# GLOBAL RIGIDITY OF ANOSOV HOMEOMORPHISMS

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Contents



## 1. Introduction

Given a homeomorphism $f : M \to M$ of a manifold, we want to understand the structure of the orbits of $f$. A natural question to ask here is what other homeomorphisms have the same orbit structure as $f$. This question leads to the notion of topological conjugacy: two homeomorphisms, $f : M \to M$ and $g : M \to M$ have the same orbit structure if they are topologically conjugate, i.e. if there is a homeomorphism $h : M \to M$ such that $h \circ f = g \circ h$. Thus, given a homeomorphism $f : M \to M$, we can study the topological dynamics of $f$ by studying the dynamics of any other homeomorphism in the same conjugacy class as $f$. The benefit of this approach is that if we can find a "simpler" homeomorphism, $g$, in the same conjugacy class as $f$, we'll then be able to understand the orbit structure of $f$ simply by understanding the orbit structure of $g$. [1] This prompts us to ask the following question:

**Question 1.1.** Given a homeomorphism $f : M \to M$, what kinds of 'simpler' homeomorphisms exist in the same conjugacy class as $f$?

The answer to this question will depend entirely on $f$. However, for specific types of homeomorphisms, we'll be able to provide an answer. In this proposal, we'll explain how to Question 1.1 for several different types of homeomorphisms.

First, we'll give an answer to this questions for Anosov diffeomorphisms on nilmanifolds giving by giving a result by Franks and Manning that states that an Anosov diffeomorphism on a nilmanifold is topologically conjugate to a

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[1] By 'simpler', we mean a map whose dynamics are more tractable. For example linear as opposed to non-linear.

algebraic model. We'll provide a more precise statement of this result, along with the necessary definitions, in Section 2.

The second class of homeomorphisms for which we'll answer Question 1.1 will be motivated by this first result. We'll next study the set of Anosov homeomorphisms. An Anosov homeomorphism is a expansive homeomorphism with the shadowing property. The definitions of expansiveness and of the shadowing property will both be given in Section 3. These two properties capture much of the essential behavior of Anosov diffeomorphisms, and as a result Anosov homeomorphisms will share many of the properties of Anosov diffeomorphisms. We'll see that we can answer Question 1.1 for Anosov homeomorphisms on nilmanifolds analogously to how we answered it for Anosov diffeomorphisms. The first step in this generalization was done by Hiraide, who proved an analogue of Franks-Manning for Anosov homeomorphisms on tori. In this proposal, we'll generalize Hiraide's result to nilmanifolds, thus proving an analogue of Franks-Manning for Anosov homeomorphisms. This discussion will take place over the course of Sections 3-4.

Note that these two results only apply on nilmanifolds. This suggests the obvious question of whether these results can be extended for more general classes of manifolds. The answer to this question is unknown [4]. However, it is known for a related class of covering maps, namely the set of expanding maps. This topic proposal will conclude with a brief discussion of the expanding maps conjecture, which will answer Question 1.1 for expanding maps.

## 2. Global Rigidity for Anosov Diffeomorphisms

We begin this section by recalling the definition of an Anosov diffeomorphism.

**Definition 2.1.** Let $M$ be a complete Riemannian manifold. A diffeomorphism $f : M \to M$ is *Anosov* if there exists a continuous splitting of the tangent bundle $TM$ into $df$-invariant subbundles the following two conditions are satisfied

- There exists a continuous, $df$-invariant splitting of the tangent bundle $TM = E^s \oplus E^u$.
- There exists a Riemannian metric on $M$ with respect to which the map $df$ is contracting on $E^s$ and is expanding on $E^u$. More precisely, there exist constants $C > 0$, $C' > 0$ and $0 < \lambda < 1$ and a Riemannian metric on $M$ such that for all $n \geq 0$,

$$\|df^n(v)\| \leq C\lambda^n \|v\| \quad \text{for } v \in E^s$$
$$\|df^{-n}(v)\| \leq C'\lambda^n \|v\| \quad \text{for } v \in E^u$$

Anosov diffeomorphisms are of great interest in dynamics because they exhibit a great deal of stability and rigidity. Of particular interest is Question 1.1 for Anosov diffeomorphisms on compact manifolds. This question of classifying Anosov diffeomorphisms up to topological conjugacy was originally asked by Smale in [19, Problem 3.5]. Franks provided an answer for Anosov diffeomorphisms on tori by proving

**Theorem 2.2** ([6])**.** *An Anosov diffeomorphism of a torus is topologically conjugate to a hyperbolic toral automorphism.*

Recall that a map $g : T^n \to T^n$ is a *hyperbolic toral autoomorphism* if it has a lift $\tilde{g} : \mathbb{R}^n \to \mathbb{R}^n$ that is a hyperbolic linear transformation in $\mathrm{GL}_n(\mathbb{Z})$. Later, Manning generalized Franks' result to Anosov diffeomorphisms on nilmanifolds. Before stating Manning's generalization of Theorem 2.2, we'll recall the definition of a nilmanifold and provide an example of an Anosov diffeomorphism of a nilmanifold.

**Definition 2.3.** A *nilmanifold* is a compact homogeneous space of the form $N/\Gamma$, where $N$ is a simply connected nilpotent Lie group and $\Gamma$ is a cocompact lattice in $N$. (This definition of $\Gamma$ is technically redundant. By [17, Theorem 2.1], all lattices in a nilpotent Lie group are cocompact, so we could actually just assume that $\Gamma$ is a lattice in $N$.)

As we'll see Appendix B, nilmanifolds are natural generalizations of tori, which explains why they provide a natural way to generalize Franks' result. We'll now give two examples to illustrate these concepts. First, we'll give a basic example of a nilmanifold. This will be an extremely useful example to keep in mind when working with nilmanifolds.

**Example 2.4** (The Heisenberg manifold)**.** The *Heisenberg group* is group of $3 \times 3$ matrices,

$$H = H_3(\mathbb{R}) = \left\{ A_{(x,y,z)} = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix};\ x, y, z \in \mathbb{R} \right\}$$

under matrix multiplication. Note that $H$ is 2-step nilpotent Lie group and is diffeomorphic to $\mathbb{R}^n$. The Lie algebra of $H$ is given by

$$\mathrm{Lie}(H) = \left\{ B_{(x,y,z)} = \begin{bmatrix} 0 & x & z \\ 0 & 0 & y \\ 0 & 0 & 0 \end{bmatrix};\ x, y, z \in \mathbb{R} \right\}$$

and is generated by the matrices, $X = B_{(1,0,0)}$, $Y = B_{(0,1,0)}$, and $Z = B_{(0,0,1)}$. The Lie bracket is given by letting $[X, Y] = Z$ and letting all other brackets of generators be zero. The Heisenberg group is an extremely useful example when studying nilpotent Lie groups.

The *Heisenberg manifold* is obtained by quotienting the Heisenberg group by the lattice $H_3(\mathbb{Z}) = \left\{A_{(x,y,z)};\ x, y, z \in \mathbb{Z}\right\}$.

We'll now give an example of an Anosov diffeomorphism of a nilmanifold. This example originated with Smale [19]. Our presentation of this example is based on that given in [13, Chapter 17.3].

**Example 2.5.** Let $G = H \times H$, where $H$ is the Heisenberg group. To begin, we'll define a hyperbolic automorphism, $F$, of $G$. We'll then find a $F$-invariant lattice $\Gamma$ in $G$, which will let us conclude that $F$ descends to an Anosov diffeomorphism of the nilmanifold $G/\Gamma$.

First, we construct a hyperbolic automorphism, $F$, of $G$. We do this by constructing a hyperbolic Lie algebra automorphism, $F' : \mathrm{Lie}(G) \to \mathrm{Lie}(G)$, which we'll then project down to $G$ using the exponential map to obtain $F$. Observe that the Lie algebra of $G$ can be viewed as

$$\mathrm{Lie}(G) = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix};\ A, B \in \mathrm{Lie}(H) \right\}$$

and is generated by two copies, $X_1, Y_1, Z_1$ and $X_2, Y_2, Z_2$, of the standard basis for $\mathrm{Lie}(H)$. The bracket is given by $[X_i, Y_i] = Z_i$ and letting all other brackets of generators be zero.

We begin by letting $\lambda_1 = \frac{3+\sqrt{5}}{2}$ and $\lambda_2 = \frac{3-\sqrt{5}}{2}$. Note that $0 < \lambda_2 < 1$ and $\lambda_1 = \lambda_2^{-1}$. Now, we define the hyperbolic Lie algebra automorphism, $F' : \mathrm{Lie}(G) \to \mathrm{Lie}(G)$ by

$$\begin{aligned} X_1 &\mapsto \lambda_1 X_1, & X_2 &\mapsto \lambda_1^{-1} X_2, \\ Y_1 &\mapsto \lambda_1^2 Y_1, & Y_2 &\mapsto \lambda_1^{-2} Y_2, \\ Z_1 &\mapsto \lambda_1^3 Z_3, & Z_2 &\mapsto \lambda_1^{-3} Z_2. \end{aligned}$$

We let $F = \exp \circ F'$, where $\exp : \mathrm{Lie}(G) \to G$ is the Lie exponential map. (Note that the exponential map is a diffeomorphism since $G$ is a simply-connected nilpotent Lie group.)

We now construct a lattice, $\Gamma$, in $G$ that is preserved by $F$. We'll construct $\Gamma$ by finding a lattice $\Gamma'$ in $\mathrm{Lie}(G)$ that is preserved by $F'$ and then projecting $\Gamma'$ down to $G$ using the exponential map. Recall from Galois theory that the field $\mathbb{Q}(\sqrt{5})$ has a single nontrivial automorphism, $\sigma$, which is defined by $\sigma(a + b\sqrt{5}) = a - b\sqrt{5}$ for $a, b \in \mathbb{Q}$. We define the lattice $\Gamma' \subset \mathrm{Lie}(G)$ by

$$\Gamma' = \left\{ \begin{pmatrix} A & 0 \\ 0 & \sigma(A) \end{pmatrix};\ A \in \mathrm{Lie}(H) \text{ with entries algebraic integers in } \mathbb{Q}(\sqrt{5}) \right\}$$

Since the subgroup of algebraic integers in $\mathbb{Q}(\sqrt{5})$ is generated by $\left\{1, \frac{1+\sqrt{5}}{2}\right\}$, we see that $\Gamma'$ is a lattice in $\mathrm{Lie}(G)$. Also note that since $\lambda \in \mathbb{Z}\left[\frac{1+\sqrt{5}}{2}\right]^{\times}$, we get that $F'(\Gamma') = \Gamma'$. Thus, the lattice $\Gamma = \exp(\Gamma')$ in $G$ is $F$-invariant. We've now shown that $F$ projects down to an Anosov diffeomorphism of the nilmanifold $G/\Gamma$.

An Anosov diffeomorphism of a nilmanifold constructed in the manner of Example 2.5 is known as a *hyperbolic nilmanifold automorphism*, which we'll define now.

