Bloch's lecture 10.27.03, Beilinson's office hours 10.30.03.
Subject: Quillen's two constructions of K-groups.
References: articles by Quillen and Quillen-Grayson in Springer Lecture Notes 341, 551.

A. The +-construction. Let X be a connected topological space (which means, say, CW complex), $N \subset \pi_1(X)$ a normal subgroup (one need not specify a base point due to normality). Suppose N is perfect, i.e., coincides with its commutant.

Claim. There is a morphism $\phi : X \to X^+$ which satisfies properties (o)–(iii) below. Here $\tilde{X}^+$ is a universal covering of X^+ and $\tilde{X}$ its pull-back to X.

(o) X^+ is connected,

(i) $\pi_1(X)/N \tilde{\to} \pi_1(X^+)$,

(ii) $\tilde{X}^+/\tilde{X} := \mathcal{C}one(\tilde{\phi} : \tilde{X} \to \tilde{X}^+)$ is contractible. Here $\tilde{X}^+$ is a universal covering of X^+ and $\tilde{X}$ its pull-back to X.

One can replace (ii) by either of the equivalent properties (ii)', (ii)". Below $H.(?) := H.(?, \mathbb{Z})$.

(ii)' $H.(\tilde{X}) \tilde{\to} H.(\tilde{X}^+)$.

(ii)" $H.(X, \phi^* L) \tilde{\to} H.(X^+, L)$ for every local system L of abelian groups on X^+.

Construction. We define $X \hookrightarrow Y \hookrightarrow X^+$ where Y is obtained from X by gluing 2-cells, and X^+ from Y by gluing 3-cells. The sets of 2- and 3-cells are in a canonical bijection.

Take any set $\{\gamma_\alpha\}$ of loops that generate N (as a normal subgroup). Get Y by gluing 2-cells B^2_α by γ_α.

The composition $\pi_2(Y) \to H_2(Y) \to H_2(Y/X)$ is surjective. Proof of surjectivity: Let $\tilde{Y}$ be a universal covering of Y, $\tilde{X}$ its pull-back to X. Then our arrow is the composition $\pi_2(Y) \tilde{\leftarrow} \pi_2(\tilde{Y}) \tilde{\to} H_2(\tilde{Y}) \twoheadrightarrow H_2(\tilde{Y}/\tilde{X}) \twoheadrightarrow H_2(Y/X)$. Here the second arrow is an isomorphism since $\pi_1(\tilde{Y}) = 0$, the third one is surjective since $H_1(\tilde{X}) = 0$ (for $\pi_1(\tilde{X}) = N$ is perfect).

Choose any section $\oplus \mathbb{Z}_\alpha = H_2(Y/X) \to \pi_2(Y)$. The images of the generators form a family of maps $\xi_\alpha : S^2 \to Y$. Glue 3-cells B^3_α to Y by means of ξ_α. This is X^+. Properties (0)–(ii) are evident. □

Universality property. Let $\psi : X \to Z$ be a map which kills N. Then for any X^+ as above, there is a unique morphism $\psi^+ : X^+ \to Z$ in the homotopy category such that $\psi = \psi^+ \phi$ (the equality should hold in the homotopy category).

Proof. It suffices to consider the X^+ built by the above construction (for a morphism from this X^+ to any other version of X^+, which exists by universality, is a homotopy equivalence).

The set P_1 of homotopy classes of extensions ψ_Y of ψ to Y is non-empty, and it is naturally a $\prod \pi_2(Z, z_\alpha)$-torsor. Here $z_\alpha \in Z$ is a point on the loop $\psi(\gamma_\alpha)$ (which is homotopic to 0). The obstruction $c(\psi_Y)$ to the existence of an extension of a given ψ_Y to X^+ lies in $\prod \pi_2(Z, z_\alpha)$. The map $\psi_Y \mapsto c(\psi_Y)$ is affine with respect to the $\prod \pi_2(Z, z_\alpha)$-translations. Thus there is a unique ψ_Y that can be extended to X^+.

