $c_A \otimes c_B$. In particular, for B equal to the $\overset{F}{\otimes}$-inverse to A, one has $\kappa_A + \kappa_B = 0$ and $c_A \otimes c_B = 1$, which implies that $\kappa_A(\gamma_1, \gamma_2) < \infty$ and $c_A^{\gamma_1,\gamma_2} : \lambda_A^{\gamma_1} \otimes \lambda_A^{\gamma_2} \to \lambda_A^{\gamma_1+\gamma_2} \otimes \omega^{\otimes \kappa_A(\gamma_1,\gamma_2)}$ is always an isomorphism.

LEMMA. *$(\kappa_A, \lambda_A, c_A)$ is a θ-datum.*

Proof. For $\gamma_1, \gamma_2, \gamma_3 \in \Gamma$ consider the restriction of the triple ope (see 3.5.12) to the line bundle $A_0^{\ell\gamma_1} \boxtimes A_0^{\ell\gamma_2} \boxtimes A_0^{\ell\gamma_3}$. At each diagonal $x_i = x_j$ it has zero of order $\kappa_A(\gamma_i, \gamma_j)$. On the other hand, its top term at, say, the diagonal $x_1 = x_2$ is equal to $\circ_A(c_A(\gamma_1, \gamma_2), id_{A_0^{\ell\gamma_3}})$, so it has zero of order $\kappa_A(\gamma_1 + \gamma_2, \gamma_3)$. Therefore $\kappa_A(\gamma_1 + \gamma_2, \gamma_3) = \kappa_A(\gamma_1, \gamma_3) + \kappa_A(\gamma_2, \gamma_3)$, so κ_A is bilinear. It is obviously symmetric. The associativity of c_A follows from the associativity property of the ope. The twisted commutativity of c_A follows from the commutativity of the ope, since the transposition symmetry acts on the subquotient $A^\ell \otimes \omega^{\kappa_A(\gamma_1,\gamma_2)}$ of $\tilde{\Delta}_* A^\ell$ as multiplication by $(-1)^{\kappa_A(\gamma_1,\gamma_2)}$. □

We define the Picard functor $\mathcal{P}^{Heis}(X, \Gamma) \to \mathcal{P}^\theta(X, \Gamma)$ from (3.10.4.1) as $A \mapsto \theta_A := (\kappa_A, \lambda_A, c_A)$. Its restriction to $\mathcal{P}^{Heis}(X, \Gamma)^0$ is the composition of (3.10.2.1) and the inverse to (3.10.3.1), so our functor is fully faithful. To prove that our functor is an equivalence, it suffices to construct its right inverse. We will do this in 3.10.8 after reinterpreting the notion of θ-datum in 3.10.7.

3.10.7. Denote by $\mathcal{D}iv(X)$ a functor which assigns to a quasi-compact scheme Z the group of Cartier divisors in $X \times Z/Z$ proper over Z. This is a sheaf with respect to the flat topology. Set $\mathcal{D}iv(X, \Gamma) := \mathcal{D}iv(X) \otimes \Gamma$; we call its points Γ-*divisors*.

Let λ be a super line bundle on $\mathcal{D}iv(X, \Gamma)$. Therefore λ is a rule that assigns to any Z and a Γ-divisor $D \in \mathcal{D}iv(X, \Gamma)(Z)$ a super line bundle λ_D on Z in a way compatible with the base change.

DEFINITION. A *factorization structure* on λ is a rule that assigns to any Z and a pair of Γ-divisors $D_1, D_2 \in \mathcal{D}iv(X, \Gamma)(Z)$ whose supports do not intersect an

[99] We will see in a moment that this morphism is non-zero, i.e., $\kappa_A(\gamma_1, \gamma_2) \neq \infty$.

isomorphism $c_{D_1,D_2} : \lambda_{D_1} \otimes \lambda_{D_2} \xrightarrow{\sim} \lambda_{D_1+D_2}$. We demand c to be compatible with the base change and to be associative and commutative:

(i) For every $D_1, D_2, D_3 \in \mathcal{D}iv(X,\Gamma)(Z)$ whose supports are pairwise disjoint, the two morphisms $c_{D_1+D_2,D_3}(c_{D_1,D_2} \otimes id_{\lambda_{D_3}})$ and $c_{D_1,D_2+D_3}(id_{\lambda_{D_3}} \otimes c_{D_1,D_2})$: $\lambda_{D_1} \otimes \lambda_{D_2} \otimes \lambda_{D_3} \xrightarrow{\sim} \lambda_{D_1+D_2+D_3}$ coincide.

(ii) One has $c_{D_1,D_2} = c_{D_2,D_1}\sigma$, where $\sigma : \lambda_{D_1} \otimes \lambda_{D_2} \xrightarrow{\sim} \lambda_{D_2} \otimes \lambda_{D_1}$ is the commutativity constraint.

Pairs (λ, c) as above form naturally a Picard groupoid (the tensor product is $(\lambda_1, c_1) \otimes (\lambda_2, c_2) := (\lambda_1 \otimes \lambda_2, c_1 \otimes c_2)$); denote it by $\mathrm{Pic}^f(\mathcal{D}iv(X,\Gamma))$.

PROPOSITION. *There is a natural equivalence of Picard groupoids*

$$Pic^f(\mathcal{D}iv(X,\Gamma)) \xrightarrow{\sim} \mathcal{P}^\theta(X,\Gamma). \tag{3.10.7.1}$$

Proof. For $\gamma \in \Gamma$ let $i^\gamma : X \to \mathcal{D}iv(X,\Gamma)$ be the morphism $x \mapsto (x) \otimes \gamma$; here (x) is the Cartier divisor that corresponds to x.

For $(\lambda, c) \in \mathrm{Pic}^f(\mathcal{D}iv(X,\Gamma))$ the corresponding $\theta = (\kappa, \lambda^\gamma, c^{\gamma_1,\gamma_2}) \in \mathcal{P}^\theta(X,\Gamma)$ is defined as follows. We set $\lambda^\gamma := i^{\gamma *}\lambda$. For $\gamma_1, \gamma_2 \in \Gamma$ let $\lambda^{\gamma_1,\gamma_2}$ be the pull-back of λ by the morphism $i^{\gamma_1,\gamma_2} : X \times X \to \mathcal{D}iv(X,\Gamma)$, $(x_1, x_2) \mapsto i^{\gamma_1}(x_1) + i^{\gamma_2}(x_2)$. On the complement U to the diagonal Δ our c yields an isomorphism $\lambda^{\gamma_1} \boxtimes \lambda^{\gamma_2}|_U \xrightarrow{\sim} \lambda^{\gamma_1,\gamma_2}|_U$. So we have an isomorphism $\lambda^{\gamma_1} \boxtimes \lambda^{\gamma_2} \xrightarrow{\sim} \lambda^{\gamma_1,\gamma_2}(-\kappa(\gamma_1,\gamma_2)\Delta)$ for certain $\kappa(\gamma_1,\gamma_2) \in \mathbb{Z}$. Pulling it back to the diagonal, we get an isomorphism c^{γ_1,γ_2} : $\lambda^{\gamma_1} \otimes \lambda^{\gamma_2} \xrightarrow{\sim} \lambda^{\gamma_1+\gamma_2} \otimes \omega^{\kappa(\gamma_1,\gamma_2)}$ of super line bundles on X.

Let us check that $\theta = (\kappa, \lambda^\gamma, c^{\gamma_1,\gamma_2})$ so defined is, indeed, a θ-datum. The commutativity property of c^{γ_1,γ_2} is evident. Now notice that for any $I \in \mathcal{S}$ and $(\gamma_i) \in \Gamma^I$ we have a super line bundle $\lambda^{(\gamma_i)}$ on X^I defined as the pull-back of λ by the map $i^{(\gamma_i)} : X^I \to \mathcal{D}iv(X,\Gamma)$, $(x_i) \mapsto \sum i^{\gamma_i}(x_i)$, and c yields an isomorphism

$$\boxtimes\lambda^{\gamma_i} \xrightarrow{\sim} \lambda^{(\gamma_i)}(-\sum \kappa(\gamma_i,\gamma_{i'})\Delta_{i,i'}) \tag{3.10.7.2}$$

(the summation is over the set of unordered pairs $i \neq i' \in I$ and $\Delta_{i,i'}$ is the diagonal divisor $x_i = x_{i'}$). One checks the bilinearity of κ and the associativity of c^{γ_1,γ_2} looking at (3.10.7.2) on $X \times X \times X$.

The functor $\mathrm{Pic}^f(\mathcal{D}iv(X,\Gamma)) \to \mathcal{P}^\theta(X,\Gamma)$ we have constructed is evidently a morphism of Picard groupoids. It is faithful: this follows from (3.10.7.2) since any point of $\mathcal{D}iv(X,\Gamma)$ factors flat locally through $i^{(\gamma_i)}$ for some $(I,(\gamma_i))$. It remains to prove that our functor is essentially surjective.