**Definition 2.6.** Let $M = N/\Gamma$ be a nilmanifold. A smooth map $f : N/\Gamma \to N/\Gamma$ is called a *nilmanifold endomorphism* if $f$ lifts to an endomorphism $\tilde{f} : N \to N$. If in addition, the derivative of $\tilde{f}$ at the identity, $D_e\tilde{f}$ is hyperbolic (i.e. has no eigenvalues of modulus one), then $f$ is called a *hyperbolic nilmanifold endomorphism.*

Armed with these definitions, we're now ready to state Manning's generalization of Theorem 2.2.

**Theorem 2.7** ([15, Theorem C], [4, Theorem 5.1])**.** *An Anosov diffeomorphism of a nilmanifold is topologically conjugate to a hyperbolic nilmanifold automorphism.*

*Remark* 2.8. Manning originally proved this theorem for infra-nilmanifolds, which are a generalization of nilmanifolds. However, Dekimpe later found that an error the proof for infra-nilmanifolds and produced a counterexample of an Anosov diffeomorphism on an infra-nilmanifold that was not conjugate to a hyperbolic nilmanifold automorphism (although Manning's argument does still works for nilmanifolds) [4, Section 5]. Dekimpe conjectured that a modified version of Theorem 2.7 holds on infra-nilmanifolds [4, p.133-134].

## 3. Anosov Homeomorphisms

We now begin the topic that will occupy the majority of this proposal – that is a generalization of the results of Section 2 to a larger class of homeomorphisms, known as *Anosov homeomorphisms*. This section will be devoted to a definition of Anosov homeomorphisms and a discussion of their properties. Our ultimate goal is to prove an analogue of Theorem 2.7 for these maps, which we'll do in Section 4.

First, we'll explain the origins and motivations for the definition of an Anosov homoeomorphism. Anosov diffeomorphisms are of great interest largely due to their structural stability, which is rooted in the 'stability' of orbits under perturbation [2],[20]. The property of 'stability' of orbits under perturbation is made precise in the *shadowing property*, which is defined as follows.

**Definition 3.1.** Let $f : X \to X$ be a homeomorphism of a metric space. A sequence of points $\{x_i\}_{i\in\mathbb{Z}} \subset X$ is called a *$\delta$-pseudo-orbit* if $d(f(x_i), x_{i+1}) < \delta$ for all $i \in \mathbb{Z}$. A point $z \in X$ *$\varepsilon$-shadows* a sequence of points, $\{x_i\}_{i\in\mathbb{Z}}$, if $d(f^i(z), x_i) < \varepsilon$ for all $i \in \mathbb{Z}$. We say that $f$ has the *shadowing property* if for any $\varepsilon > 0$, there exists $\delta > 0$ such that any $\delta$-pseudo-orbit is $\varepsilon$-shadowed by a point in $X$.

All Anosov diffeomorphisms exhibit this property. In fact, they have a strengthened version of the shadowing property: For small $\varepsilon > 0$ and sufficiently small $\delta > 0$, any $\delta$-pseudo-orbit is $\varepsilon$-shadowed by a unique point $x$. The uniqueness in this strengthened version of the shadowing property follows from the following property, which holds for all Anosov diffeomorphisms.

**Definition 3.2.** We say that a homeomorphism of a metric space $f : X \to X$ is *expansive* if there exists a constant $c > 0$ such that $d(f^n(x), f^n(y)) < c$ for all $n \in \mathbb{Z}$ implies $x = y$. $c$ is called the *expansive constant* for $f$.

These two properties capture much of the stability of Anosov diffeomorphisms. For example, the following structural stability theorem for Anosov diffeomorphisms) follows relatively easily from these two properties.

**Theorem 3.3.** *Let $M$ be a closed manifold. If two Anosov diffeomorphisms $f : M \to M$ and $g : M \to M$ are sufficently close in the $C^0$-topology, then they are topologically conjugate.*

*Proof.* This follows from the unique shadowing property for Anosov diffeomorphisms via the observation that if $d_{C^0}(f, g) < \delta$, then any orbit of $g$ is a $\delta$-pseudo-orbit for $f$. For more details, see [2, Chapter 2] and [20, Theorem 4]. □

Note that both the shadowing property and expansiveness are topological properties, we can use them to define a more general set of maps than Anosov diffeomorphisms. Since these two properties capture much of the stability of Anosov diffeomorphisms, the hope is that this new set of maps, called Anosov homeomorphisms, will exhibit many of the same stability properties as Anosov diffeomorphisms.

**Definition 3.4.** Let $X$ be a compete metric space. A homeomorphism, $f : X \to X$ is called an *Anosov homeomorphism* if it is expansive and has the shadowing property.

A first stability result for Anosov homeomorphisms is easy. Since the proof of Theorem 3.3 relied only on the shadowing property and expansiveness, it will hold for Anosov homeomorphisms. Our ultimate goal will be to prove a global stability theorem, analogous to Theorem 2.7, for Anosov homeomorphisms on nilmanifolds (Theorem 4.2). This will answer Question 1.1 for Anosov homeomorphisms on nilmanifolds. The proof of Theorem 4.2 will follow a similar structure to the proof of Theorem 2.7. Manning's proof of Theorem 2.7 relied on a number of established properties of Anosov diffeomorphisms. We will therefore need to prove analogues of these properties for Anosov homeomorphisms before we can proceed with the proof of Theorem 4.2. Throughout these discussions, we'll let $M$ be a closed Riemannian manifold. We'll let $d$ be the distance function on $M$ induced by the Riemannian metric.

### 3.1. Stable and unstable sets.

Given a homeomorphism $f : X \to X$, of a metric space, we begin by defining the stable and unstable sets of $f$.

**Definition 3.5.** For $x \in X$, the *stable set*, $W^s(x)$, and the *unstable set*, $W^u(x)$, are

$$W^s(x) := \left\{ y \in X; d(f^n(x), f^n(y)) \overset{n\to\infty}{\longrightarrow} 0 \right\}$$
$$W^u(x) := \left\{ y \in X; d(f^{-n}(x), f^{-n}(y)) \overset{n\to\infty}{\longrightarrow} 0 \right\}$$

As we will see, the stable and unstable sets of an Anosov diffeomorphism, $f : M \to M$, will form transverse foliations. These will give extra structure to $M$ and will be useful in the study of the dynamical behavior of $f$. This section will begin with a brief overview of the properties of the stable and unstable sets of an Anosov diffeomorphism. We'll then explain how these properties can be generalized to Anosov homeomorphisms.

We begin with a discussion of the stable and unstable sets for an Anosov diffeomorphism, $f : M \to M$. Proofs of all these facts can be found in most textbooks on hyperbolic dynamics, e.g. [21, Chapter 4.4]. For each $x \in M$, the stable and unstable sets, $W^s(x)$ and $W^u(x)$ are both immersed smooth submanifolds of $M$. For this reason the stable (resp. unstable) sets of an Anosov diffeomorphism are known as stable (resp. unstable) manifolds. The collection of stable (resp. unstable) manifolds forms a continuous, $f$-invariant foliation with smooth leaves, known as the stable (resp. unstable) foliation of $f$ and denoted by $\mathscr{F}^s_f$ (resp. $\mathscr{F}^u_f$). (For a very brief review of foliations see Appendix A.1.) Additionally, the stable and unstable foliations, $\mathscr{F}^s_f$ and $\mathscr{F}^u_f$, are transverse. This provides a local decomposition of $M$ (alternatively a local product structure on $M$).

To explain more precisely what we mean by "local product structure", we define the *local stable and unstable sets* of $f$.

**Definition 3.6.** Let $f : X \to X$ be a homeomorphism of a metric space. For $\varepsilon > 0$ and $x \in X$, the *local stable set*, $W^s_\varepsilon(x)$, and the *local unstable set*, $W^u_\varepsilon(x)$, are

$$W^s_\varepsilon(x) := \{y \in X; d(f^n(x), f^n(y)) \leq \varepsilon,\ \forall n \geq 0\}$$
$$W^u_\varepsilon(x) := \{y \in X; d(f^{-n}(x), f^{-n}(y)) \leq \varepsilon,\ \forall n \geq 0\}$$

For sufficiently small $\varepsilon$, local stable and unstable sets of an Anosov diffeomorphism are embedded submanifolds of $M$. Their relationship to the global stable and unstable manifolds is given by the following proposition.

**Proposition 3.7** ([14])**.** *Let $f : X \to X$ be an expansive homeomorphism of the compact metric space $X$ with expansive constant $c > 0$. Let $0 < \varepsilon \leq c$. Then, for $x \in X$, we have*

$$W^s(x) = \bigcup_{n\geq 0} f^{-n} W^s_\varepsilon(f^n(x)) \quad \text{and} \quad W^u(x) = \bigcup_{n\geq 0} f^{n} W^u_\varepsilon(f^{-n}(x))$$

We can therefore view, for sufficiently small $\varepsilon$, the local stable (resp. unstable) manifold at $x$, $W^s_\varepsilon(x)$ (resp. $W^u_\varepsilon(x)$), as giving a connected neighborhood of $x$ in the global stable (resp. unstable) manifold at $x$. We can now explain precisely what we mean by saying that the stable and unstable foliations of the Anosov diffeomorphism $f$ give $M$ a local product structure.

**Proposition 3.8.** *Let $f : M \to M$ be an Anosov diffeomorphism. There exists $\varepsilon > 0$ such that for all $x \in X$, there exists an embedding $\phi_x : W^s_\varepsilon(x) \times W^u_\varepsilon(x) \to M$ where $\phi_x(x_1, x_2)$ is the unique point of intersection of $W^s_\varepsilon(x_1)$ and $W^u_\varepsilon(x_2)$.*

This proposition follows immediately from the transversality of the stable and unstable foliations for $f$. This concludes our discussion of the stable and unstable manifolds for an Anosov diffeomorphism.

We now turn our attention to generalizing these results to Anosov homeomorphisms. In order to do this, we'll need the notion of a generalized foliation. A generalized foliation is essentially a foliation where the leaves are homology manifolds instead of manifolds. For a precise definition of generalized foliations and a discussion of their properties, see Appendix A.2.

We can now explain how to generalize the above results to Anosov homeomorphisms. These results are all due to Hiraide [12, Section 2]. Let $f : M \to M$ be an Anosov homeomorphism. We begin by observing that the collection of stable (resp. unstable) sets for $f$, which we'll denote by $\mathscr{F}^s_f$ (resp. $\mathscr{F}^u_f$), is a $f$-invariant decomposition of $M$. These collections form transverse generalized foliations. [2]

**Theorem 3.9** ([12, Proposition A])**.** *If $f : M \to M$ is an Anosov homeomorphism of the closed manifold $M$, then the collections*

$$\mathscr{F}^\sigma_f = \{W^\sigma(x);\ x \in M\}, \quad \sigma \in \{s, u\}$$

*are transverse generalized foliations of $M$.*

The rest of this subsection will be dedicated to sketching the proof of this theorem. For more details see [12, Section 2]. To prove Theorem 3.9, we just need to find canonical coordinate charts and to show that the leaves of $\mathscr{F}^s_f$ and $\mathscr{F}^u_f$ are path-connected. In the course of doing this, we'll show how the stable and unstable generalized foliations give $M$ a local product structure, providing an analogue of Proposition 3.8.