Take such ψ_Y (as a map). The homotopy classes of its extensions to X^+ form a $\prod \pi_3(Z, z_\alpha)$-torsor P_2. The group of (homotopy classes of) self-homotopies of ψ_Y leaving ψ fixed acts on P_2. One easily checks that this action is transitive, so all possible ψ^+ are homotopic (with the restriction to X fixed). □

Definition. For an associative algebra A one defines $K_0(A)$ as the K-group of the category of finitely generated projective A-modules. For $i \geq 1$ we set $K_i(A) := \pi_i(BGL_\infty(A)^+)$. Here $GL_\infty(A) := \cup GL_n(A)$, $BGL_\infty(A)$ is the corresponding Eilenberg-MacLane space, and the $+$-construction is taken with respect to the commutant $E(A)$ of $GL_\infty(A)$ (it is known that $E(A)$ is the subgroup generated by elementary matrices, so it is perfect).

Exercise. Choose some $BGL_\infty(\mathbb{Z}) \hookrightarrow BGL_\infty(\mathbb{Z})^+$. Then for any A the space $BGL_\infty(A) \underset{BGL_\infty(\mathbb{Z})}{\sqcup} BGL_\infty(\mathbb{Z})^+$ is $BGL_\infty(A)^+$. (The condition to check is that $E(A) \subset GL_\infty(A)$ is generated *as a normal subgroup* by elementary matrices with integral coefficients). By the way, one can get $BGL_\infty(\mathbb{Z})^+$ by adding to $BGL_\infty(\mathbb{Z})$ only one 2-cell and one 3-cell.

Denote by $\mathcal{P} = \mathcal{P}(A)$ the groupoid of finitely-generated projective A-modules (the morphisms are isomorphisms). Notice that $\mathcal{P}$ is a symmetric monoidal category with respect to the direct sum operation. Thus the classifying space $B\mathcal{P}$ is an H-space. It is highly non-connected: $\pi_0\mathcal{P}$ is the semi-group of isomorphism classes of finitely-generated projective A-modules. Its group completion is $K_0(A)$.

Suppose we have an H-space T such that $\pi_0 T$ is a commutative group and a morphism of H-spaces $\nu : B\mathcal{P} \to T$.

Lemma. Such ν yields $\nu_i : K_i(A) \to \pi_i T$.

Proof. The group completion of the semi-group $\pi_0 B\mathcal{P}$ equals $K_0(A)$, hence we get ϕ_0. To get $\phi_{>0}$ we will define a (canonical up to homotopy) morphism $\bar\nu : BGL_\infty(A) \to T^0$ where T^0 is the neutral component of T. By the universality property of $+$-construction (and commutativity of $\pi_1(T^0)$) it extends then to a morphism $BGL_\infty(A)^+ \to T^0$. Our ν_i are the corresponding morphisms of the homotopy groups.

Since $\pi_0(T)$ is a group, our T is a group in the homotopy category; denote by $-$ the "substraction" morphism $T \times T \to T$. For $P \in \mathcal{P}$ we have a map $\bar\nu_P : B\mathrm{Aut}P \to T^0$, $\bar\nu_P := \nu|_{B\mathrm{Aut}P} - \nu(P)$. Here $\nu(P) \in T$ is the ν-image of $P \in B\mathcal{P}$. In particular, we have maps $\bar\nu_{A^n} : BGL_n(A) \to T^0$. It is clear that the restriction of $\bar\nu_{A^n}$ to $BGL_{n-1}(A) \hookrightarrow BGL_n(A)$ (the standard embedding) is canonically homotopic to $\bar\nu_{A^{n-1}}$. Applying the telescope construction, we get the promised $\bar\nu : BGL_\infty(A) \to T^0$. □

B. The Q-construction. Let $\mathcal{M}$ be an exact category. Thus it is an additive category equipped with a class of "exact sequences" satisfying some axioms. They say, essentially, that $\mathcal{M}$ can be realized as a full subcategory in an abelian category closed under extensions; the exact sequences are those coming from exact sequences in the abelian category. The essential example is the category of finitely-generated projective A-modules. Below "injections" and "surjection" in $\mathcal{M}$ mean morphisms that can be included (appropriately) in the "exact sequences" (Quillen's terms are "admissible monomorphism" and admissible epimorphism"). Notation: $\hookrightarrow$, $\twoheadrightarrow$.