Consider the Picard groupoid of extensions $\mathrm{Ext}(\mathcal{D}iv(X,\Gamma),\mathbb{G}_m)$. Such an extension is the same as a pair (λ, c) as above except that c_{D_1,D_2} are defined for every D_1, D_2 (the supports may intersect). So we have an evident morphism of Picard groupoids $\mathrm{Ext}(\mathcal{D}iv(X,\Gamma),\mathbb{G}_m) \to \mathrm{Pic}^f(\mathcal{D}iv(X,\Gamma))$. Its composition with (3.10.7.1) takes values in $\mathcal{P}(X,\Gamma) = \mathrm{Tors}(X, T^\vee)$ (see (3.10.3.1)). The functor

$$\mathrm{Ext}(\mathcal{D}iv(X,\Gamma),\mathbb{G}_m) \to \mathrm{Tors}(X, T^\vee) \tag{3.10.7.3}$$

is an equivalence. To see this, notice that our objects are Γ-linear, so we can assume that $\Gamma = \mathbb{Z}$ and our functor is $\mathrm{Ext}(\mathcal{D}iv(X),\mathbb{G}_m) \to \mathrm{Pic}(X)$. Now the inverse functor assigns to $\mathcal{L} \in \mathrm{Pic}(X)$ the line bundle $\lambda_D := \mathrm{Nm}_{D/Z}(p_X^*\mathcal{L})$ on $\mathcal{D}iv(X)$ with evident c; here $p_X : X \times Z \to X$ is the projection.[100]

To finish the proof, it remains to present for every symmetric bilinear form κ on Γ some $(\lambda^\kappa, c^\kappa) \in \mathrm{Pic}^f(\mathcal{D}iv(X,\Gamma))$ which yields a κ-twisted θ-datum.

Suppose for a moment that $\Gamma = \mathbb{Z}$. For $D \in \mathcal{D}iv(X)(Z)$ choose (locally on Z) some effective divisor $D' \in \mathcal{D}iv(X, Z)$ such that $D \geq -D'$. Set

$$\lambda_D := \det(p_{Z*}\mathcal{O}_{X\times Z}(D)/\mathcal{O}_{X\times Z}(-D')) \otimes \det{}^{\otimes -1} p_{Z*}\mathcal{O}_{D'}. \tag{3.10.7.4}$$

Here $p_Z : X \times Z \to Z$ is the projection, $\mathcal{O}_{D'} := \mathcal{O}_{X\times Z}/\mathcal{O}_{X\times Z}(-D')$; notice that $p_{Z*}\mathcal{O}_{X\times Z}(D)/\mathcal{O}_{X\times Z}(-D')$ and $p_{Z*}\mathcal{O}_{D'}$ are vector bundles on Z. Our λ_D is independent of the auxiliary choice of D' in the usual way, so we have defined a super line bundle λ on $\mathcal{D}iv(X)$. It carries a natural factorization structure c coming from the fact that $\mathcal{O}_{X\times Z}(D_1)/\mathcal{O}_{X\times Z}(-D_1') \oplus \mathcal{O}_{X\times Z}(D_2)/\mathcal{O}_{X\times Z}(-D_2') = \mathcal{O}_{X\times Z}(D_1 + D_2)/\mathcal{O}_{X\times Z}(-D_1' - D_2')$ and $\mathcal{O}_{D_1'} \oplus \mathcal{O}_{D_2'} = \mathcal{O}_{D_1'+D_2'}$ if our divisors do not intersect. The form κ defined by (λ, c) is the product pairing on $\mathbb{Z}$.

Suppose Γ is arbitrary. Any $\gamma^\vee \in \Gamma^\vee$ yields a bilinear form $\kappa_{\gamma^\vee} : (\gamma_1, \gamma_2) \mapsto \gamma^\vee(\gamma_1)\gamma^\vee(\gamma_2)$. Every integral symmetric bilinear form is an integral linear combination of the $\kappa_{\gamma^\vee}$'s, so we can assume that $\kappa = \kappa_{\gamma^\vee}$. Consider the pull-back of λ from (3.10.7.4) by the projections $\mathcal{D}iv(X,\Gamma) \to \mathcal{D}iv(X)$ defined by $\gamma^\vee$. The bilinear form of the corresponding θ-datum equals κ, and we are done. □

3.10.8. From θ-data to lattice algebras. To finish the proof of the theorem in 3.10.4, we will construct the functor inverse to (3.10.4.1).

Let Z be a quasi-compact scheme and S an effective Cartier divisor in $X \times Z/Z$ proper over Z. Let $\mathcal{D}iv(X)_S$ be the functor on the category of quasi-compact Z-schemes that assigns to Y/Z the subgroup $\mathcal{D}iv(X)_S(Y) \subset \mathcal{D}iv(X)(Y)$ of all Cartier divisors whose support is contained (set-theoretically) in (the pull-back of) S. Set $\mathcal{D}iv(X,\Gamma)_S = \mathcal{D}iv(X,\Gamma)_{S,Z} := \mathcal{D}iv(X)_S \otimes \Gamma$.

LEMMA. *$\mathcal{D}iv(X)_S$ is a formally smooth ind-scheme over Z which can be represented as the inductive limit of a directed family of subschemes which are finite and flat over Z. The same is true for $\mathcal{D}iv(X,\Gamma)_S$.*

Proof. Follows from the fact that the functor which assigns to a test scheme Y the set of all effective Cartier divisors in $X \times Y/Y$ of given degree n is representable by the scheme $\mathrm{Sym}^n X$ (see [SGA 4] Exp. XVII 6.3.8) which is smooth. □

Therefore $\mathcal{D}iv(X,\Gamma)_S = \varinjlim \mathrm{Spec}\, R_\alpha$ where $\{R_\alpha\}$ is a directed system of $\mathcal{O}_Z$-algebras connected by surjections which are locally free $\mathcal{O}_Z$-modules of finite rank.

REMARK. Notice that $\mathcal{D}iv(X,\Gamma)_{S,Z}$ depends only on the reduced scheme $S_{red} \subset X \times Z$, so $\mathcal{D}iv(X,\Gamma)_{S,Z}$ carries a canonical action of the universal formal groupoid on Z (whose space is the formal completion of $Z \times Z$ at the diagonal; cf. the proof of the proposition in 3.4.7). If Z is smooth, this means that the ind-Z-scheme $\mathcal{D}iv(X,\Gamma)_{S,Z}$ carries a canonical flat connection ∇.

For $\theta \in \mathcal{P}^\theta(X,\Gamma)$ let $(\lambda, c) \in \mathrm{Pic}^f(\mathcal{D}iv(X,\Gamma))$ be the super line bundle with factorization structure that corresponds to θ by (3.10.7.1). For any (S, Z) the pull-back of λ to $\mathcal{D}iv(X,\Gamma)_S$ can be seen as a projective system of invertible R_α-modules

[100] Recall that for effective D the norm functor $\mathrm{Nm}_{D/Z} : \mathrm{Pic}\,(X \times Z) \to \mathrm{Pic}\,(Z)$ comes from the norm homomorphism $p_{Z*}\mathcal{O}_D^\times \to \mathcal{O}_Z^\times$; one extends it to the whole $\mathcal{D}iv(X)$ demanding $\mathrm{Nm}_{D/Z}$ to be multiplicative with respect to D. For details see [SGA 4] Exp. XVII 6.3, Exp. XVIII 1.3.5.

$\lambda_{S,Z} = \varinjlim \lambda_{R_\alpha}$. Therefore we have a locally projective[101] $\mathcal{O}_Z$-module

$$A^\ell_{\theta\, S,Z} := \mathcal{H}om_{\mathcal{O}_Z}(\lambda^*_{S,Z}, \mathcal{O}_Z) = \bigcup \lambda_{R_\alpha} \underset{R_\alpha}{\otimes} \mathcal{H}om_{\mathcal{O}_Z}(R_\alpha, \mathcal{O}_Z). \tag{3.10.8.1}$$

It is clear that $A^\ell_{\theta\, S,Z}$ depends on $(S,Z) \in \mathcal{C}(X)$ (we use the notation from 3.4.6) in a functorial way; i.e., they form a structure from (i) in 3.4.6. The factorization structure c on λ yields in the obvious way a factorization structure (see (ii) in 3.4.6) on A^ℓ_θ which we denote also by c. The pair (A^ℓ_θ, c) evidently satisfies properties (a) and (b) in loc. cit., so it is a factorization algebra. We denote it simply by A^ℓ_θ or $A^\ell_{(\Gamma,\theta)}$. Notice that for $\theta \in \mathcal{P}(X,\Gamma) \subset \mathcal{P}^\theta(X,\Gamma)$ our A^ℓ_θ is commutative.

PROPOSITION. *A^ℓ_θ is naturally a lattice chiral algebra whose θ-datum identifies canonically with θ.*

Proof. Notice that $\mathcal{P}^\theta(X) \to \mathcal{FA}(X)$, $(\Gamma,\theta) \mapsto A^\ell_{(\Gamma,\theta)}$, is naturally a tensor functor (we use the notation from Remark (i) in 3.10.3). Let $1 = 1_\Gamma \in \mathcal{P}^\theta(X,\Gamma)$ be the unit θ-datum. Looking at the "diagonal" morphisms $(\Gamma,1) \to (\Gamma,1)\oplus(\Gamma,1)$ and $(\Gamma,\theta) \to (\Gamma,1)\oplus(\Gamma,\theta)$ in$\mathcal{P}^\theta(X)$, we get the morphisms of factorization algebras $\delta : A^\ell_1 \to A^\ell_1 \otimes A^\ell_1$ and $\delta_\theta : A^\ell_\theta \to A^\ell_1 \otimes A^\ell_\theta$. The first arrow is a coproduct which makes A^ℓ_1 a commutative and cocommutative counital Hopf factorization algebra.[102] The second arrow is a counital coaction of A^ℓ_1 on A^ℓ_θ.

The diagonal morphism $(\Gamma,1) \to (\Gamma,\theta) \oplus (\Gamma,\theta^{\otimes -1})$ yields a morphism $A^\ell_1 \to A^\ell_\theta \overset{A_1}{\otimes} A^\ell_{\theta^{\otimes-1}} \subset A^\ell_\theta \otimes A^\ell_{\theta^{\otimes-1}}$ (see 3.4.16 for the notation). One easily checks that it is an isomorphism, so A_θ is an A_1-cotorsor.