The bulk of this proof will involve finding a canonical coordinate chart about an arbitrary point, $x \in M$ (c.f. Definition A.6). The motivation behind the upcoming construction is that the canonical coordinate chart about $x$

[2] The definition of transverse generalized foliations originated with Hiraide and is given in Appendix A.2.

should be given by intersecting stable and unstable sets for points near $x$. Keeping this in mind, the details all involve finding a domain on which we can define the map and proving that it satisfies the properties in Definition A.6.

We begin by defining a map, $\alpha$, that takes two points in a subset of $M \times M$ and gives the intersection of their local stable and unstable sets. To see why we can do this, we observe that for sufficiently small $\varepsilon_0 > 0$, if $x, y \in M$ are sufficiently close (i.e. within $\delta_0$ of each other), then the local stable set at $x$, $W^s_{\varepsilon_0}(x)$, and the local unstable set at $y$, $W^u_{\varepsilon_0}(y)$, will intersect at a single point, which we'll call $\alpha(x, y)$. Note that $\alpha$ is defined on a neighborhood, $\Delta(\delta_0) = \{(x, y) \in M \times M;\ d(x, y) < \delta_0\}$, of the diagonal in $M \times M$. We've therefore defined a continuous map $\alpha : \Delta(\delta_0) \to M$ that takes a pair of points to the intersection of their local stable and unstable sets [11, Lemmas 1-2].

We'll now will use $\alpha$ to define a canonical coordinate chart, $\alpha_x$ at an arbitrary point $x \in M$. First, we define the range of $\alpha$. Take $\varepsilon_0 > 0$ as in the last paragraph. For sufficiently small $\delta > 0$ and $\sigma = s, u$, we define $D^\sigma_x$ to be the connected component of $x$ in the set $\{y \in W^\sigma_{\varepsilon_0}(x);\ d(x, y) < \delta\}$. By choosing $\delta$ small enough, we have that $\alpha$ is defined on $D^s_x \times D^u_x \subset \Delta(\delta_0)$. We'll define $\alpha_x$ to be the restriction of $\alpha$ to $D^s_x \times D^u_x$, i.e. $\alpha_x = \alpha|_{D^s_x \times D^u_x} : D^s_x \times D^u_x \to N_x := \alpha(D^s_x \times D^u_x)$. It turns out that $N_x \subset M$ is a connected, open neighborhood of $x$ and that $\alpha_x$ is a homeomorphism [12, Proposition 2.1]. The rest of the argument involves showing that $\alpha_x$ satisfies the properties given in Definition A.6 and will be omitted.

3.2. **Lifts of stable and unstable sets.** For analogous reasons to those we'll discuss in Section 4, much of Franks' proof of 2.2 and Mannings' proof of Theorem 2.7 took place using maps lifted to the universal cover. These arguments exploited the facts that Anosov diffeomorphisms lift to Anosov diffeomorphisms whose stable and unstable sets are lifts of the original stable and unstable manifolds. We'll now give versions of these facts for Anosov homeomorphisms, which will be used in our proof of Theorem 4.2.

We begin with the following set-up. Let $M$ be a closed Riemannian manifold and let $p : \tilde{M} \to M$ be a smooth covering map for $M$. By lifting the Riemannian metric on $M$, we see that $\tilde{M}$ is a complete Riemannian manifold.

We can now generalize the previous results about lifts of Anosov diffeomorphisms to Anosov homeomorphims. These generalizations are due to Hiraide. For more details on them and their proofs, see [12, Section 3]. Let $f : M \to M$ be an Anosov homeomorphism. The map $f$ lifts to a homeomorphism $\tilde{f} : \tilde{M} \to \tilde{M}$. Just as Anosov diffeomorphisms lift to Anosov diffeomorphism, we observe that an Anosov homeomorphism lifts to an Anosov homeomorphism.

Next, we'll discuss the relationship between the stable and unstable sets of $f$ and $\tilde{f}$. For $\tilde{x} \in \tilde{M}$ and $\varepsilon > 0$, we let $\tilde{W}^s_\varepsilon(\tilde{x})$ and $\tilde{W}^u_\varepsilon(\tilde{x})$ be the local stable and unstable sets of $\tilde{f}$ at $\tilde{x}$. We let $\tilde{W}^s(\tilde{x})$ and $\tilde{W}^u(\tilde{x})$ be the stable and unstable sets of $\tilde{f}$ at $\tilde{x}$. Just as for an Anosov diffeomorphism, the stable and unstable sets for $\tilde{f}$ project down to the stable and unstable sets for $f$. In fact, locally this projection is an isometry.

**Lemma 3.10** ([12, Lemma 3.2])**.** *For sufficiently small $\varepsilon > 0$, the projection $p : \tilde{W}^\sigma(\tilde{x}) \to W^\sigma(p(\tilde{x}))$ is an isometry.*

We next claim that the local product structure on $M$ given by $\alpha_x$, from Section 3.1, lifts to give a local product structure on $\tilde{M}$. First, we observe that the map $\alpha : \Delta(\delta_0) \to M$ from Section 3.1 lifts to a map $\tilde{\alpha} : \tilde{\Delta}(\delta_0) = \{(\tilde{x}, \tilde{y}) \in \tilde{M} \times \tilde{M};\ \tilde{d}(\tilde{x}, \tilde{y}) < \delta_0\} \to \tilde{M}$ such that $\tilde{\alpha}(\tilde{x}, \tilde{y}) = \tilde{W}^s_{\varepsilon_0}(\tilde{x}) \cap \tilde{W}^u_{\varepsilon_0}(\tilde{y})$. As before, we get canonical coordinate charts around a point $\tilde{x}$ by restricting $\tilde{\alpha}$ to the appropriate domain. In this case, we define $\tilde{D}^s_{\tilde{x}}$ and $\tilde{D}^u_{\tilde{x}}$ by intersecting, respectively, the lifts of $D^s_{p(\tilde{x})}$ and $D^u_{p(\tilde{x})}$ with a small ball in $\tilde{M}$ centered at $x$. We obtain the connected neighborhood $\tilde{N}_{\tilde{x}}$ about $\tilde{x}$ similarly. We then get that the map $\tilde{a}_{\tilde{x}} : \tilde{D}^s_{\tilde{x}} \times \tilde{D}^u_{\tilde{x}} \to \tilde{N}_{\tilde{x}}$ is a canonical coordinate chart about $\tilde{x}$. This gives us a local product structure on $\tilde{M}$.

It then follows that the collection of stable (resp.) unstable sets of $\tilde{f}$ forms a generalized foliation, denoted $\mathscr{F}^s_{\tilde{f}}$ (resp. $\mathscr{F}^u_{\tilde{f}}$), and that the stable and unstable generalized foliations for $\tilde{f}$ are transverse.

3.3. **Indices of fixed points.** Let $f : M \to M$ be an Anosov diffeomorphism. The index of $f$ at any fixed point $x$, denoted $\mathrm{Ind}_x(f)$, will be either $\pm 1$ since $d_x f : T_x M \to T_x M$ is hyperbolic. The sign of $\mathrm{Ind}_x(f)$ will depend on the orientation of the stable and unstable subspaces, $E^s_x$ and $E^u_x$, at $x$. Thus, if the unstable bundle, $E^u$, of $f$ is orientable (which implies that the unstable foliation for $f$ is orientable), we can make the fixed point index globally constant, i.e. for all $x, x' \in \mathrm{Fix}(f)$, $\mathrm{Ind}_x(f) = \mathrm{Ind}_{x'}(f)$. This along with the Lefschetz fixed point theorem tells us that the absolute value of the Lefschetz number of $f$, denoted $L(f)$, is equal to the number of fixed points of $f$. This fact is relied upon in the proofs of Theorems 2.2 and 2.7.

The purpose of this section is to give the following similar result about the fixed point index of an Anosov homeomorphism, which will allow us to use the Lefschetz number to count fixed points.

**Theorem 3.11** ([12], Proposition B)**.** *Let $f : M \to M$ be an Anosov homeomorphism of the closed manifold $M$. If the generalized unstable foliation $\mathscr{F}^u_f$ is orientable, then for sufficiently large $m$, all the fixed points of $f^m$ have the same index, which is either $1$ or $-1$.*

Note that the assumption in this theorem (i.e. that the generalized unstable foliation be orientable) is analogous to the assumption we made in the Anosov case. For the definition of orientability for a generalized foliation, see Appendix A.2. The proof of Theorem 3.11 can be found in [12, Section 5].

3.4. **The spectral decomposition.** A useful approach to studying the dynamics of a map on a larger set is to break the set up into smaller invariant sets on which the map exhibits simpler behavior. The spectral decomposition gives us a way to do this for the non-wandering set of an Anosov diffeomorphism. We begin by recalling the definition of the non-wandering set.

**Definition 3.12.** Let $f : X \to X$ be a homeomorphism of a compact metric space $X$. A point $x \in X$ is called *nonwandering* if for any neighborhood $U$ of $x$, $\exists n \geq 1$ such that $f^n(U) \cap U \neq \emptyset$. The nonwandering set of $f$, denoted $\Omega(f)$ is the set of nonwandering points of $f$.

Note that the non-wandering set of a homeomorphism $f$ is both compact and $f$-invariant. Also note that the set of periodic points of an Anosov diffeomorphism, $f$, denoted $\mathrm{Per}(f)$, is dense in the non-wandering set. This fact is used in the proof of the spectral decomposition. Before we state the spectral decomposition, we recall one more definition.

**Definition 3.13.** A continuous map $f : X \to X$ is *topologically mixing* if for any open sets $U, V \subset X$, there exists an integer $N$ such that $f^n(U) \cap V \neq \emptyset$ for all $n \geq N$.

We now can state the spectral decomposition for an Anosov diffeomorphism: the non-wandering set of an Anosov diffeomorphism can be broken up as follows.

**Theorem 3.14** (Spectral Decomposition)**.** *Let $f : M \to M$ be an Anosov diffeomorphism of a compact manifold $M$. There exist closed, pairwise disjoint sets $X_1, ..., X_k$ and a permutation $\sigma \in S_k$ such that*
*(a) $\Omega(f) = \bigcap_{i=1}^k X_i$,*
*(b) $f(X_i) = X_{\sigma(i)}$, and*
*(c) if for $a > 0$, $\sigma^a(i) = i$, then $f^a|_{X_i}$ is topologically mixing.*

Smale originally proved the spectral decomposition for Axiom A systems, of which Anosov diffeomorphisms are an example [19, Theorem 6.2]. The form of the spectral decomposition in Theorem 3.14 was proved by Bowen. Its proof can be found in [3, Section 3.B]. For completeness, we'll state Smale's spectral decomposition (in the Anosov diffeomorphism case) as a corollary. However, before doing so, we recall that a homeomorphism is *topologically transitive* if it has a dense periodic orbit. In future, we'll use the spectral decomposition to refer to both Theorem 3.14 and the following corollary.