Quillen's category $Q = Q\mathcal{M}$ has the same objects as $\mathcal{M}$, and a morphism $X \to Y$ in Q is an identification of X with a subquotient of Y. Thus we have a filtration $Y_0 \hookrightarrow Y_1 \hookrightarrow Y$ and an isomorphism $X \xrightarrow{\sim} Y_1/Y_0$. The composition is evident (the notion of subquotient is transitive).

The direct sum in $\mathcal{M}$ defines a symmetric monoidal structure on Q. Since BQ is connected (the set of morphisms from 0 to any object is non-empty), in the next definition we need not refer to the base point for π_i's, and π_1 is commutative:

Definition. $K_i(\mathcal{M}) := \pi_{i+1}(BQ)$.

Special subcategories. Below $\mathcal{P} \subset Q$ is the groupoid whose objects are the same as those of Q and morphisms are all isomorphisms in Q.

Remark. An isomorphism $X \xrightarrow{\sim} Y$ in $\mathcal{M}$ can be considered as an identification of X with a subobject of Y, or of Y with a quotient object of X. Thus, depending on your fancy, $\mathcal{P}$ identifies canonically either with the groupoid $\mathcal{M}^{iso}$ of isomorphisms in $\mathcal{M}$, or with its inverse.

Among subquotients there are subobjects and quotient objects, These define two subcategories $Q^+, Q^- \subset Q$. They have the same objects as Q. Clearly Q^+ is the subcategory of injections in $\mathcal{M}$, and Q^- is opposite to the subcategory of surjections in $\mathcal{M}$. One has $\mathcal{P} = Q^+ \cap Q^-$.

Two important facts:

(a) The subcategories Q^+, Q^- are contractible (since 0 is the initial object in either of them).

(b) Any subquotient can be considered either as a quotient of a subobject, or a subobject of a quotient object. Thus any morphism f in Q can be uniquely represented as the composition of a morphism in Q^+ and a morphism in Q^- in either order, i.e., written as the diagonal in a commutative square with columns in Q^+ and rows in Q^-. We refer to it as the canonical commutative square of f.

Adding to $\mathcal{P} \subset Q^+$ all morphisms from the initial object $0 \in Q^+$, we get a "cone" subcategory $C^+\mathcal{P} \subset Q^+$. Similarly, we get a cone subcategory $C^-\mathcal{P} \subset Q^-$. Set $S\mathcal{P} := C^+\mathcal{P} \cup C^-\mathcal{P} \subset Q$. This is a subcategory in Q whose classifying space is the suspension of $B\mathcal{P}$, i.e., $BS\mathcal{P} = SB\mathcal{P}$. By adjunction, the embedding $S\mathcal{P} \subset Q$ yields a map

$$\nu : B\mathcal{P} \to \Omega BQ.$$

The monoidal structure on Q makes BQ is an H-space. So ΩBQ carries two natural H-structures, which coincide by a standard general result from algebraic topology. The same monoidal structure makes $B\mathcal{P}$ an H-space, and ν is a morphism of H-spaces (clear for one of the H-structures).

Proposition. The map $\pi_0\nu : \{\text{isomophism classes of objects in } \mathcal{M}\} \to \pi_0\Omega BQ = K_0(\mathcal{M})$ identifies $K_0(\mathcal{M})$ with the quotient of the free abelian group generated by the isoorphism classes modulo the usual relation coming from exact sequences.