To see that A_θ is a lattice chiral algebra, it remains to identify A_1 and F, i.e., to define an isomorphism of the Hopf $\mathcal{D}_X$-algebras $A^\ell_{1\,X} \xrightarrow{\sim} F^\ell_X$. There is a natural pairing of group $\mathcal{D}_X$-ind-schemes

$$\langle\ \rangle : \mathcal{D}iv(X,\Gamma)_{\Delta,X} \times \mathcal{T}^\vee_X \to \mathbb{G}_{mX}, \tag{3.10.8.2}$$

$\langle D\otimes\gamma, f^{\otimes\gamma^\vee}\rangle := \mathrm{Nm}_{D/Z}(f)^{\langle\gamma,\gamma^\vee\rangle}$. Here $\Delta \subset X\times X$ is the diagonal divisor, Z is a test X-scheme, $D \in \mathcal{D}iv(X)_\Delta(Z)$, and f is a section of $\mathcal{J}\mathbb{G}_{mX}$ realized, as in 2.3.3, as an invertible function of the formal neighborhood of the pull-back of Δ in $X\times Z$. An immediate computation shows that the morphism of Hopf $\mathcal{D}_X$-algebras $A^\ell_{1\,X} \to F^\ell_X$ that corresponds to (3.10.8.2) is an isomorphism.

It is clear from the construction that the θ-datum that corresponds to the lattice chiral algebra A_θ equals θ. We are done. □

3.10.9 The Heisenberg Lie* subalgebra. We use the notation from 3.10.5 and 3.10.6. For $\gamma \in \Gamma$ set $\alpha(\gamma) := d\chi_\gamma/\chi_\gamma \in F^0 = F^0_X$; we extend it to $\alpha : \mathfrak{t}_{\mathcal{D}} \to F^0$ by $\mathcal{D}$-linearity. Our α takes values in the submodule of Lie algebra elements with respect to the coalgebra structure, and it yields an isomorphism of $\mathcal{D}_X$-algebras $\mathrm{Sym}\, \mathfrak{t}^\ell_{\mathcal{D}} \xrightarrow{\sim} F^{\ell 0}$.

Let A be any lattice chiral algebra and $\theta = (\kappa,\lambda,c) \in \mathcal{P}^\theta(X,\Gamma)$ its θ-datum. Consider the composition $A^\ell \boxtimes A^\ell \xrightarrow{\circ_A} \Delta^{\sim}_* A^\ell \twoheadrightarrow \Delta^{\sim}_*(A^\ell/\mathcal{O}_X 1_A)$. For any $\gamma \in \Gamma$ its restriction to $\lambda^\gamma \boxtimes \lambda^{-\gamma}$ has zero of order $-\kappa(\gamma,\gamma)+1$ at the diagonal. The top coefficient is a morphism $\lambda^\gamma \otimes \lambda^{-\gamma} \to (A^{0\ell}/\mathcal{O}_X 1_A)\otimes \omega^{\otimes -\kappa(\gamma,\gamma)+1}$; since the

[101] In fact, it is locally free.

[102] The counit $A^\ell_1 \to \mathcal{O}$ comes from the (unique) morphism $(\Gamma, 1_\Gamma) \to (0, 1_0)$ in $\mathcal{P}^\theta(X)$.

left-hand side is identified, via $c^{\gamma,-\gamma}$, with $\omega^{\otimes-\kappa(\gamma,\gamma)}$, we can consider it as a section $\alpha^\theta(\gamma)$ of $A/\omega 1_A$.

Consider the coaction morphism $\delta_A : A^\ell \to F^\ell \otimes A^\ell$. For $l_\gamma \in \lambda^\gamma$ one has $\delta_A(l_\gamma) = \chi_\gamma \otimes l_\gamma$, so, looking at the ope, we see that

$$\delta_A(\alpha^\theta(\gamma)) = \alpha(\gamma)\otimes 1_A + 1_F \otimes \alpha^\theta(\gamma) \in F^\ell \otimes A/\omega 1_{F\otimes A}. \tag{3.10.9.1}$$

Since A is an invertible F^ℓ-comodule, this property determines the section $\alpha^\theta(\gamma)$ uniquely. In particular, it depends on γ in the additive way. We extend α^θ to a morphism of $\mathcal{D}_X$-modules $\mathfrak{t}_\mathcal{D} \to A/\omega 1_A$. Let $\mathfrak{t}^\theta_\mathcal{D} \to A$ be the corresponding morphism of the ω-extensions; we denote it also by α^θ.

PROPOSITION. (i) *The image of* $\mathfrak{t}_\mathcal{D}$ *is a commutative Lie* subalgebra of* $A/\omega 1_A$, *so* $\mathfrak{t}^\theta_\mathcal{D}$ *is a central extension of the commutative Lie* algebra* $\mathfrak{t}_\mathcal{D}$. *The Lie* bracket* $\mathfrak{t}_\mathcal{O}\times\mathfrak{t}_\mathcal{O} \to h(\omega)$ *is the Heisenberg bracket* $(\gamma_1\otimes f_1, \gamma_2\otimes f_2)\mapsto \kappa(\gamma_1,\gamma_2)f_1 df_2$ *(see 2.5.9).*

(ii) *The adjoint action of* $h(\mathfrak{t}_\mathcal{D}) = \mathfrak{t}_\mathcal{O}$ *on any* $\lambda^\eta \subset A^\ell$, $\eta\in\Gamma$, *is multiplication by the character* $\kappa(\eta) := \kappa(\eta,\cdot)\in\Gamma^\vee$.

(iii) *For* $l\in\lambda^\eta$ *the image of* $\nabla(l)$ *in* $A/\omega\lambda^\eta$ *coincides with the 0th term of the ope expansion* $l\circ\alpha^\theta(\eta)$.[103]

(iv) *As an* $\mathcal{O}_X$*-module equipped with chiral* $U(\mathfrak{t}_\mathcal{D})^\theta$*-action,* A *is equal to the induced* $U(\mathfrak{t}_\mathcal{D})^\theta$*-module* $Ind^\theta A_0$ *where* $\mathfrak{t}^\theta_\mathcal{D}$ *acts on* $A_0 = \sum\lambda^\eta$ *according to* (ii).[104] *In particular,* $\alpha^\theta : U(\mathfrak{t}_\mathcal{D})^\theta \xrightarrow{\sim} A^0$.

(v) *The extension* $\mathfrak{t}^\theta_\mathcal{D}$ *depends on* θ *in the additive way, i.e.,* $\theta\mapsto\mathfrak{t}^\theta_\mathcal{D}$ *is a Picard functor. The automorphisms of* θ *act on* $\mathfrak{t}^\theta_\mathcal{D}$ *according to the homomorphism* $T^\vee(\mathcal{O}_X) = \mathcal{O}_X^\times\otimes\Gamma^\vee \to Aut\,\mathfrak{t}^\theta_\mathcal{D} = \mathfrak{t}^*\otimes\omega$, $f^{\otimes\gamma^\vee}\mapsto\gamma^\vee\otimes df/f$.

Proof. We check (ii) first. Let x be a local parameter on X. For $\gamma,\eta\in\Gamma$ choose generators $l_\eta\in\lambda^\eta$ and $l_{\pm\gamma}\in\lambda^{\pm\gamma}$ such that $c(l_\gamma,l_{-\gamma}) = 1_A dx^{-\kappa(\gamma,\gamma)}$. Consider the triple ope $\phi := (x_1-x_2)^{\kappa(\gamma,\gamma)} l_\gamma(x_1)l_{-\gamma}(x_2)l_\eta(x_3) \in \tilde{\Delta}^{(3)}_* A$ (see 3.5.12). The orders of zero of the double ope are known by 3.10.6, so one has $\phi = (\frac{x_1-x_3}{x_2-x_3})^{\kappa(\gamma,\eta)}\psi$ where $\psi\in A_{x_3}[[x_1-x_2,x_2-x_3]]$. Let $\psi_0\in A_{x_3}$ be the constant term of the Taylor power series ψ. Consider ϕ as an element of $A_{x_3}((x_2-x_3))((x_1-x_2))$; then $\phi = (1+\frac{x_1-x_2}{x_2-x_3})^{\kappa(\gamma,\eta)}\psi = (1+\kappa(\gamma,\eta)\frac{x_1-x_2}{x_2-x_3})\psi_0$ modulo $(x_1-x_2)P$ where $P := A_{x_3}[[x_2-x_3]] + (x_1-x_2)A_{x_3}((x_2-x_3))[[x_1-x_2]]$. Computing ϕ as $(x_1-x_2)^{\kappa(\gamma,\gamma)}(l_\gamma(x_1)l_{-\gamma}(x_2))l_\eta(x_3)$, we see that $(1+\kappa(\gamma,\eta)\frac{x_1-x_2}{x_2-x_3})\psi_0 = l_\eta(x_3) + (x_1-x_2)\alpha^\theta(\gamma)(x_2)l_\eta(x_3)$ modulo P. Therefore $\psi_0 = l_\eta$ and $\alpha^\theta(\gamma)(x_2)l_\eta(x_3) = \kappa(\gamma,\eta)(x_2-x_3)^{-1}l_\eta(x_3)+$ regular terms, which is (ii).