**Corollary 3.15.** *Let $f$ be as in Theorem 3.14. Then there exist pairwise disjoint, closed, $f$-invariant sets, $\Omega_1, ..., \Omega_s$ such that $\Omega(f) = \bigcup_{i=1}^s \Omega_i$ and $f|_{\Omega_i}$ is topologically transitive.*

We now proceed to give a generalization of the spectral decomposition for Anosov homeomorphisms. Before stating the spectral decomposition for Anosov homeomorphisms, we note that, just as we saw for Anosov diffeomorphisms, the set of periodic points of an Anosov homeomorphism is dense in the non-wandering set. We now give the generalization of Theorem 3.14 for Anosov homeomorphisms.

**Theorem 3.16** (Spectral Decomposition)**.** *Let $f : M \to M$ be an Anosov homeomorphism of a compact manifold $M$. Then, there exist closed, pairwise disjoint sets $X_1, ..., X_k$ and a permutation $\sigma \in S_k$ such that*
*(a) $\Omega(f) = \bigcap_{i=1}^k X_i$,*
*(b) $f(X_i) = X_{\sigma(i)}$, and*
*(c) if for $a > 0$, $\sigma^a(i) = i$, then $f^a|_{X_i}$ is topologically mixing.*

The arguments given by Bowen to prove Theorem 3.14 rely only on the shadowing property and expansiveness. They therefore apply to Anosov homeomorphisms, as was shown by Aoki [1].

3.5. **Growth of periodic orbits.** We know from the previous section that the set of periodic points for an Anosov diffeomorphism (or for an Anosov homeomorphism) is dense in the non-wandering set. The following theorem shows us that the number of $n$-periodic orbits of an Anosov diffeomorphism grows exponentially in $n$. Before stating this theorem, we give some terminology. For $n \in \mathbb{N}$, we let $P_n(f) = |\,\mathrm{Fix}(f^n)|$ to be the number of $n$-periodic points of $f$.

**Theorem 3.17.** *Let $M$ be a closed topological manifold, and let $f : M \to M$ be an Anosov homeomorphism. Then, there exists $k \in \mathbb{N}$ and constants $c_1, c_2 > 0$ such that for $n \in \mathbb{N}$,*

$$c_1 e^{nkh_{top}(f)} \leq P_{kn}(f) \leq c_2 e^{nkh_{top}(f)}$$

The proof of this theorem relies on the spectral decomposition along with the specification property, which is a stronger type of shadowing property. For details see [13, Chapters 18.3 and 18.5]. This property can be generalized to Anosov homeomorphisms as follows.

**Theorem 3.18.** *Let $M$ be a closed topological manifold, and let $f : M \to M$ be an Anosov homeomorphism. Then there exists $k \in \mathbb{N}$ and constants $c_1, c_2 > 0$ such that for $n \in \mathbb{N}$,*

$$c_1 e^{nkh_{top}(f)} \leq P_{kn}(f) \leq c_2 e^{nkh_{top}(f)} \tag{3.1}$$

The proof of this generalization is identical to the proof of Theorem 3.17 once one shows that the specification property holds for Anosov homeomorphisms. This is not hard to do directly using the shadowing property and expansiveness.

## 4. Global Rigidity for Anosov Homeomorphisms

Our goal in this section is to answer Question 1.1 for Anosov homeomorphisms. We begin, just as we did in Section 2 with Anosov homeomorphisms on tori. Hiraide proved that Franks' answer to Question 1.1 for Anosov diffeomorphisms on tori generalizes to Anosov homeomorphisms on tori.

**Theorem 4.1** ([12])**.** *An Anosov homeomorphism of a torus is topologically conjugate to a hyperbolic toral automorphism.*

Just as Manning extended Franks' proof of Theorem 2.2, we'll extend Hiraide's proof of Theorem 4.1 to nilmanifolds.

**Theorem 4.2.** *An Anosov homeomorphism of a nilmanifold is topologically conjugate to a hyperbolic nilmanifold automorphism.*

Section 4 will be devoted to the proof of this theorem.

4.1. **Overview of the proof of Theorem 4.2.** In this subsection, we'll give an overview of the proof of Theorem 4.2. Subsections 4.2-4.5 will then be devoted to filling in the details of this overview.

Let $f : M \to M$ be an Anosov homeomorphism of the nilmanifold $M = N/\Gamma$. We begin by finding a candidate for the hyperbolic nilmanifold automorphism in Theorem 4.2. In subsection 4.2, we'll find a hyperbolic nilmanifold automorphism, $A : M \to M$, that is homotopic to $f$. As a consequence of the existence of $A$, we'll get that $f$ fixes a point $x_0\Gamma \in M$. By conjugating $f$ by a translation, we can assume without loss of generality that $x_0\Gamma = e\Gamma$.

The goal of the rest of the proof will be to construct a conjugacy between $A$ and $f$. To do this, we'll first construct a semiconjugacy between $A$ and $f$. More precisely, we'll find a continuous map $h : M \to M$ that is homotopic to the identity, that satisfies $A \circ h = h \circ f$, and that fixes $e\Gamma$. This will occur in subsection 4.3.

We'll complete the proof of Theorem 4.2 by proving that the semiconjugacy $h$ is actually a conjugacy. To do this, we just need to show that $h$ is a homeomorphism. The main ingredient in this argument will be showing that $h$ is a local homeomorphism. This combined with the facts that $h$ is surjective (since it's homotopic to the identity) and is a proper map (since $M$ is compact) will imply that $h : (M, e\Gamma) \to (M, e\Gamma)$ is a covering map. Then, since $h$ is homotopic to the identity, we'll get that the covering spaces $h : (M, e\Gamma) \to (M, e\Gamma)$ and $\mathrm{id} : (M, e\Gamma) \to (M, e\Gamma)$ are isomorphic, i.e there is a homeomorphism $g : M \to M$ such that $h = \mathrm{id} \circ g$ [9, Theorem 1.38]. This will complete the argument that $h$ is a homeomorphism, and thus gives a conjugacy between $A$ and $f$.

We'll then just need to show that the map $h$ is a local homeomorphism. We'll do this by showing that its lift, $\tilde{h} : (N, e) \to (N, e)$ [3] is a local homeomorphism. Recall that Brower's theorem on invariance of domain states that a locally injective continuous map between two manifolds without boundary is a local homeomorphism. Thus, we'll be done if we can show that $\tilde{h}$ is locally injective. In fact, we'll show that $\tilde{h}$ is injective.

First, we note that $f$ lifts to an Anosov homeomorphism $\tilde{f} : (N, e) \to (N, e)$. We recall from Section 3.2 that the stable and unstable sets for $\tilde{f}$ are transverse generalized foliations on $N$. The first step in the argument that $\tilde{h}$ is injective will be showing that it suffices to prove that $\tilde{h}$ is injective on stable and unstable leaves of $\tilde{f}$. This will occur in subsection 4.4. Finally, in subsection 4.5, we'll show that $\tilde{h}$ is injective on stable and unstable leaves. This will complete the proof that $\tilde{h}$ is injective, and thus complete the proof of Theorem 4.2.

[3] When we take this lift, we lift the point $e\Gamma \in M$ to the point $e \in N$. In the rest of this section, we'll be lifting $e\Gamma \in M$ to $e \in N$ unless otherwise noted.

4.2. **Linear models of homeomorphisms on nilmanifolds.** The purpose of this section is to find a candidate for the hyperbolic nilmanifold automorphism in Theorem 4.2. We'll do this by finding a linear model of $f$, which we'll then show is hyperbolic. Our linear model of $f$ will be a nilmanifold endomorphism $A$ that is homotopic to $f$. To construct $A$, we'll show that the induced action of $f$ on $\pi_1(M, e\Gamma)$ can be lifted to an automorphism of $\Gamma$. We'll then extend this automorphism to all of $N$ to get our linear model.

Let $f_* : \pi_1(M, e\Gamma) \to \pi_1(M, f(e\Gamma))$ be the homomorphism that $f$ induces on the fundamental group of $M$. We can view $\pi_1(M, e\Gamma)$ and $\pi_1(M, f(e\Gamma))$ as subgroups of $N$. To do this, we first identify $\pi_1(M, e)$ with $\Gamma$ (via the endpoints of the lifts of the loops in the fundamental group). Recall that changing basepoint in the fundamental group is the same as conjugating by some path in $M$. So in the universal cover of $M$ (i.e. $N$), the identification that takes $\pi_1(M, e\Gamma)$ to $\Gamma$ will take $\pi_1(M, f(e\Gamma))$ to $x^{-1}\Gamma x$ for some $x \in N$. By lifting to $N$, we can view $f_*$ as a homomorphism $\Gamma \to x^{-1}\Gamma x$.

Since we want a homomorphism $\Gamma \to \Gamma$, we compose $f_*$ with conjugation by $x^{-1}$, which gives us our automorphism of $\Gamma$. To summarize, we've shown that we can lift $f_* : \pi_1(M, e\Gamma) \to \pi_1(M, f(e\Gamma))$ to an automorphism of $\Gamma$, which is defined up to an inner automorphism of $N$. We can uniquely extend $f_* : \Gamma \to \Gamma$ to an automorphism $\tilde{A} : N \to N$ [17, Corollary 1 of Theorem 2.11]. Since $\tilde{A}$ preserves $\Gamma$, it descends to a nilmanifold automorphism, $A : M \to M$. Note that $f$ is homotopic to $A$ since they induce conjugate maps on $\pi_1(M)$ and $M$ is a $K(\pi, 1)$.

We now claim that the linearization $A$ is hyperbolic. This will follow immediately from the following proposition.

**Proposition 4.3.** *Let $f : M \to M$ be an Anosov homeomorphism of a nilmanifold $M = N/\Gamma$. If $A : M \to M$ is a nilmanifold automorphism that is homotopic to $f$, then $A$ is hyperbolic.*

*Proof.* By passing to a double cover of $M$, it suffices to consider the case where the unstable generalized foliation of $f$, $\mathscr{F}^u_f$, is orientable. The goal of this proof is to show that $A$ is hyperbolic. More formally, we need to show that $D_eA$ has no eigenvalues of absolute value one. Let $\lambda_1, ..., \lambda_n$ be the eigenvalues of $D_eA$ (counted with multiplicity).

The first step in this proof will be to relate the number of $m$-periodic points of $f$, for $m \in \mathbb{N}$, to the eigenvalues of $D_eA$. We'll do this using the Lefschetz fixed point theorem. First, recall that since $f^m$ and $A^m$ are homotopic, their Lefschetz numbers are the same, i.e. $L(f^m) = L(A^m)$. The Lefschetz fixed point theorem says that the Lefschetz number of $f^m$ is

$$L(f^m) = \sum_{x \in \mathrm{Fix}(f^m)} \mathrm{Ind}_{f^m}(x)$$

Then, since $\mathscr{F}^u_f$ is orientable, we can apply Theorem 3.11 to get that for some $\ell \in \mathbb{N}$, all fixed points of $f^m$ have the same index (of either $\pm 1$), for all $m \geq \ell$. Thus, the homeomorphism $f^m$ has $N(f^m)$ fixed points, where $N(f^m) = |L(f^m)|$. Now, Theorem B.10 gives another expression for $L(f^m) = L(A^m)$, namely that $L(f^m) = \prod_{i=1}^n (1 - \lambda_i^m)$. We've therefore shown that, for $m \geq \ell$, the number of fixed points of $f^m$ is given by

$$P_m(f) = N(f^m) = \prod_{i=1}^n |1 - \lambda_i^m| \,. \tag{4.1}$$

The rest of the proof will be devoted to showing that this equation cannot hold if $A$ is not hyperbolic. We do this in two steps.