Proof. Contracting $BQ^- \subset BQ$ to a point, we get a space BQ/BQ^- homotopically equivalent to Q. It has a single vertex, thus its fundamental group is generated by 1-simplices with relations coming from 2-simplices. Each 1-simplex comes from a morphism in Q. Looking at its canonical square (see (b) above) we see that our 1-simplex is (canonically) homotopic to one $Z \hookrightarrow Y$ from Q^+. Then the canonical square for the morpism $0 \to Y$ corresponding to the subquotient Y/Y, identifies it with $0 \hookrightarrow Y/Z$. This loop equals $\pi_0\nu(Y/Z) \in K_0(\mathcal{M})$, so $K_0(\mathcal{M})$ is generated by those elements. The relations come from composition of morphisms in Q. The corresponding canonical squares compose in an evident way, and this yields the usual relations coming from exact sequences. □

C. Comparison. Suppose that $\mathcal{M}$ is the category of finitely-generated projective A-modules. Then ν fits into the setting of the Lemma from Section A, so we get

$$\nu_i : K_i(A) \to K_i(\mathcal{M}).$$

Theorem. These are isomorphisms.

Idea of the proof. (a) The first step is to describe ν in a slightly different way.

Let us decompose the embedding $Q^+ \hookrightarrow Q$ into a composition $Q^+ \xrightarrow{i} \tilde{Q}^+ \xrightarrow{p} Q$ where i is fully faithful, $\tilde{Q}^+$ is contractible, and p is surjective on morphisms.

Objects of $\tilde{Q}^+$ are injections $X \hookrightarrow Y$ in $\mathcal{M}$. A morphism $(X \hookrightarrow Y) \to (X' \hookrightarrow Y')$ in $\tilde{Q}^+$ is a pair of injections $X' \hookrightarrow X$, $Y \hookrightarrow Y'$ such that the composition $X' \hookrightarrow X \hookrightarrow Y \hookrightarrow Y'$ is equal to our morphism $X' \hookrightarrow Y'$ (so we get a commutative "trapezoid"). The functors are $i(Y) := (0 \hookrightarrow Y)$ and $p(X \hookrightarrow Y) := Y/X$. The contractibility of $\tilde{Q}^+$ comes from canonical morphisms $(X \hookrightarrow Y) \to (0 \hookrightarrow Y) \leftarrow (0 \hookrightarrow 0)$ in $\tilde{Q}^+$.

Notice that the fiber $p^{-1}(0) \subset \tilde{Q}^+$ is formed by $X \xrightarrow{\sim} Y$, i.e., it identifies canonically with $\mathcal{M}^{iso} = \mathcal{P}$. Contracting this fiber in $\tilde{Q}^+$ into a point (say, using the above contraction), we embed a cone over $B\mathcal{P}$ into $B\tilde{Q}^+$. Projecting it down by p (so that the bottom of our cone contracts to the point), we get a map $SB\mathcal{P} \to BQ$. The map $B\mathcal{P} \to \Omega BQ$ defined by adjunction is homotopic to ν.

(b) If $p : B\tilde{Q}^+ \to BQ$ were a fibration then, by contractibility of $\tilde{Q}^+$, our ν would have been a homotopy equivalence. This is not the case, but one can modify the picture to make it become true.

Notice that direct sum makes $\tilde{Q}^+$ a symmetric monoidal category, and p is a symmetric monoidal functor. The monoidal category $p^{-1}(0)$, which is our old friend $\mathcal{P}$ from Section A, acts then along the fibers of p. Quillen explains how one can modify canonically an action of any monoidal category on another category so that the translation functors become homotopy equivalences. Applying this to our situation (the monoidal category $\mathcal{P}$ acting on $\tilde{Q}^+$) we get a new category over Q. It is still contractible, and the projection to Q is a fibration. The morphism of H-spaces $B\mathcal{P} \to T$ where T is the new fiber over 0 has the property that the corresponding maps ν_i from the Lemma from Section A are isomorphisms. We are done. □