To check (i), we need to compute the polar part of the ope product of $\alpha^\theta(\gamma)$ and $\alpha^\theta(\eta)$. Consider the ope $\mu := (x_1-x_2)^{\kappa(\gamma,\gamma)}(x_3-x_4)^{\kappa(\eta,\eta)} l_\gamma(x_1)l_{-\gamma}(x_2)l_\eta(x_3)l_{-\eta}(x_4)$. It equals $(\frac{(x_1-x_3)(x_2-x_4)}{(x_1-x_4)(x_2-x_3)})^{\kappa(\gamma,\eta)}\nu$ where ν is a Taylor power series with the constant term 1_A. As an element of $A_{x_4}((x_2-x_4))[[x_1-x_2,x_3-x_4]]$, our μ is equal to $(1+\kappa(\gamma,\eta)(x_1-x_2)(x_3-x_4)(x_2-x_4)^{-2})1_A$ modulo terms which are Taylor power series or lie in the cube of the maximal ideal of $k((x_2-x_4))[[x_1-x_2,x_3-x_4]]$. On the other hand, one has $\mu := (x_1-x_2)^{\kappa(\gamma,\gamma)}(x_3-x_4)^{\kappa(\eta,\eta)}(l_\gamma(x_1)l_{-\gamma}(x_2))(l_\eta(x_3)l_{-\eta}(x_4)) =$

[103] Notice that, by (ii), the ope $l\circ\alpha^\theta(\eta)$ has pole of first order with the principal part in λ^η, so the image in $A/\omega\lambda^\eta$ of its 0th term is well defined.

[104] For the induced modules, see 3.7.15 and 3.7.21.

$(1+\alpha(\gamma)(x_2)(x_1-x_2)+\cdots)(1+\alpha(\eta)(x_4)(x_3-x_4)+\cdots)$. Thus $\alpha(\gamma)(x_2)\alpha(\eta)(x_4) = \kappa(\gamma,\eta)(x_2-x_4)^{-2}1_A$ + regular terms, and we are done.

We leave (iii) to the reader. To deduce (iv), we have to show that the canonical morphism $\mathrm{Ind}^\theta(A_0) \to A$ is an isomorphism. This follows easily from (3.10.9.1); the details are left to the reader.

The first statement in (v) is evident. Thus it suffices to check the second statement only for the trivial θ-datum, where it is immediate. □

Remarks. (i) Let $\{\gamma_i\} \subset \Gamma$ be a subset that generates Γ as a semigroup. Then $\sum \lambda^{\gamma_i}$ generates A as a chiral algebra.[105]

(ii) Suppose that A is symmetric (see Remark in 3.10.1). The action of the involution on $\mathfrak{t}^\kappa_{\mathcal{D}}$ yields then a $\mathcal{D}$-module splitting $\mathfrak{t}_{\mathcal{D}} \to \mathfrak{t}^\theta_{\mathcal{D}}$ (that identifies $\mathfrak{t}_{\mathcal{D}}$ with the -1-eigenspace of the involution). Therefore $\mathfrak{t}^\theta_{\mathcal{D}}$ is identified with the Heisenberg extension $\mathfrak{t}^\kappa_{\mathcal{D}}$ from 2.5.9.

If A is commutative, then $\mathfrak{t}^\theta_{\mathcal{D}}$ is the twist of the trivial extension by the $T^\vee$-torsor corresponding to A (see 3.10.2) according to the $T^\vee(\mathcal{O}_X)$-action described in (iv) of the above proposition.

3.10.10. The boson-fermion correspondence. Suppose that $\Gamma = \mathbb{Z}$ and κ is the product pairing. Let $\theta := (\lambda, c)$ be any κ-twisted θ-datum. Set $\lambda_+ := \lambda^{+1} \otimes \omega$ and $\lambda_- := \lambda^{-1} \otimes \omega$; these are odd line bundles and $c^{+1,-1}$ yields a pairing $c : \lambda_+ \otimes \lambda_- \xrightarrow{\sim} \omega$. Then $(\lambda_{+\mathcal{D}} \oplus \lambda_{-\mathcal{D}})^\flat := \lambda_{+\mathcal{D}} \oplus \lambda_{-\mathcal{D}} \oplus \omega$ carries a Lie* bracket with the only non-zero component being the pairing $\in P_2^*(\{\lambda_{+\mathcal{D}}, \lambda_{-\mathcal{D}}\}, \omega)$ defined by c. Thus $(\lambda_{+\mathcal{D}} \oplus \lambda_{-\mathcal{D}})^\flat$ is an ω-extension of the abelian Lie* algebra $\lambda_{+\mathcal{D}} \oplus \lambda_{-\mathcal{D}}$. Let $\mathcal{C}l_\theta$ be the corresponding twisted enveloping algebra (see 3.7.20); this is a Clifford chiral algebra (see 3.8.6).

Proposition. *There is a natural isomorphism of chiral algebras*

$$\mathcal{C}l_\theta \xrightarrow{\sim} A_\theta. \tag{3.10.10.1}$$

Proof. The canonical embeddings $\lambda^{\pm 1} \hookrightarrow A^\ell_\theta$ define the morphisms of $\mathcal{D}$-modules $\iota : \lambda_{+\mathcal{D}} \oplus \lambda_{-\mathcal{D}} \to A_\theta$ and $\iota^\flat := \iota \oplus 1_{A_\theta} : (\lambda_{+\mathcal{D}} \oplus \lambda_{-\mathcal{D}})^\flat \to A_\theta$. The latter is a morphism of Lie* algebras; it yields the promised morphism of chiral algebras $\mathcal{C}l_\theta \to A_\theta$. Our morphism is injective since $\mathcal{C}l_\theta$ is a simple chiral algebra (due to irreducibility of the Clifford modules); it is surjective by Remark (i) in 3.10.9. □

3.10.11. Remark. According to [**Ro**], an arbitrary lattice chiral algebra A can be reconstructed from its θ-datum $\theta = (\kappa, \lambda, c) \in \mathcal{P}^\theta(X, \Gamma)$ as follows.

Let $N = N(\theta)$ be the direct sum of λ^γ, $\gamma \in \Gamma$, and let $P = P(\theta)$ be the sum of $\lambda^{\gamma_1} \boxtimes \lambda^{\gamma_2}(\kappa(\gamma_1,\gamma_2)\Delta) \subset j_*j^*N \boxtimes N$. Let $\tilde{A}^\ell$ be the factorization algebra freely generated by (N, P) (see 3.4.14). So we have the universal morphisms $\tilde{\iota} : N \to \tilde{A}^\ell_X$ and $P \to \tilde{A}^\ell_{X\times X}$. Let $\tilde{\iota}^\gamma : \lambda^\gamma \to \tilde{A}^\ell_X$ be the components of $\tilde{\iota}$ and let $\tilde{c}^{\gamma_1,\gamma_2} : \lambda^{\gamma_1} \otimes \lambda^{\gamma_2} \to \tilde{A}^\ell_X \otimes \omega_X^{\otimes \kappa(\gamma_1,\gamma_2)}$ be the pull-back to the diagonal of the morphism[106] $\lambda^{\gamma_1} \boxtimes \lambda^{\gamma_2} \hookrightarrow P(-\kappa(\gamma_1,\gamma_2)\Delta) \to \tilde{A}^\ell_{X\times X}(-\kappa(\gamma_1,\gamma_2)\Delta)$. Let $A(\theta)$ be the quotient of $\tilde{A}$ modulo the ideal generated by the sections $\tilde{\iota}^0(1) - 1_{\tilde{A}} \in \tilde{A}^\ell_X$ and

[105] Indeed, the chiral subalgebra generated by $\sum \lambda^{\gamma_i}$ contains all λ^γ (as the top coefficients of appropriate ope), hence it contains the image of α^θ, and we are done by (iii) of the proposition.

[106] Recall that $\Delta^* \tilde{A}^\ell_{X\times X} = \tilde{A}^\ell_X$ (see (3.4.2.1)) and $\Delta^* \mathcal{O}_{X\times X}(n\Delta) = \omega_X^{\otimes -n}$.

$\tilde{c}^{\gamma_1,\gamma_2} - \tilde{\iota}^{\gamma_1+\gamma_2} c^{\gamma_1,\gamma_2} \in A_X^\ell \otimes \omega_X^{\otimes \kappa(\gamma_1,\gamma_2)} \otimes (\lambda^{\gamma_1})^{\otimes -1} \otimes (\lambda^{\gamma_2})^{\otimes -1}$. Let $\iota^\gamma = \iota_\theta^\gamma : \lambda^\gamma \to A(\theta)^\ell$ be the induced morphisms; they satisfy an evident universality property.

By 3.10.6 and 3.4.14 there is a unique morphism of chiral algebras $A(\theta) \to A$ which sends ι^γ to the embedding $\lambda^\gamma = A_0^{\ell\gamma} \hookrightarrow A^\ell$ (see 3.10.6). As shown in [**Ro**], this is an isomorphism.

3.10.12. Group ind-scheme basics. We are going to identify modules over a lattice algebra with representations of a certain Heisenberg group. Let us recall first the relevant definitions.

(a) Let G be an ind-affine group ind-scheme. To avoid unnecessary complications, we assume it to be the inductive limit of countably many affine schemes, so $G = \operatorname{Spf} P := \bigcup \operatorname{Spec} P_\alpha$ where P is a topological algebra, $P_\alpha = P/I_\alpha$, and I_α are open ideals that form a base of the topology. The product on G makes P a counital topological Hopf algebra.

Example. Let $x \in X(k)$ be a point on our curve and $O_x \subset K_x$ the algebra of Taylor formal power series and the field of Laurent power series. Any affine group scheme T yields a group ind-scheme that we denote (by abuse of notation) by $T(K_x)$, $T(K_x)(R) := T(R\hat{\otimes}K_x)$ (cf. 2.4.8 and 2.4.9). It contains a group subscheme $T(O_x)$, $T(O_x)(R) := T(R\hat{\otimes}O_x)$.