First, we'll show that none of the $\lambda_i$ are roots of unity. Suppose wlog that $\lambda_1$ is a $q$th root of unity, i.e. $\lambda_1^q = 1$. We know from the spectral decomposition that $\mathrm{Per}(f) \neq \emptyset$. So, for some $k \in \mathbb{N}$, $P_k(f) \geq 1$, which implies that $P_{jk}(f) \geq 1$ for all $j \in \mathbb{N}$. However, for $m = qk$, the right hand side of (4.1) is 0. We've therefore shown that none of the eigenvalues of $D_eA$ are roots of unity.

Now, we show that $A$ is hyperbolic. Recall from Theorem 3.18 that $\exists k \in \mathbb{N}$ such that $P_{mk}(f)$ grows exponentially in $m$. Combining this with (4.1) tells us that there exists a constant $c > 0$ such that for all $m \in \mathbb{N}$,

$$ce^{kh_{top}(f)} \leq \frac{P_{(m+1)k}(f)}{P_{mk}(f)} = \prod_{i=1}^n \frac{|1 - \lambda_i^{(m+1)k}|}{|1 - \lambda_i^{mk}|} \tag{4.2}$$

Now, suppose that $A$ is not hyperbolic. We'll show that the lim inf of the right hand side is zero, which will be a contradiction. Let $\lambda_1, ..., \lambda_s$ are the eigenvalues of $D_eA$ of modulus 1. We write,

$$\prod_{i=1}^n \frac{|1 - \lambda_i^{(m+1)k}|}{|1 - \lambda_i^{mk}|} = \left( \prod_{|\lambda_i|=1} \frac{|1 - \lambda_i^{(m+1)k}|}{|1 - \lambda_i^{mk}|} \right) \left( \prod_{|\lambda_i|>1} \frac{|1 - \lambda_i^{(m+1)k}|}{|1 - \lambda_i^{mk}|} \right) \left( \prod_{|\lambda_i|<1} \frac{|1 - \lambda_i^{(m+1)k}|}{|1 - \lambda_i^{mk}|} \right)$$

We consider each of these terms separately. Note that

$$\prod_{|\lambda_i|>1} \frac{|1-\lambda_i^{(m+1)k}|}{|1-\lambda_i^{mk}|} \overset{m\to\infty}{\longrightarrow} \prod_{|\lambda_i|>1} |\lambda_i|^k \qquad \text{and} \qquad \prod_{|\lambda_i|<1} \frac{|1-\lambda_i^{(m+1)k}|}{|1-\lambda_i^{mk}|} \overset{m\to\infty}{\longrightarrow} 1$$

Thus,

$$\liminf_{m\to\infty} \prod_{i=1}^{n} \frac{|1-\lambda_i^{(m+1)k}|}{|1-\lambda_i^{mk}|} = \left( \prod_{|\lambda_i|>1} |\lambda_i|^k \right) \left( \liminf_{m\to\infty} \prod_{|\lambda_i|=1} \frac{|1-\lambda_i^{(m+1)k}|}{|1-\lambda_i^{mk}|} \right)$$

We'll now show that

$$\liminf_{m\to\infty} \prod_{|\lambda_i|=1} \frac{|1-\lambda_i^{(m+1)k}|}{|1-\lambda_i^{mk}|} = 0$$

Since $\lambda_i$ is of modulus 1 and is not a root of unity, for $1 \leq i \leq s$, by density of irrational rotations of the circle, we can find an increasing sequence of natural numbers $\{m_j\}$ such that $\lambda_i^{m_j k} \to \lambda_i^{-k}$ as $j \to \infty$ for all $1 \leq i \leq s$. Thus, we have that

$$\prod_{i=1}^{s} |1-\lambda_i^{(m_j+1)k}| \overset{j\to\infty}{\longrightarrow} 0$$

Also note that

$$\prod_{i=1}^{s} |1-\lambda_i^{m_j k}| \overset{j\to\infty}{\longrightarrow} \prod_{i=1}^{s} |1-\lambda_i^{k}| \neq 0$$

since none of the $\lambda_i$ are roots of unity. We therefore get that,

$$0 \leq \liminf_{m\to\infty} \prod_{|\lambda_i|=1} \frac{|1-\lambda_i^{(m+1)k}|}{|1-\lambda_i^{mk}|} \leq \lim_{j\to\infty} \prod_{|\lambda_i|=1} \frac{|1-\lambda_i^{(m_j+1)k}|}{|1-\lambda_i^{m_j k}|} = 0$$

This completes the proof that

$$\liminf_{m\to\infty} \prod_{i=1}^{n} \frac{|1-\lambda_i^{(m+1)k}|}{|1-\lambda_i^{mk}|} = 0$$

Taking lim infs on both sides of (4.2) then gives $0 < ce^{kh_{top}(f)} \leq 0$, which is a contradiction. ☐

Recall that when we defined the linearization $A$ of an Anosov homeomorphism $f : M \to M$ of a nilmanifold, we only were able to define $A$ up to an inner automorphism of $N$ because we didn't know whether $f$ had any fixed points. We are now equipped to show that $f$ does indeed have fixed points, which we can assume without loss of generality include the identity, $e\Gamma$.

**Corollary 4.4.** *An Anosov homeomorphism of a nilmanifold has at least one fixed point.*

*Proof.* This follows immediately from the Lefschetz fixed point theorem and Proposition 4.3. ☐

4.3. **Building the semiconjugacy.** The goal of this section is to show that if $f : N/\Gamma \to N/\Gamma$ is a homeomorphism whose linearization $A$ is hyperbolic, then $f$ is semiconjugate to $A$. Combining this with Proposition 4.3 will give the first step in constructing the conjugacy in Theorem 4.2.

**Proposition 4.5.** *Let $M = N/\Gamma$ be a nilmanifold, and let $f : M \to M$ be a homeomorphism that fixes the point $e\Gamma \in M$. If $f$ is freely homotopic to a hyperbolic nilmanifold automorphism $A : M \to M$, then there exists a continuous map $h : M \to M$ (freely) homotopic to the identity such that $A \circ h = h \circ f$ and $h(e\Gamma) = e\Gamma$. Furthermore, the map $h$ is unique.*

*Remark* 4.6. Note that the requirement that $h(e\Gamma) = e\Gamma$ is necessary for uniqueness of $h$. Also, note that the map $h$ from Proposition 4.5 is surjective since it's homotopic to the identity. Thus, $f$ is semiconjugate to $A$ via $h$.

The rest of this subsection is dedicated to an overview of the proof of Proposition 4.5. The big idea in this proof is that because $M$ is a $K(\pi, 1)$, "everything" is controlled (up to homotopy) by the fundamental group. To make this precise, we recall the following two facts from Algebraic Topology:

**Fact 4.7** ([16, top of p.423])**.** *Let $X$ be a $K(\pi, 1)$. Any two maps $g$ and $g'$ from $X \to X$ that preserve a base point, say $x \in X$, are freely homotopic if and only if they induce conjugate homomorphisms on the fundamental group $\pi_1(X, x)$ (i.e. there exists $\alpha \in \pi_1(X, x)$ such that for all $\beta \in \pi_1(X, x)$, we have that $g_*(\beta) = \alpha^{-1} g'_*(\beta)\alpha$).*

**Fact 4.8** ([9, Proposition 1B.9])**.** *Let $X$ be a $K(\pi,1)$. Every homomorphism $\phi : \pi_1(X,x) \to \pi_1(X,x)$ is induced by a unique (up to homotopy fixing $x$) continuous map $g : (X,x) \to (X,x)$.*

The proof of Proposition 4.5 breaks down into two major parts/lemmas. First, we use the fact that $f$ and $A$ are freely homotopic to construct a conjugacy of the induced homomorphisms $f_*$ and $A_*$ on the fundamental group, $\pi_1(M, e\Gamma)$. We'll see that this conjugacy is induced by a base point preserving map of $M$ that is freely homotopic to the identity. This will take place in Lemma 4.9 and will rely on Facts 4.7 and 4.8 along with the fact that $A$ is hyperbolic. Next, we show that this conjugacy between the induced homomorphisms $f_*$ and $A_*$ is induced by an actual semiconjugacy between $A$ and $f$. This will take place in Lemma 4.10. This step is the harder of the two and is where we really use the fact that $M$ is a nilmanifold.

We now give a precise statement of the first step.

**Lemma 4.9.** *Under the same assumptions as Proposition 4.5, there exists a continuous, basepoint preserving map $h_0 : (M, e\Gamma) \to (M, e\Gamma)$ that is (freely) homotopic to the identity such that*

$$\begin{array}{ccc} \pi_1(M,e\Gamma) & \xrightarrow{\phi} & \pi_1(M,e\Gamma) \\ \big\downarrow{\scriptstyle f_*} & & \big\downarrow{\scriptstyle A_*} \\ \pi_1(M,e\Gamma) & \xrightarrow{\phi} & \pi_1(M,e\Gamma) \end{array} \tag{4.3}$$

*commutes, where $\phi = (h_0)_*$.*

*Proof of Lemma 4.9.* By Fact 4.7, Since $f$ and $A$ are (freely) homotopic, they induce maps on $\pi_1(M, e\Gamma)$ that differ by an inner automorphism. Then, the hyperbolicity of $A$ allows us to use this conjugacy to define a homomorphism $\phi : \pi_1(M, e\Gamma) \to \pi_1(M, e\Gamma)$ that makes (4.3) commute. □

We've show that the induced homomorphisms $f_*$ and $A_*$ of $\pi_1(M, e\Gamma)$ are conjugate via a homomorphism, $\phi$, that is induced by map $h_0 : (M, e\Gamma) \to (M, e\Gamma)$ that is freely homotopic to the identity. Now, we'll complete the proof of the Proposition 4.5 by using the map $h_0$ and the conjugacy it induces between $f_*$ and $A_*$ to construct a base point preserving semiconjugacy between $A$ and $f$ that also induces $\phi$ (and thus is homotopic to the identity).

**Lemma 4.10.** *Let $M = N/\Gamma$ be a nilmanifold, let $f : M \to M$ be a homeomorphism with fixed point $e\Gamma \in M$, and let $A : M \to M$ be a hyperbolic nilmanifold automorphism. Suppose that there exists a continuous map $h_0 : (M, e\Gamma) \to (M, e\Gamma)$ such that (4.3) commutes, where $\phi = (h_0)_*$. Then, there exists a continuous map $h : M \to M$ that is homotopic to $h_0$, that fixes $e\Gamma$ and such that $A \circ h = h \circ f$. Further, the map $h$ is unique.*

*Proof of Lemma 4.10.* This proof is based on the proof of Theorem 2.2. in [7]. Since $M$ is a $K(\pi,1)$, it suffices to find a continuous map $h : (M, e\Gamma) \to (M, e\Gamma)$ that satisfies the equation $A \circ h = h \circ f$ and that induces the homomorphism $\phi$ on $\pi_1(M, e\Gamma)$. We'll find this map by constructing its lift $\tilde{h} : (N, e) \to (N, e)$.

The advantage of working with the lifted maps is that the requirement that $h$ induces the homomorphism $\phi$ on $\pi_1(M, e\Gamma)$ has a more concrete interpretation when working with the lifted maps. To see this, recall that a map $\tilde{g} : (N, e) \to (N, e)$ is the lift of a map $g : (M, e\Gamma) \to (M, e\Gamma)$ if and only if for all $x \in N,\ \gamma \in \Gamma$, we have that $\tilde{g}(x\gamma) = \tilde{g}(x)g_*(\gamma)$ [4]. Thus, our goal is to find a map $\tilde{h} : (N, e) \to (N, e)$ that will satisfy the equation

$$\tilde{A} \circ \tilde{h} = \tilde{h} \circ \tilde{f}, \tag{4.4}$$

where $\tilde{A}$ is the hyperbolic automorphism of $N$ that descends to $A$, and the equation

$$\tilde{h}(x\gamma) = \tilde{h}(x)\phi(\gamma). \tag{4.5}$$

The second equation ensures that $\tilde{h}$ descends to a map $h : M \to M$ that is homotopic to $h_0$.

Our strategy will be to obtain $\tilde{h}$ by multiplying the lift, $\tilde{h}_0 : (N, e) \to (N, e)$, of $h_0$ by a "correction". More precisely, we want $\tilde{h}$ to be a function of the form $\tilde{h} = \hat{h}\tilde{h}_0$, [5] where $\hat{h} : (N, e) \to (N, e)$ is $\Gamma$-periodic (i.e. $\hat{h}(x\gamma) = \hat{h}(x)$). Note if we can find a function $\tilde{h}$ of this form that satisfies (4.4), we'll be done since (4.5) holds for all functions of the form $\hat{h}\tilde{h}_0$. The rest of this proof will be dedicated to finding the function $\hat{h}$.

[4] The map $g_* : \pi_1(M, e\Gamma) \to \pi_1(M, e\Gamma)$ is the homomorphism induced by $g$ on the fundamental group. Recall that since $N$ is the universal cover of $M = N/\Gamma$, we can identify $\pi_1(M, e\Gamma)$ with $\Gamma$ as follows: Each loop $\alpha$ in $M$ based at $e\Gamma$ lifts to a unique path $\tilde{\alpha} : [0,1] \to N$ with $\tilde{\alpha}(0) = e$. Note that $\tilde{\alpha}(1) \in \Gamma = p^{-1}(e\Gamma)$ (where $p : N \to M = N/\Gamma$ is the usual projection) depends only on the homotopy class of $\alpha$ in the fundamental group. Then, we identify $[\alpha] \in \pi_1(M, e\Gamma)$ with $\tilde{\alpha}(1) \in \Gamma$. This identification allows us to view the map $f_* : \pi_1(M, e\Gamma) \to \pi_1(M, e\Gamma)$ as a homomorphism $\Gamma \to \Gamma$ and shows that $\tilde{f}|_\Gamma = f_*$ as a map from $\Gamma \to \Gamma$.

[5] The function $\hat{h}\tilde{h}_0$ is given by pointwise multiplication, i.e. $\left(\hat{h}\tilde{h}_0\right)(x) = \hat{h}(x)\tilde{h}_0(x)$.

Note that any function $\tilde{h}$ that satisfies (4.4) will be a fixed point of the function $F_0 : C(N,N) \to C(N,N)$ given by $F_0(k) = \tilde{A} \circ k \circ \tilde{f}^{-1}$. The fixed points of $F_0$ are precisely the points mapped to the identity by the function $T : C(N,N) \to C(N,N)$ given by $T(k)(x) = k(x)^{-1} F_0(k)(x)$. Solving the equation $T(k) \equiv e$, rather than (4.4), will allow us to use the fact that $\tilde{h}$ is of the form $\hat{h}\tilde{h}_0$ to get an equation for $\hat{h}$ in terms of $\tilde{h}_0$. We'll show that $T$ maps a function of the form $\tilde{h} = \hat{h}\tilde{h}_0$ to the identity if and only if the equation

$$T(\tilde{h}_0^{-1}) = T(\hat{h}) \tag{4.6}$$

holds, where $\tilde{h}_0^{-1}(x) = \tilde{h}_0(x)^{-1}$. Thus, we'll be done if we find a $\Gamma$-periodic function $\hat{h} \in C(N,N)$ that fixes $e$ and that solves the equation $T(\tilde{h}_0^{-1}) = T(\hat{h})$. We'll now formalize this.

We will solve (4.6) using the following spaces of functions.

- The space $Q_0 := C(N,N)$ of continuous functions from $N$ to $N$ with the compact-open topology, and
- its subset $Q := \{k \in Q_0;\ k(e) = e \text{ and } k \text{ is } \Gamma - \text{periodic}\}$ of continuous, $\Gamma$-periodic functions from $N$ to itself that fix $e \in N$. Note that $Q$ is a closed subgroup of $Q_0$. In fact, $Q$ is a nilpotent Banach Lie group. [6]

Recall that our goal is to find a function $\hat{h} \in Q$ such that the function $T : Q_0 \to Q_0$ sends the map $\tilde{h} := \hat{h}\tilde{h}_0$ to the identity in $Q_0$, which is just the map that sends everything to the identity. (For this reason, we'll denote the identity in $Q_0$ as $e$, the same as the identity in $N$.) We claim that $T$ maps a function, $\tilde{h}$, of this form, to the identity iff (4.6) holds. This follows immediately from the fact that $T(\hat{h}\tilde{h}_0) = \tilde{h}_0^{-1}\hat{h}^{-1}F_0(\hat{h})F_0(\tilde{h}_0^{-1})$.

Thus, our goal is to find a function $\hat{h} \in Q$ such that $T\left(\tilde{h}_0^{-1}\right) = T\left(\hat{h}\right)$. To do this, first observe $T\left(\tilde{h}_0^{-1}\right) \in Q$. Now, all that remains is to show that the the image of $T$ restricted to $Q$ is all of $Q$. This will prove the existence of a map $\hat{h}$ with the desired properties.

We'll see that $T(Q) = Q$ in the following two steps, which we'll just sketch. For full details see [7, Theorem 2.2]. First, we note that $T(Q)$ contains an neighborhood of the identity via an application of the inverse function theorem to $T|_Q : Q \to Q$ at the identity. We then can use induction on the nilpotency class of $Q$ to see that the image of $T$ is closed under multiplication. These two facts along with the fact that $Q$ is a connected Lie group show that $T(Q) = Q$. $\square$

4.4. **Global Product Structure.** Before jumping into this section, let's summarize where we are in the proof of Theorem 4.2. Let $f : M \to M$ be an Anosov homeomorphism of the nilmanifold $N = M/\Gamma$. Assume without loss of generality that $e\Gamma \in M$ is a fixed point for $f$. We've found a hyperbolic nilmanifold automorphism, $A : M \to M$, that is homotopic to $f$, and we've constructed a map $h : M \to M$ that is homotopic to the identity, that fixes $e\Gamma$, and that satisfies the equation, $A \circ h = h \circ f$.

Now, we just need to show that $h$ is a semiconjugacy, in other words, that $h$ is a homeomorphism. Recall from subsection 4.1 that, to do this, we just need to show that $\tilde{h}$ is injective, where $\tilde{h} : (N,e) \to (N,e)$ is the lift of $h$. (As usual, we lift as in [3].)

Recall from Section 3.2 that the stable and unstable sets for $\tilde{f}$ form transverse generalized foliations, which we call the stable and unstable foliations of $\tilde{f}$ and denote by $\mathscr{F}^s_{\tilde{f}}$ and $\mathscr{F}^u_{\tilde{f}}$. In this subsection, we'll prove to show that $\tilde{h}$ is injective, it suffices to show that $\tilde{h}$ is injective on leaves of the stable and unstable foliations. This will follow by proving that the stable and unstable generalized foliations establish a global product structure on $N$. More precisely that,

**Proposition 4.11.** *For any points $x, y \in N$ the stable leaf through $x$ and the unstable leaf through $y$ intersect at exactly one point, i.e. the set $\tilde{W}^s(x) \cap \tilde{W}^u(y)$ contains exactly one point.*

Before going through the proof of Proposition 4.11, we'll show how this proposition implies that injectivity of $\tilde{h}$ follows from injectivity on stable and unstable leaves. Take $x, y \in N$ such that $\tilde{h}(x) = \tilde{h}(y)$. By Proposition 4.11, we can define a point $z := \tilde{W}^s(x) \cap \tilde{W}^u(y)$ to be the intersection of the stable leaf through x and the unstable leaf through $y$. If we show that $\tilde{h}(x) = \tilde{h}(y) = \tilde{h}(z)$, then injectivity of $\tilde{h}$ will follow from injectivity of the stable and unstable leaves. Thus, it suffices to show that $\tilde{h}(y) = \tilde{h}(z)$.

To prove $\tilde{h}(y) = \tilde{h}(z)$, recall that since $\tilde{A}$ is a hyperbolic automorphism of $N$, for arbitrary $M_1 > 0$, the map $\tilde{A}$ is expansive with expansive constant $M_1$. Thus, to show that $\tilde{h}(y) = \tilde{h}(z)$, it suffices to show that there exists a

[6] To see that $Q$ is a Banach Lie group, let $\Delta := \{k' \in \Delta_0;\ k'(e) = 0 \text{ and } k \text{ is } \Gamma - \text{periodic}\}$ be the set of continuous, $\Gamma$-periodic functions from $N$ to its Lie algebra $\mathfrak{n}$ that send $e \in N$ to $0 \in \mathfrak{n}$. Note that $\Delta$ is a vector space with norm, $\|\cdot\|$, given by $\|k'\| := \sup_{x \in N} |k'(x)|$, where $|\cdot|$ is the norm on $\mathfrak{n}$. This norm makes $\Delta$ into a Banach space. We can then define a chart $\text{Exp} : \Delta \to Q$ by composition with the Lie exponential map from $\mathfrak{n} \to N$. This chart along with the group structure inherited from $Q_0$ makes $Q$ into a Banach Lie group [8, Book I, Chapter 4.3.1].The fact that $Q$ is nilpotent follows from the fact that $N$ is nilpotent.

constant $M_1 > 0$ such that for all $n \in \mathbb{Z}$,

$$d\left(\tilde{A}^n \circ \tilde{h}(z), \tilde{A}^n \circ \tilde{h}(y)\right) \leq M_1. \tag{4.7}$$

To see this, first recall that since $\tilde{A} \circ \tilde{h} = \tilde{h} \circ \tilde{f}$ and $\tilde{h}(x) = \tilde{h}(y)$, we have that for all $n \in \mathbb{Z}$,

$$d\left(\tilde{A}^n \circ \tilde{h}(z), \tilde{A}^n \circ \tilde{h}(y)\right) = d\left(\tilde{h} \circ \tilde{f}^n(z), \tilde{h} \circ \tilde{f}^n(y)\right) = d\left(\tilde{h} \circ \tilde{f}^n(z), \tilde{h} \circ \tilde{f}^n(x)\right)$$

In light of these two equations, to prove 4.7, it'll be sufficient to prove that there exists a constant $M_1 > 0$ such that for all $n \geq 0$, the following two inequalities hold.

$$d\left(\tilde{h} \circ \tilde{f}^{-n}(z), \tilde{h} \circ \tilde{f}^{-n}(y)\right) \leq M_1$$
$$d\left(\tilde{h} \circ \tilde{f}^{n}(z), \tilde{h} \circ \tilde{f}^{n}(x)\right) \leq M_1$$

These inequalities follow immediately from the following two observations,

- Since $h$ is homotopic to the identity, the map $\tilde{h}$ is a bounded distance away from the identity, i.e. there exists a constant $M_0 > 0$ such that $\forall w \in N$, $d\left(\tilde{h}(w), w\right) \leq M_0$.
- The facts that $z \in \tilde{W}^s(x)$ and $z \in \tilde{W}^u(y)$ imply that there exists a constant $C > 0$ such that for all $n \geq 0$,

$$d\left(\tilde{f}^n(x), \tilde{f}^n(z)\right) \leq C \quad \text{and} \quad d\left(\tilde{f}^{-n}(y), \tilde{f}^{-n}(z)\right) \leq C$$

Now, all that remains is to prove that the stable and unstable generalized foliations give a global product structure on $N$, i.e. Proposition 4.11. We'll do this in four steps:

First, we claim that the map $\tilde{f}$ has exactly one fixed point.