For a discrete vector space V a G-action on V is a rule that assigns to each test commutative algebra R an action of the group $G(R)$ on the R-module $V_R := V \otimes R$ which is natural with respect to morphisms of the R's. Equivalently, this a counital P-coaction $V \to P\hat{\otimes}V := \varprojlim(P/I_\alpha) \otimes V$. A discrete vector space equipped with a G-action is referred to as a G-module or a representation of G; these objects form a tensor abelian category Gmod.

Consider the topological dual $P^* := \bigcup P_\alpha^*$. This is an associative unital algebra. It carries a topology whose base is formed by all left ideals whose intersection with each profinite-dimensional vector space P_α^* is open. The completion is a topological associative algebra (in the sense of 3.6.1) called the *group algebra* of G; we denote it by $k[G]$. A discrete $k[G]$-module is the same as a G-module: one has Gmod= $k[G]$mod. So $k[G]$ is the topological algebra of endomorphisms of the functor which assigns to a G-module the underlying vector space.

Our $k[G]$ is naturally a counital cocommutative topological Hopf algebra. Denote by $k[G]^{\mathfrak{gr}}$ the functor which assigns to a test algebra R the group of the group-like invertible elements of the topological Hopf R-algebra $k[G]\hat{\otimes}R$. There is an evident natural group homomorphism $G \to k[G]^{\mathfrak{gr}}$, $g \mapsto \delta_g$.

Remark. The latter map need not be an isomorphism: for example, if H is a semisimple simply connected algebraic group, then $k[H(K)_x] = k$.

(b) The above picture has a twisted version:

For a group ind-scheme G its *super extension* $G^\flat$ is an extension of G by the Picard groupoid of super line bundles.[107] Explicitly, such a $G^\flat$ is a rule that assigns to any $g \in G(R)$ an invertible super R-module $\lambda_g^\flat$ in a functorial manner, and to any pair $g_1, g_2 \in G(R)$ a natural identification $c^\flat_{g_1,g_2} : \lambda^\flat_{g_1} \otimes \lambda^\flat_{g_2} \xrightarrow{\sim} \lambda^\flat_{g_1 g_2}$ which is associative in the obvious sense. We have the parity homomorphism $\epsilon^\flat : G \to \mathbb{Z}/2$, where $\epsilon^\flat_g$ is the parity of $\lambda^\flat_g$. The super extensions of G form a Picard groupoid in the obvious way.

[107] For details see, e.g., the appendix to section 2 in [**BBE**].

REMARK. In the above definition we used only the *monoidal* structure of the Picard groupoid (the commutativity constraint is irrelevant). Therefore a super extension amounts to a pair that consists of an extension of G by the Picard groupoid of line bundles and a parity homomorphism $G \to \mathbb{Z}/2$; the former structure is the same as a central $\mathbb{G}_m$-extension of G.

For commuting $g_1, g_2 \in G(R)$, we set $\{g_1, g_2\}^\flat := c^\flat_{g_1,g_2}/c^\flat_{g_2,g_1}\sigma \in R^\times$ where $\sigma : \lambda^\flat_{g_1} \otimes \lambda^\flat_{g_2} \xrightarrow{\sim} \lambda^\flat_{g_2} \otimes \lambda^\flat_{g_1}$ is the commutativity constraint. Thus $\{g, g\}^\flat = (-1)^{\epsilon^\flat_g}$. If G is commutative, then the *commutator pairing* $\{\ \}^\flat : G\times G \to \mathbb{G}_m$ is bimultiplicative.

A $G^\flat$*-module* is a super vector space V together with a rule that assigns to $g \in G(R)$ an isomorphism of super R-modules $\lambda^\flat_g \otimes V \xrightarrow{\sim} V_R$ and satisfies the usual properties; equivalently, this is an action of $G^\flat$ on V, considered as a mere group ind-scheme and a mere vector space, which changes the $\mathbb{Z}/2$-grading on V according to $\epsilon^\flat$. Denote by $G^\flat$mod the abelian category of $G^\flat$-modules.

Consider $\lambda^{-\flat} := (\lambda^\flat)^{\otimes -1}$ as a super line bundle over G. Its sections form a topological super P-module $\lambda^{-\flat}_P = \varprojlim \lambda^{-\flat}_{P_\alpha}$. The topological dual $(\lambda^{-\flat}_P)^* = \bigcup (\lambda^{-\flat}_{P_\alpha})^*$ is naturally an associative super algebra. It carries a natural topology formed by all left ideals whose intersection with each profinite-dimensional vector space $(\lambda^{-\flat}_{P_\alpha})^*$ is open. The completion is the topological associative super algebra $k[G]^\flat$ called the $\flat$*-twisted* group algebra of G. A $G^\flat$-module is the same as a discrete $k[G]^\flat$-module: one has $G^\flat\text{mod} = k[G]^\flat\text{mod}$.

The diagonal morphism $G^\flat \to G \times G^\flat$ yields a morphism of topological algebras $\delta^\flat : k[G]^\flat \to k[G]\hat{\otimes} k[G]^\flat$ which is a counital $k[G]$-coaction on $k[G]^\flat$.

(c) The Cartier duality establishes an anti-equivalence between the category of commutative affine group schemes and that of commutative group ind-finite ind-schemes.[108] It extends readily to a self-duality on the category of commutative group ind-schemes that can be represented as an extension of a group ind-finite ind-scheme by an affine group scheme. On the level of topological Hopf algebras, the Cartier duality interchanges the topological algebras of functions and the group algebras.

Let G be such a group ind-scheme and G' its Cartier dual, so $G' = \operatorname{Spf} k[G]$. The canonical morphism $G \to k[G]^{\mathfrak{gr}}$ (see (a)) is an isomorphism.

For a topological associative (super) algebra Q a counital $k[G]$-coaction $\delta_Q : Q \to k[G]\hat{\otimes} Q$ amounts to a G'-action on Q (which is a rule that assigns to a test algebra R a $G(R)$-action on the topological R-algebra $Q\hat{\otimes} R$ natural with respect to morphisms of the R's). The category of such pairs (Q, δ_Q) is naturally a tensor category with respect to the tensor product $\overset{k[G]}{\otimes}$, where $Q_1 \overset{k[G]}{\otimes} Q_2 \subset Q_1 \hat{\otimes} Q_2$ is the topological subalgebra of elements on which the G'-actions along Q_1 and along Q_2 coincide. We say that (Q, δ_Q) is a $k[G]$*-cotorsor* if it is invertible with respect to $\overset{k[G]}{\otimes}$. The category of $k[G]$-cotorsors is a naturally Picard groupoid.

LEMMA. *The above Picard groupoid identifies naturally with the Picard groupoid of super extensions of* G.

Sketch of a proof. For a super extension $G^\flat$ of G the pair $(k[G]^\flat, \delta^\flat)$ is a $k[G]$-cotorsor. Conversely, let (Q, δ_Q) be a $k[G]$-cotorsor. For $g \in G(R)$ consider the

[108] See, e.g., section 4 in Chapter II of [**Dem**].

corresponding group-like element $\delta_g \in k[G]\hat{\otimes}R$. Then $\lambda_g := \{q \in Q\hat{\otimes}R : \delta_Q(q) = \delta_g \otimes q\}$ is an invertible super R-module, and the product in Q provides a morphism $c^\flat_{g_1,g_2} : \lambda^\flat_{g_1} \otimes \lambda^\flat_{g_2} \to \lambda^\flat_{g_1 g_2}$ which is an isomorphism. This datum is the super extension $G^\flat$ that corresponds to Q. The details are left to the reader. □

3.10.13. The Heisenberg groups. We use the notation of 3.10.12; see especially the example in loc. cit.

Consider the commutative group ind-scheme $K_x^\times := \mathbb{G}_m(K_x)$; its Lie algebra equals K_x, and the connected component of the reduced ind-scheme is the group scheme $O_x^\times := \mathbb{G}_m(O_x)$. So an R-point of $K_x^\times$ is the same as an invertible element f of $K_x\hat{\otimes}R$.

Contou-Carrère defines in [**CC**][109] a canonical skew-symmetric bimultiplicative *symbol* pairing

$$\{\ \}_x : K_x^\times \times K_x^\times \to \mathbb{G}_m. \tag{3.10.13.1}$$

The corresponding pairing of the groups of k-points is the tame symbol map $(f,g) \mapsto (-1)^{v(f)v(g)}(f^{v(g)}/g^{v(f)})(x)$ (here $v(f)$ is the order of zero of f at x); the derivative with respect to the first variable is the pairing $K_x \times K_x^\times \to \mathbb{G}_a$, $(a,g) \mapsto \mathrm{Res}_x a d\log g$; the Lie algebra pairing $K_x \times K_x \to k$ is $(a,b) \mapsto \mathrm{Res}_x adb$. If $f \in O_x^\times(R) = (O_x\hat{\otimes}R)^\times$, then

$$\{f,g\}_x = \mathrm{Nm}_{\mathrm{div} g/R} f. \tag{3.10.13.2}$$

The symbol pairing is non-degenerate: it identifies $K_x^\times$ with its own Cartier dual. This means that the corresponding morphism of the group ind-schemes $K_x^\times \to \mathrm{Spf}\, k[K_x^\times]$ is an isomorphism.

Consider, as in 3.10.1, our lattice Γ and the corresponding torus $T := \Gamma \otimes \mathbb{G}_m$. Any symmetric bilinear form $\kappa : \Gamma \times \Gamma \to \mathbb{Z}$ defines a skew-symmetric pairing $\{\ \}^\kappa_x : T(K_x) \times T(K_x) \to \mathbb{G}_m$, $\{f_1^{\otimes\gamma_1}, f_2^{\otimes\gamma_2}\}^\kappa_x := \{f_1, f_2\}_x^{\kappa(\gamma_1,\gamma_2)}$.