**Lemma 4.12.** *Let $f : M \to M$ be an Anosov homeomorphism of the nilmanifold $M = N/\Gamma$. Let $\tilde{f} : N \to N$ be a lift of $f$ to $N$, and let $\tilde{A} : N \to N$ be a hyperbolic automorphism of $N$. If the $C^0$-distance between $\tilde{A}$ and $\tilde{f}$ is bounded, then $\tilde{f}$ has exactly one fixed point.*

*Proof.* This proof is a slight modification of the proof given in [12, Lemma 6.5]. The main ingredients in this proof are the Lefschetz fixed point theorem and the homotopy invariance of the Lefschetz number. Since we're working in a space that isn't compact, we need to be careful when applying these properties. [7] Since $\tilde{A}$ is a hyperbolic automorphism, we know that its Lefschetz number is $L(\tilde{A}) = \pm 1$. The Lefschetz number of $\tilde{f}$ is defined because it is a bounded distance away from $\tilde{A}$.

Next, we observe that since $N$ is contractible, we can construct a homotopy between $\tilde{f}$ and $\tilde{A}$ that does not introduce fixed points outside of a compact set. Thus, $L(\tilde{f}) = L(\tilde{A}) = \pm 1$, which implies that $\tilde{f}$ has at least one fixed point.

To see that $\tilde{f}$ has at most one fixed point, we show that for some $m \geq 1$, $\tilde{f}^m$ has at most one fixed point. Since all fixed points of $\tilde{f}$ are fixed points of $\tilde{f}^m$, this will show that $\tilde{f}$ has at most one fixed point. Since we know that $L(\tilde{f}^m) = L(\tilde{A}^m) = 1$ for all $m$, it suffices to show that for some $m \geq 1$, the number of fixed points of $\tilde{f}^m$, $N(\tilde{f}^m)$, is equal to the Lefschetz number of $\tilde{f}^m$. To prove this, we just need to show that there exists $m$ such that all fixed points of $\tilde{f}^m$ have the same fixed point index of $\pm 1$, which follows from Theorem 3.11. ☐

We now prove that the non-wandering set of $f$ is the whole nilmanifold.

**Lemma 4.13.** *The nonwandering set of an Anosov homeomorphism $f : M \to M$ of a nilmanifold $M = N/\Gamma$ is the entire nilmanifold, i.e. $\Omega(f) = M$.*

*Proof.* This follows from the same argument given by Hiraide in [12, Proposition 6.6]. ☐

We now can begin showing that the stable and unstable generalized foliations of $\tilde{f}$ give a global product structure on $N$.

**Lemma 4.14.** *Given the set up from the beginning of this subsection, for $x, y \in N$, the stable manifold of $\tilde{f}$ at $x$, $\tilde{W}^s(x)$, and the unstable manifold of $\tilde{f}$ at $y$, $\tilde{W}^u(y)$, intersect at at most one point.*

*Proof.* This follows from the previous two lemmas along with the spectral decomposition. The details are exactly the same as those in [12, Lemma 6.7]. ☐

[7] Recall that the Lefschetz number of a map $g : X \to X$ is only defined if the set of fixed points $\mathrm{Fix}(g)$ is compact. Two maps have the same Lefschetz number if they are homotopic via a map that does not introduce fixed points out of a compact set. [5]

We can now finally complete the proof of Proposition 4.11, i.e. that the stable and unstable generalized foliations of $\tilde{f}$ give a global product structure on $N$. All that remains is to prove that $\tilde{W}^s(x)$ and $\tilde{W}^u(y)$ actually intersect for each $x, y \in N$. This follows from using fact that $\mathscr{F}^s_{\tilde{f}}$ and $\mathscr{F}^u_{\tilde{f}}$ endow $N$ with a local product structure and gluing local product neighborhoods together using the arguments given in [6, Lemma 1.6].

4.5. **Distance in the stable and unstable leaves.** All that remains in the proof of Theorem 4.2 is to show that $\tilde{h}$ is injective on the stable and unstable leaves of $\tilde{f}$. We begin by sketching the idea of the proof that $\tilde{h}$ is injective on unstable leaves. [8] Using the fact that $\tilde{h}$ is homotopic to the identity, we can show that if $z \in \tilde{W}^u(y)$ and $\tilde{h}(y) = \tilde{h}(z)$, then the 'distance' in the unstable leaf $\tilde{W}^u(y)$ between $y$ and $z$ must be hounded independently of $y$ and $z$. This along with expansiveness of $f$ will imply that $y = z$. Note that since we don't know that the unstable leaves are manifolds, we don't have an intrinsic notion of distance on the unstable leaves. We therefore need to define a new notion of distance to make this argument work.

To define our new notion of distance, fix $\varepsilon_0 = \frac{c}{4}$, where $c$ is an expansive constant for $\tilde{f}$. We define,

$$\begin{aligned} &\text{for } z \in W^s(x), \quad d(x, z; \tilde{W}^s(x)) = \min\left\{m \geq 0;\ \tilde{f}^m(z) \in \tilde{W}^s_{\varepsilon_0}(\tilde{f}^m(x))\right\}, \text{ and} \\ &\text{for } z \in W^u(y), \quad d(y, z; \tilde{W}^u(y)) = \min\left\{m \geq 0;\ \tilde{f}^{-m}(z) \in \tilde{W}^u_{\varepsilon_0}(\tilde{f}^{-m}(y))\right\}. \end{aligned} \tag{4.8}$$

Note that $d(x, z; \tilde{W}^s(x)) < \infty$ and $d(y, z; \tilde{W}^u(y)) < \infty$ follows from an analogue of Proposition 3.7 [12, Lemma 3.5]. This notion of distance satisfies the requirement that for all $y \in N$, there exists a constant $M_2 \in \mathbb{N}$ such that if $z \in \tilde{W}^\sigma(y)$ and $\tilde{h}(y) = \tilde{h}(z)$, then $d(y, z; \tilde{W}^\sigma(y)) \leq M_2$ [12, p.386-388]. It therefore can be used as described at the beginning of the subsection, to prove that $\tilde{h}$ is injective on stable and unstable leaves.

## Appendix A. Foliations and Generalized Foliations

This appendix is intended to give a brief overview of generalized foliations. It will begin with a review of topological foliations and will then proceed with a discussion of generalized foliations. The intent here is to show the parallels between topological foliations and generalized foliations.

A.1. **Topological Foliations.** In this section, we'll provide a brief review of topological foliations. For more details about foliations, see [10, Chapter II].

**Definition A.1.** A continuous map $F : X \to Y$ is a *topological immersion* if $\forall x \in X$, there exists a neighborhood $U$ of $x$ such that $F|_U$ is a topological embedding (i.e. a homeomorphism onto its image).

Let $M$ be an $n$-dimensional topological manifolds. We now define a topological foliation on $M$.

**Definition A.2.** A collection $\mathscr{F}$ of subsets of $M$ is a (codimension $q$) *topological foliation* of $M$ if the following properties hold:

(1) $\mathscr{F}$ is a decomposition of $M$, i.e. $\mathscr{F}$ is a collection of pairwise disjoint sets whose union is $M$.
(2) Each $L \in \mathscr{F}$ (called a *leaf*) is a connected, topologically immersed submanifold of dimension $n - q$.
(3) For each $x \in M$, there exists
   - a connected, open neighborhood $N_x \subset M$ of $x$, and
   - a homeomorphism, $\phi_x : N_x \to B_{x,\tau} \times B_{x,\pitchfork} \subset \mathbb{R}^{n-q} \times \mathbb{R}^q$

   such that for any $L \in \mathscr{F}$, $L \cap N_x$ is a (countable) union of sets of the form $\phi_x^{-1}(B_{x,\tau} \times \{y\})$ (called *plaques*).

**Example A.3.** As we saw in Section 3.1, the collections of stable and unstable manifolds for an Anosov diffeomorphisms are transverse topological foliations with smooth leaves.

A.2. **Generalized Foliations.** In this section, we'll discuss a way to generalize the concept of a foliation. To the best of my knowledge, this concept originated with Hiriade in [12]. Our discussion is based on that given in [12]. We begin with the definition of a generalized foliation. In this section, let $M$ be a connected, topological manifold without boundary, and let $n$ be the dimension of $M$.

**Definition A.4.** A collection, $\mathscr{F}$, of subsets of $M$ is a *generalized foliation* of $M$ if the following properties hold:

(1) $\mathscr{F}$ is a decomposition of $M$.
(2) Each $L \in \mathscr{F}$ (called a *leaf*) is path-connected.
(3) For each $x \in M$, there exist
   - nontrivial, connected subsets $D_x, K_x \subset M$ with $D_x \cap K_x = \{x\}$,

[8] The idea behind the proof of injectivity on stable leaves is identical.

- a connected, open neighborhood $N_x \subset M$ of $x$,
- a homeomorphism $\phi_x : D_x \times K_x \to N_x$ (called *local coordinates around* $x$)

such that

(a) $\phi_x(x, x) = x$,
(b) $\phi_x(y, x) = y \ \forall y \in D_x$ and $\phi_x(x, z) = z \ \forall z \in K_x$,
(c) For any $L \in \mathscr{F}$, there is at most a countable set $B \subset K_x$ such that $N_x \cap L = \phi_x(D_x \times B)$.