Definition. A *Heisenberg κ-extension* is a super extension of $T(K_x)$ whose commutator pairing equals $\{\ \}_x^{-\kappa}$. A *Heisenberg extension* is a super extension which is a Heisenberg κ-extension for some κ.

The Baer product of Heisenberg κ- and κ'-extensions is a Heisenberg $(\kappa+\kappa')$-extension, so the Heisenberg extensions form a Picard groupoid $\mathcal{H}eis\, T(K_x)$. It is a disjoint union of the groupoids $\mathcal{H}eis^\kappa T(K_x)$ of the κ-extensions. So $\mathcal{H}eis^0 T(K_x)$ is the Picard subgroupoid $\mathrm{Ext}(T(K_x), \mathbb{G}_m)$ of commutative $\mathbb{G}_m$-extensions of $T(K_x)$, and each $\mathcal{H}eis^\kappa T(K_x)$ is a $\mathcal{H}eis^0 T(K_x)$-torsor (it is non-empty: see below).

Since every commutative $\mathbb{G}_m$-extension of $T(K_x)$-splits, $\mathrm{Ext}(T(K_x), \mathbb{G}_m)$ identifies naturally with the Picard groupoid of $\mathrm{Hom}(T(K_x), \mathbb{G}_m)$-torsors. The symbol pairing yields an isomorphism $\mathrm{Hom}(T(K_x), \mathbb{G}_m) = T^\vee(K_x)$, so these objects are the same as $T^\vee(K_x)$-torsors.

Let $T(K_x)^\flat$ be a Heisenberg κ-extension. Let us describe the category of $T(K_x)^\flat$-modules.

Let $Z = Z_\kappa \subset T(K_x)$ be the kernel of the $\{\ \}^\kappa_x$-pairing. So Z lies in the kernel of the parity homomorphism $\epsilon^\kappa : T(K_x) \to \mathbb{Z}/2$, $f^{\otimes\gamma} \mapsto v(f)\kappa(\gamma,\gamma) \mathrm{mod} 2$, and we

[109] See also [**D2**] 2.9 and [**BBE**] 3.1–3.3.

have the central $\mathbb{G}_m$-extension $Z^\flat \subset T(K_x)^\flat$. The Cartier dual $(Z^\flat)'$ is an extension of $\mathbb{Z}$ by Z'. Let $\operatorname{Spf} R \subset (Z^\flat)'$ be the preimage of $1 \in \mathbb{Z}$; this is a Z'-torsor.

Any $T(K_x)^\flat$-module is automatically a discrete R-module, so the category $T(K_x)^\flat$mod is an R-category.

PROPOSITION. *$T(K_x)^\flat$mod is equivalent, as an R-category, to the category Rmod of discrete R-modules.*

Proof. We will define a functor $T(K_x)^\flat\text{mod} \to R\text{mod}$ leaving the construction of the inverse to the reader. The construction (for $\kappa \neq 0$) depends on auxiliary choices. Different choices yield (non-canonically) isomorphic functors.

Choose a decomposition $\Gamma = \Gamma_0 \times \Gamma_1$ where Γ_0 is the kernel of κ; we write $T_i := \mathbb{G}_m \otimes \Gamma_i$, etc. Then $Z = Z_0 \times Z_1$ where $Z_0 = T_0(K_x)$ and Z_1 is the kernel of the morphism $T_1 \to T_1^\vee$ defined by κ which is a finite group of order $|\det(\kappa|_{\Gamma_1})|$. Then $Z_1 \subset T_1(O_x)$, so we have the embedding $Z_1^\flat \subset T_1(O_x)^\flat$ of the central $\mathbb{G}_m$-extensions. For each character $\chi : Z_1^\flat \to \mathbb{G}_m$ whose restriction to $\mathbb{G}_m \subset Z_1^\flat$ equals $id_{\mathbb{G}_m}$ choose its extension to a character $\tilde{\chi}$ of $T_1(O_x)^\flat$.

Now let V be a $T(K_x)^\flat$-module. For any χ as above let $V_{\tilde{\chi}} \subset V$ be the subspace of all vectors on which $T_1(O_x)^\flat \subset T(K_x)^\flat$ acts by the character $\tilde{\chi}$. This is a $Z^\flat$-submodule of V, hence a discrete R-module. Our functor $T(K_x)^\flat\text{mod} \to R\text{mod}$ is $V \mapsto \bigoplus_{\chi} V_{\tilde{\chi}}$. □

3.10.14. Let us return to the lattice chiral algebras. Let $\theta \in \mathcal{P}^\theta(X,\Gamma)^\kappa$ be any θ-datum. Let A_θ be the corresponding chiral lattice algebra (see (3.10.4.1)).

THEOREM. *$A^{as}_{\theta x}$ is the group algebra of a certain Heisenberg κ-extension $T(K_x)^\theta$ naturally defined by A_θ.*

REMARK. We see that the category of A_θ-modules supported at x identifies naturally with the category of $T(K_x)^\theta$-modules. By the above proposition, for a non-degenerate κ our category is semisimple with $|\det\kappa|$ isomorphism classes of irreducibles. This result (in the case when κ is even and positive definite) was first established by Dong [**Don**].

Proof. (a) If $\theta = 1$, then $A^{as}_{1x} = F^{as}_x$ is the topological Hopf algebra of functions on the ind-group $T^\vee(K_x)$ (see 2.3.2 and (2.4.9.1)). It identifies canonically with $k[T(K_x)]$ via the Cartier duality $T(K_x) \times T^\vee(K_x) \to \mathbb{G}_m$, $f^{\otimes\gamma}, g^{\otimes\gamma^\vee} \mapsto \{f,g\}_x^{\langle\gamma,\gamma^\vee\rangle}$.

(b) For an arbitrary θ the F-coaction $\delta_\theta : A^\ell_\theta \to F^\ell \otimes A^\ell_\theta$ yields, by transport of structure, an F^{as}_x-coaction on $A^{as}_{\theta x}$ (see 3.6.8).

LEMMA. *$A^{as}_{\theta x}$ is an F^{as}_x-cotorsor.*

Proof of Lemma. We want to show that the morphism $F^\ell \to A^\ell_\theta \otimes A^\ell_{\theta^{-1}}$ yields an isomorphism $F^{as}_x \xrightarrow{\sim} A^{as}_{\theta x} \overset{F^{as}_x}{\otimes} A^{as}_{\theta^{-1}x} \subset A^{as}_{\theta^{-1}x} \hat{\otimes} A^{as}_{\theta^{-1}x}$.

Let $\Xi_\theta \subset \Xi^{as}_x(A_\theta)$ be the subset of subalgebras $A_{\theta\xi}$ which are invariant with respect to the $\mathcal{T}^\vee$-action, i.e., satisfy the property $\delta_\theta(A_{\theta\xi}) \subset F^\ell \otimes A^\ell_{\theta\xi}$. This subset is cofinal in the Ξ^{as}_x-topology: indeed, for any $B \in \Xi^{as}_x(A_\theta)$ one has $B \cap \delta_\theta^{-1}(F^\ell \otimes B) \in \Xi_\theta$. Thus $A^{as}_{\theta x} = \varprojlim A^\ell_{\theta\xi x}$, $A_{\theta\xi} \in \Xi_\theta$.

In fact, the set Ξ_θ does not depend on θ. Namely, the maps $\Xi_{\theta_1} \to \Xi_{\theta_2}$, $A_{\theta_1\xi} \mapsto A_{\theta_2\xi} := \delta^{-1}(A_{\theta_1\xi} \otimes A_{\theta_2/\theta_1})$, form a transitive system of bijections between the Ξ_θ's; here $\delta : A_{\theta_1} \xrightarrow{\sim} A_{\theta_2} \overset{F}{\otimes} A_{\theta_2/\theta_1} \hookrightarrow A_{\theta_1} \otimes A_{\theta_2/\theta_1}$ is the "coproduct" morphism. So we write Ξ instead of Ξ_θ.

For an $\mathcal{O}_X$-module N equipped with a counital F^ℓ-coaction $\delta_N : N \to F^\ell \otimes N$ set $N_x^\wedge := \varprojlim_{\Xi} (\delta_\theta^{-1} F_\xi^\ell \otimes N)_x$; this is a topological vector space equipped with a counital F_x^{as}-coaction. If N is an F^ℓ-cotorsor, then $N_x^\wedge$ is an F_x^{as}-cotorsor (since all $\mathcal{O}_X$-module F^ℓ-cotorsors are equivalent Zariski locally at x, we can assume that $N = F^\ell$ where the statement is evident). Moreover, if N' is the inverse F^ℓ-cotorsor, so that we have $F^\ell \xrightarrow{\sim} N \overset{F^\ell}{\otimes} N' \subset N \otimes N'$, then $N'^\wedge_x$ is the F_x^{as}-cotorsor inverse to $N_x^\wedge$, and the corresponding map $F^\ell{}^\wedge_x \to N_x^\wedge \hat{\otimes} N'^\wedge_x$ yields an isomorphism $F_x^{as} = F^\ell{}^\wedge_x \xrightarrow{\sim} N_x^\wedge \overset{F_x^{as}}{\otimes} N'^\wedge_x \subset N_x^\wedge \hat{\otimes} N'^\wedge_x$ (by the same argument).

Now we can prove the lemma. We have seen that $A_x^{as} = A^\ell{}^\wedge_x$ as a topological vector space equipped with an F_x^{as}-action. So the last isomorphism for $N = A_\theta^\ell$, $N' = A_{\theta^{-1}}^\ell$ is the equality we wished to check; q.e.d. □

(c) By the lemma from (c) in 3.10.12, we have a super extension $T(K_x)^\theta$ of $T(K_x)$ such that $A_{\theta x}^{as} = k[T(K_x)]^\theta$. It remains to check that $T(K_x)^\flat$ is a Heisenberg κ-extension, i.e., that its commutator pairing $\{\ \}^\theta$ equals $\{\ \}_x^\kappa$. Consider the difference $< >^\kappa := \{\ \}^\theta - \{\ \}_x^\kappa$. This pairing depends only on κ (since $\{\ \}^\theta$ depends only on the isomorphism class of θ near x, which is κ) and is invariant with respect to automorphisms of K_x (for the same reason).

As follows from (3.10.9.1), the Lie algebra of $T(K_x)^\theta$ equals $\hat{h}_x(j_{x*}j_x^*\mathfrak{t}_{\mathcal{D}}^\theta) \overset{\alpha^\theta}{\hookrightarrow} A_{\theta x}^{as}$, so the Lie algebra pairing that comes from $< >^\kappa$ vanishes by (i) of the proposition in 3.10.9.

Let us show that the pairing between $T(K_x)_{red}$ and the Lie algebra $\mathfrak{t}(K_x)$ that comes from $< >^\kappa$ vanishes. We know that it vanishes on $T(O_x) \times \mathfrak{t}(K_x)$ by the above and on $T(K_x)_{red} \times \mathfrak{t}(O_x)$ by (ii) in the proposition in 3.10.9. So $< >^\kappa$ comes from a pairing $\Gamma \times \mathfrak{t}(K_x)/\mathfrak{t}(O_x) = T(K_x)_{red}/T(O_x) \times \mathfrak{t}(K_x)/\mathfrak{t}(O_x) \to k$. Consider the action of a group of homotheties on K_x (with respect to a parameter in K_x). It is trivial on Γ and has trivial coinvariants on K_x/O_x. Since it preserves $< >^\kappa$, we get the promised vanishing.

We have shown that $< >^\kappa$ vanishes on $T(K_x) \times T(K_x)^0$ where $T(K_x)^0$ is the connected component of $T(K_x)$. Therefore it comes from a pairing $\Gamma \times \Gamma = T(K_x)/T(K_x)^0 \times T(K_x)/T(K_x)^0 \to \mathbb{G}_m$. Since $< >^\kappa$ depends on κ in the additive way, it suffices to consider the case when $\gamma = \mathbb{Z}$ and κ is the product pairing. Here $\{\ \}^\theta$ and $\{\ \}_x^\kappa$ are controlled by the parity, so $< >^\kappa = 0$, and we are done. □

3.10.15. A twisted version. Suppose we have a finite group H acting on Γ, an H-equivariant lattice algebra A_θ (see Remark in 3.10.1), and an H-torsor P on $U_x := X \setminus \{x\}$. Consider the twisted chiral algebra $A_\theta(P)$ (see 3.4.17); it is an $F(P)$-cotorsor. We also have the twisted torus $T(P)$, which is a group scheme over U_x, and the ind-scheme $T(P)(K_x)$ of its K_x-points. Notice that κ, which is an H-invariant bilinear form on Γ, defines the pairing $\{\ \}_x^\kappa : T(P)(K_x) \times T(P)(K_x) \to \mathbb{G}_m$, so, as in 3.10.13, we have the notion of a Heisenberg κ-extension of $T(P)(K_x)$.

THEOREM. *$(A_\theta(P))_x^{as}$ is the group algebra of a certain Heisenberg κ-extension $T(P)(K_x)^\theta$ of $T(P)(K_x)$.*

Proof. Same as the proof of the non-twisted version (see 3.10.14). □

EXERCISE. Formulate and prove the main results of [**BaK**].

3.10.16. Chiral monoids. Let us highlight the geometric structure underlying the construction of 3.10.8. This section will not be used later in the book.

For us, "ind-algebraic space" means a functor on the category of affine schemes representable by the inductive limit of a directed system of quasi-compact algebraic spaces connected by closed embeddings. For a quasi-compact scheme Z an *ind-algebraic Z-space* is an ind-algebraic space $\mathcal{G}$ equipped with a morphism $\mathcal{G} \to Z$; denote by $\mathcal{I}_Z$ the category of these objects with morphisms being closed embeddings. All the $\mathcal{I}_Z$'s form a fibered category $\mathcal{I}$ over the category of quasi-compact schemes (with the fibered product as the pull-back functor).

Suppose we have a pair $(\mathcal{G}, c)$ where:

(i) $\mathcal{G}$ is a morphism from the fibered category $\mathcal{C}(X)$ (see 3.4.6) to $\mathcal{I}$. Thus $\mathcal{G}$ is a rule that assigns to each $S \in \mathcal{C}(X)_Z$ an ind-algebraic Z-space $\mathcal{G}_S = \mathcal{G}_{S,Z}$; for $S' \leq S$ we have a canonical closed embedding $\mathcal{G}_{S',Z} \hookrightarrow \mathcal{G}_{S,Z}$; everything is compatible with the base change.

In particular, the universal divisors yield ind-algebraic $\mathrm{Sym}^n X$-spaces $\mathcal{G}_{\mathrm{Sym}^n X}$.

(ii) c is a rule that assigns to every pair of *mutually disjoint* divisors $S_1, S_2 \in \mathcal{C}(X)_Z$ an identification $c_{S_1,S_2} : \mathcal{G}_{S_1,Z} \underset{Z}{\times} \mathcal{G}_{S_2,Z} \xrightarrow{\sim} \mathcal{G}_{S_1+S_2,Z}$. We demand that these identifications be commutative and associative in the obvious manner and that they be compatible with the natural morphisms from (i).

Assume that the following conditions hold:

(a) For every n the closure in $\mathcal{G}_{\mathrm{Sym}^n X}$ of the complement to the preimage of the discriminant divisor in $\mathrm{Sym}^n X$ equals $\mathcal{G}_{\mathrm{Sym}^n X}$. Equivalently, $\mathcal{G}_{\mathrm{Sym}^n X}$ can be represented as the inductive limit of a directed family $\{\mathcal{G}_\alpha\}$ of algebraic $\mathrm{Sym}^n X$-spaces and closed embeddings such that each $\mathcal{G}_\alpha$ has no non-zero local functions supported over the discriminant divisor of $\mathrm{Sym}^n X$.

(b) One has $\mathcal{G}_{\mathrm{Sym}^0 X} \neq \emptyset$.

(c) The ind-algebraic spaces $\mathcal{G}_{\mathrm{Sym}^n X}$ are separable.

DEFINITION. Such a $(\mathcal{G}, c)$ is called *chiral*, or *factorization*, *monoid* on X. Chiral monoids form a category which we denote by $\mathcal{CM}(X)$.

A chiral monoid is said to be *commutative* if c can be extended to a morphism $c_{S_1,S_2} : \mathcal{G}_{S_1} \underset{Z}{\times} \mathcal{G}_{S_2} \to \mathcal{G}_{S_1+S_2}$ defined for *arbitrary* $S_i \in \mathcal{C}(X)_Z$ and natural with respect to morphisms in $\mathcal{C}(X)$ (by (a) and (c) such an extension is unique, if it exists).

In many aspects chiral monoids are similar to chiral algebras; the next remarks are parallel to, respectively, Remarks (i)–(iii) in 3.4.6, subsections 3.4.7, 3.4.20, and the proposition in 3.4.6.

REMARKS. (i) The associativity and commutativity of c permits us to define for any finite family of *mutually disjoint* divisors $S_\alpha \in \mathcal{C}(X)_Z$ a canonical identification $c_{\{S_\alpha\}}$ of the fibered product of $\mathcal{G}_{S_\alpha}$ over Z and $\mathcal{G}_{\Sigma S_\alpha}$.

(ii) By (b) and (i) one has $\mathcal{G}_{\mathrm{Sym}^0 X} = \operatorname{Spec} k$. So for any (S, Z) the embedding $\emptyset \subset S$ yields a canonical section $1_{\mathcal{G}} : Z = \mathcal{G}_{\emptyset,Z} \to \mathcal{G}_{S,Z}$. It is preserved by the structure embeddings $\mathcal{G}(S) \hookrightarrow \mathcal{G}(S')$ and pull-backs; this is the *unit section* of $\mathcal{G}$.

(iii) As in Remark (iii) in 3.4.6, in the above definition we can replace $\mathcal{C}(X)_Z$ by the ordered set of relative effective Cartier divisors adding the condition that all the structure embeddings $\mathcal{G}_S \hookrightarrow \mathcal{G}_{nS}$ are isomorphisms.

(iv) Suppose we have two morphisms of schemes $f, g : Z' \rightrightarrows Z$ which coincide on Z'_{red}. Then the corresponding pull-back maps $\mathcal{C}(X)_Z \to \mathcal{C}(X)_{Z'}$ coincide, so one has a canonical identification $f^*\mathcal{G}_S \xrightarrow{\sim} g^*\mathcal{G}_S$ of ind-algebraic spaces. Therefore $\mathcal{G}_{S,Z}$ carries a canonical action of the universal formal groupoid on Z (whose space is the formal completion of $Z \times Z$ at the diagonal). For $S \subset S'$ the embeddings $\mathcal{G}_S \hookrightarrow \mathcal{G}_{S'}$ are compatible with the action, as well as the pull-back identifications (in particular, $1_{\mathcal{G}}$ is fixed by the action). So we have defined a canonical integrable connection on the ind-algebraic $\mathrm{Sym}^n X$-space $\mathcal{G}_{\mathrm{Sym}^n X}$.

(v) The category $\mathcal{CM}(X)$ admits finite projective limits. A chiral monoid $\mathcal{G}$ admits a monoid structure (as an object of $\mathcal{CM}(X)$) if and only if it is commutative; such a structure is unique and the product $\mathcal{G} \times \mathcal{G} \to \mathcal{G}$ is given by the morphisms $c_{S,S} : \mathcal{G}_S \times \mathcal{G}_S \to \mathcal{G}_S$.

(vi) For a chiral monoid $\mathcal{G}$ and a finite set I let $\mathcal{G}_{X^I}$ be the pull-back of $\mathcal{G}_{\mathrm{Sym}^{|I|}X}$ by the projection $X^I \to \mathrm{Sym}^{|I|}X$. For $\pi : J \to I$ one has evident morphisms (we use the notation from 3.4.5) $\nu^{(\pi)} : \Delta^{(\pi)*}\mathcal{G}_{X^J} \to \mathcal{G}_{X^I}$, $c_{[\pi]} : j^{[\pi]*} \prod_I \mathcal{G}_{X^{J_i}} \xrightarrow{\sim} j^{[\pi]*}\mathcal{G}_{X^J}$ satisfying the obvious versions of properties (a)–(f) from 3.4.5. Conversely, any such datum $(\mathcal{G}_{X^I}, \nu^{(\pi)}, c_{[\pi]})$ defines a chiral monoid (one recovers $\mathcal{G}_{\mathrm{Sym}^n X}$ by descent).

One can also consider a weaker structure taking into consideration only non-empty I's and surjective maps between them; the resulting objects may be called *chiral semigroups*. In fact, a chiral monoid is the same as a chiral semigroup which admits a *unit section* (the definition is left to the reader); morphisms of chiral monoids are the same as morphisms of chiral semigroups that preserve the unit sections.

(vii) The functor $\mathcal{G} \mapsto \mathcal{G}_X$ on the category of chiral monoids is faithful. So a chiral monoid can be considered as an ind-algebraic X-space $\mathcal{G}_X$ equipped with an extra structure. This structure has $X_{\acute{e}t}$-local origin.

Let $\mathcal{G}$ be a chiral monoid and P an ind-algebraic space. For a point $x \in X$ set $P_{\hat{x}} := P \times \mathrm{Spf}\, O_x$; this is an ind-algebraic X-space (which lives over the formal neighbourhood of x) equipped with an evident connection ∇_P. A *chiral $\mathcal{G}$-action* on P at x is a chiral monoid structure on the disjoint union $\mathcal{G}_X \sqcup P_{\hat{x}}$ such that the embedding $\mathcal{G}_X \hookrightarrow \mathcal{G}_X \sqcup P_{\hat{x}}$ is a morphism of chiral groupoids and the connection on $P_{\hat{x}}$ defined by the chiral monoid structure (see Remark (iv) above) equals ∇_P.

DEFINITION. (i) For a chiral monoid $\mathcal{G}$ a *line bundle* on $\mathcal{G}$ is a rule that assigns to any $S \in \mathcal{C}(X)_Z$ a line bundle $\lambda_{\mathcal{G}_S}$ on $\mathcal{G}_S$, and to any morphism $(S', Z') \to (S, Z)$ in $\mathcal{C}(X)$ an identification of $\lambda_{\mathcal{G}_{S'}}$ and the pull-back of $\lambda_{\mathcal{G}_S}$ by the map $\mathcal{G}_{S'} \to \mathcal{G}_S$ in a way compatible with the composition of morphisms.

(ii) For λ as above, a *factorization structure* on λ is a rule that assigns to any *non-intersecting* $S_1, S_2 \in \mathcal{C}(X)_Z$ an identification $c : \lambda_{\mathcal{G}_{S_1}} \otimes \lambda_{\mathcal{G}_{S_2}} \xrightarrow{\sim} c^*_{S_1,S_2}\lambda_{\mathcal{G}_{S_1+S_2}}$. These isomorphisms should be compatible with the structure morphisms from (i) and commutative and associative in the obvious sense.

(iii) Suppose that $\mathcal{G}$ is commutative. The factorization structure is said to be *commutative* if the isomorphisms c from (ii) can be extended to similar isomorphisms defined for *arbitrary* $S_1, S_2 \in \mathcal{C}(X)_Z$ and compatible with the structure morphisms from (i) (by properties (a) and (c) this can be done in a unique way).

Line bundles on $\mathcal{G}$ equipped with factorization structure are also called $\mathbb{G}_m$-*extensions* of $\mathcal{G}$. Replacing mere line bundles by super line bundles, one gets the notion of *super $\mathbb{G}_m$-extension*; these objects form a Picard groupoid denoted by

$Pic^f(\mathcal{G})$. Of course, one can replace $\mathbb{G}_m$ by any commutative group scheme (or even an algebraic Picard groupoid).

EXAMPLES. (i) (The local Picard schemes). The ind-schemes $\mathcal{D}iv(X,\Gamma)_S$ (see 3.10.8) form a commutative chiral monoid. As in 3.10.8, the pull-back of any $(\lambda, c) \in \mathrm{Pic}^f(\mathcal{D}iv(X,\Gamma))$ (see 3.10.7) is naturally a super $\mathbb{G}_m$-extension of our chiral monoid. This super extension is commutative if and only if $(\lambda, c) \in \mathrm{Ext}(\mathcal{D}iv(X,\Gamma), \mathbb{G}_m) \subset \mathrm{Pic}^f(\mathcal{D}iv(X,\Gamma))$ (see (3.10.7.3)). It is not difficult to check that our functor identifies $\mathrm{Pic}^f(\mathcal{D}iv(X,\Gamma))$ with the Picard groupoid of super $\mathbb{G}_m$-extensions of our chiral groupoid.

(ii) (The affine Grassmannian). Let G be any algebraic group. Then for $S \in \mathcal{C}(X)_Z$ the functor which assigns to a Z-scheme Y the set of pairs $(\mathfrak{F}, \gamma)$ where $\mathfrak{F}$ is a G-torsor on $X \times Y$ and γ its trivialization over the complement to (the pull-back of) S is representable by an ind-scheme $\mathcal{G}_S$ of ind-finite type. These ind-schemes form naturally a chiral monoid (see [**BD**] 5.3.12). For $G = T$ we get the chiral monoid from (i). For any $x \in X$ there is a natural chiral $\mathcal{G}$-action on $G(K_x)$ at x.

REMARK. For any $\mathcal{D}_X$-scheme Y_X the corresponding multijet $\mathcal{D}_{X^I}$-schemes Y_{X^I} (see 3.4.21) define, according to 3.4.22, a canonical chiral semigroup structure on Y_X (see (vi) in the previous remarks). Notice that this chiral semigroup does not admit a unit section (unless $Y_X = X$). As was pointed out by Kapranov and Vasserot [**KV**] 3.2.4, for affine $Y_X = \mathrm{Spec}\, A^\ell$ the ind-scheme $Y_X^{mer} := \mathrm{Spec}\, A^{as}$ (see 3.6.18) of "horizontal meromorphic jets" is again a chiral semigroup, modulo the fact that property (a) from the definition of chiral monoid was not verified in loc. cit. It would be nice to check this property (in case of the usual multijets, this is the contents of the theorem in 3.4.22).

Let $\mathcal{G}$ be a chiral monoid and λ its super extension. Suppose that for each $S \in \mathcal{C}(X)_Z$, Z is affine, the corresponding $\mathcal{G}_{S,Z}$ is the inductive limit of its closed subschemes which are finite and flat over Z. In other words, $\mathcal{G}_{S,Z} = \mathrm{Spf}\, R$ where $R = \varprojlim R_\alpha$ such that R_α are $\mathcal{O}_X$-algebras which are locally free $\mathcal{O}_X$-modules of finite rank. Set

$$A^\ell_{S,Z} := \lambda \underset{R}{\otimes} R^* := \bigcup \lambda \underset{R}{\otimes} \mathcal{H}om_{\mathcal{O}_X}(R_\alpha, \mathcal{O}_X). \tag{3.10.16.1}$$

Here λ is $\lambda_{\mathcal{G}_S}$ considered as an invertible R-module. In other words, $A^\ell_{S,Z} = \pi_!(\lambda_{\mathcal{G}_S} \otimes \pi^! \mathcal{O}_Z)$; here $\pi : \mathcal{G}_{S,Z} \to Z$ is the structure projection and $\pi^!$ is the !-pull-back functor.

The factorization structures on $\mathcal{G}$ and λ define on A^ℓ the structure of the factorization algebra (see 3.4.6). In the situation of Example (i) above our A is the lattice chiral algebra from 3.10.8.

QUESTIONS. (i) Suppose that we have $(\mathcal{G}, \lambda)$ such that $\mathcal{G}_{S,Z}$ are proper over Z. Is it true that $A^\ell_{S,Z} := R\pi_!(\lambda_{\mathcal{G}_S} \otimes R\pi^! \mathcal{O}_Z)$ form a factorization DG algebra? If $\mathcal{G}$ is the affine Grassmannian for semisimple G and λ is defined by a positive integral level κ, then A should be equal to the integrable quotient of $U(\mathfrak{g}_{\mathcal{D}})^\kappa$.

(ii) Let $\mathcal{G}$ be any chiral monoid and $x \in X$ a point. Can one define a group space $\mathcal{G}_x^{as}$ such that for an ind-algebraic space P a chiral $\mathcal{G}$-action on P at x amounts to a $\mathcal{G}_x^{as}$-action on P? Will a super extension λ of $\mathcal{G}$ provide a super extension of $\mathcal{G}_x^{as}$? In the situation of Example (ii), $\mathcal{G}_x^{as}$ should be equal to $G(K_x)$.