It's clear that all foliations are generalized foliations, however the reverse may not be true. The sole difference between the definitions of a foliation and of a generalized foliation is we don't require the sets $D_x$ and $K_x$ to be manifolds in the definition of a generalized foliation. (If $D_x$ and $K_x$ are manifolds for all $x \in M$, then a generalized foliation $\mathscr{F}$ is, in fact, a topological foliation of $M$.) While the sets $D_x$ and $K_x$ may fail to be manifolds, the fact that their product, $D_x \times K_x$, is a manifold significantly restricts the ways in which $D_x$ and $K_x$ can fail to be manifolds. In other words, $D_x$ and $K_x$ (and therefore the leaves of $\mathscr{F}$), while not necessarily manifolds themselves, will behave like manifolds in many ways. In fact, the leaves of a generalized foliations are homology manifolds (also known as generalized manifolds). A homology manifolds is a topological space that looks like a manifold under homology. This is stated precisely in the following proposition.

**Proposition A.5** ([12, Lemma 4.2])**.** *Let $\mathscr{F}$ be a generalized foliation on a connected manifold without boundary. There exists $0 < p < \dim(M)$ such that any leaf $L \in \mathscr{F}$ and $x \in L$, the relative homology groups, $H_*(L, L \setminus \{x\})$ are given by*

$$H_i(L, L \setminus \{x\}) = \begin{cases} \mathbb{Z}, & \text{if } i = p \\ 0, & \text{if } i \neq p \end{cases}$$

This proposition allows us to define a notion of dimension for generalized foliation. If $\mathscr{F}$ is a generalized foliation of $M$, then the integer $p$ from Proposition A.5 is called the *dimension* of $f$.

We now give the definition of transverse generalized foliations.

**Definition A.6.** Two generalized foliations $\mathscr{F}$ and $\mathscr{F}'$ on $M$ are *transverse* if, for each $x \in M$, there exist

- nontrivial, connected subsets $D_x, D'_x \subset M$ with $D_x \cap D'_x = \{x\}$,
- a connected, open neighborhood $N_x$ of $x$ (called *a coordinate domain*),
- a homeomorphism $\phi_x : D_x \times D'_x \to N_x$ (called a *canonical coordinate chart (around* $x$*)*),

such that

(a) $\phi_x(x, x) = x$,
(b) $\phi_x(y, x) = y \ \forall y \in D_x$ and $\phi_x(x, z) = z \ \forall z \in D'_x$,
(c) for any $L \in \mathscr{F}$, there is at most a countable set $B' \subset D'_x$ such that $N_x \cap L = \phi_x(D_x \times B')$,
(d) for any $L' \in \mathscr{F}'$, there is at most a countable set $B \subset D_x$ such that $N_x \cap L' = \phi_x(B \times D'_x)$.

Observe that the canonical coordinate charts satisfy the conditions for local coordinate charts for $\mathscr{F}$ and $\mathscr{F}'$ from the definition of a generalized foliation. Also note that if $\mathscr{F}$ and $\mathscr{F}'$ are transverse generalized foliations, then $\dim(\mathscr{F}) + \dim(\mathscr{F}') = \dim(M)$.

Now, we give an idea of how to define orientability for generalized foliations. For the sake of brevity, we'll refrain from giving a precise, formal definition, and will just give an overview of the idea behind the definition. A precise definition of orientability for generalized foliations can be found in [12, Section 4]. To motivate this definition, we recall the definition of orientability for a $p$-dimensional topological manifold [9, Chapter 3.3]. To do this we recall that for each $x \in M$. the relative homology groups $H_*(M, M \setminus \{x\})$ are given by

$$H_i(M, M \setminus \{x\}) = \begin{cases} \mathbb{Z}, & \text{if } i = p \\ 0, & \text{if } i \neq p \end{cases} \tag{A.1}$$

An *orientation* on $M$ is a choice for each $x \in M$ of a generator $\mu_x$ for the groups $H_p(M, M \setminus \{x\})$, called a *local orientation* of $M$ at $x$, that is locally consistent (for more details on what we mean by locally consistent, see [9, p.234]). Since this definition only relies on the relative homology groups of the leaves having the form (A.1), by Proposition A.5, we can define orientability for generalized foliations in the same way.

## Appendix B. Nilmanifolds

Two of the main theorems in this proposal (Theorem 2.7 and Theorem 4.2) are generalizations to nilmanifolds of analogous theorems (Theorem 2.2 and Theorem 4.1) on tori. In this appendix, we'll see why a nilmanifold is a natural generalization of a torus, which will give us a way to extend proofs from tori to nilmanifolds.

We begin by reviewing the concept of a nilpotent group.

B.1. **Nilpotent Lie groups and algebras.** The idea behind this section is that many of the nice properties of abelian Lie groups and algebras hold for central extensions of abelian Lie groups and algebras. To make this precise, we'll define the notions of nilpotent Lie groups and algebras. Before we can do this, we'll define central series of a group, which gives a way to quantify how far from abelian a group is.

**Definition B.1.** Let $G$ be an abstract group.

(a) The *lower central series* of $G$ is $G = G^0 \geq G^1 \geq ...$ where $G^{s+1} = [G^s, G]$. (Note that this implies that $G^s \lhd G$.)
(b) The *upper central series* of $G$ is $G_0 = \{e\} \leq G_1 \leq ...$ where $G_{s+1} = \pi^{-1}(Z(G/G_s))$, where $\pi : G \to G/G_s$ is the quotient homomorphism. (Note that this implies that $G_s \lhd G$.)
(c) A series of subgroups $\{e\} = G_0 \leq G_1 \leq ...$ is called a *central series* for $G$ if one of the following two equivalent conditions hold:
   (i) $G_s \lhd G$ and $G_{s+1}/G_s \leq Z(G/G_s)$
   (ii) $[G_{s+1}, G] \leq G_s$.
   Alternatively, we can define a *central series* for $G$ to be a series of subgroups $G_0 = G \geq G_1 \geq \dots$ such that one of the following two equivalent conditions hold,
   (i) $G_s \lhd G$ and $G_s/G_{s+1} \leq Z(G/G_{s+1})$
   (ii) $[G_s, G] \leq G_{s+1}$.

See [18, Chapter 11.1] for proof of the equivalence of the conditions in (c).

**Definition B.2.** A group $G$ is *nilpotent* if it has a central series $G_0 = \{e\} \leq G_1 \leq \cdots \leq G_c = G$. If $G_{c-1} \neq G$, we say that $c$ is the *length* of the central series $G_0 \leq G_1 \leq \cdots \leq G_c$.

If $G$ is nilpotent, all of its central series have the same (finite) length, which we call the *nilpotency class* of $G$. In fact, we get the following relationship between the upper and lower central series: If

- $G = \gamma_0 G \geq \gamma_1 G \geq \cdots \geq \gamma_c G = \{e\}$ is the lower central series of $G$,
- $\{e\} = \zeta_0 G \leq \zeta_1 G \leq \cdots \leq \zeta_c G = G$ is the upper central series of $G$, and
- $\{e\} = G_0 \leq G_1 \leq \cdots \leq G_c = G$ is a central series for $G$,

then, $\gamma_{c-s} G \leq G_s \leq \zeta_s G$.

Now, we turn our attention to nilpotent Lie groups. When discussing central series of nilpotent Lie groups, we'll usually restrict our attention to central series made up of closed subgroups of $G$. The advantage of doing this is that it implies that all the subgroups in that central series are embedded Lie subgroups and that $G_{s+1}/G_s$ and $G/G_s$ are Lie groups. This means that we preserve the Lie group structure when working with central series. We'll now see that these remarks apply to the upper central series of a nilpotent Lie group:

**Proposition B.3.** *Let $G$ be a Lie group, and let $\{e\} = G_0 \leq G_1 \leq \dots$ be the upper central series for $G$. Then $G_s$ is closed in $G$ for each $s \in \mathbb{Z}_+$.*

The situation with the lower central series is a bit more complicated; the subgroups in the lower central series of a nilpotent Lie group are not necessarily closed. However, they will be closed in the special case of a simply-connected lie group.

**Proposition B.4.** *Let $G$ be a simply-connected Lie group, and let $G = G^0 \geq G^1 \geq \dots$ be the lower central series for $G$. Then $G^s$ is closed in $G$ for all $s \in \mathbb{Z}_+$.*

Thus, nilpotent Lie group, $G$, we can view a nilpotent Lie group, $G$, as a finite series of central extensions using the upper central series. If in addition, $G$ is simply-connected, we can do this using the lower central series.

When studying nilmanifolds, simply-connected nilpotent Lie groups are of particular interest. We've already seen that the groups in both the upper and lower central series for a simply-connected Lie group are closed. In this special case (i.e. for a simply-connected Lie group), the groups in both the upper and lower central series are simply-connected Lie subgroups. This follows from a corollary of the following theorem.

**Theorem B.5.** *If $N$ is a simply-connected nilpotent Lie group with Lie algebra $\mathfrak{n}$, then the exponential map $\exp : \mathfrak{n} \to N$ is surjective.*

**Corollary B.6.** *Let $G$ be a connected, simply-connected, nilpotent Lie group with Lie algebra $\mathfrak{g}$. For every subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, there is a closed, simply-connected Lie subgroup $H$ of $G$ with Lie algebra $\mathfrak{h}$.*

B.2. **Nilmanifolds as generalizations of tori.** We just saw that nilpotent groups were generalizations of abelian groups by viewing a nilpotent group as a finite series of central group extensions. In this subsection, we'll see that we can view nilmanifolds as generalizations of tori in a similar way. We'll see that a nilmanifold can be written as a finite sequence of torus bundles. This structure often makes it possible to extend proofs and constructions from tori to nilmanifolds using induction. Before doing this, we recall the definition of a nilmanifold:

**Definition 2.3.** A *nilmanifold* is a compact homogeneous space of the form $N/\Gamma$, where $N$ is a simply connected nilpotent Lie group and $\Gamma$ is a cocompact lattice in $N$.

A couple of remarks on this definition:

(1) Saying $\Gamma$ is a cocompact lattice in $N$ is technically redundant. By Theorem 2.1 in [17, Theorem 2.1], all lattices in a nilpotent Lie group are cocompact, so we could actually just assume that $\Gamma$ is a lattice in $N$.
(2) We'll be viewing $N/\Gamma$ as the set of left cosets, $\{n\Gamma;\ n \in N\}$, of $\Gamma$ in $N$. Everything works the same way if you use right cosets instead, but, for no particular reason, I like left cosets better.
(3) In this section, unless otherwise noted, we'll assume that $N$ is a simply-connected nilpotent Lie group of nilpotency class $c$ and that $\Gamma$ is a cocompact lattice in $N$. When we say "Let $M = N/\Gamma$ be a nilmanifold", these assumptions hold.

We now will see how a nilmanifold can be written as a finite sequence of torus bundles. This kind of result should be expected from the fact that a nilmanifold is a $K(\pi, 1)$, and thus its topology is controlled by its fundamental group $\pi_1(M, e\Gamma) \cong \Gamma$. Since $\Gamma$ is nilpotent, it can be written as a sequence of central extensions, which correspond to the fundamental groups of the desired tori.

As stated above, the we can write $M$ as a finite sequence of torus extensions of a one point space. By this we mean that there is a finite sequence of tori, $T_1, ..., T_d$, and a finite sequence of manifolds $M_0, ..., M_d$, with $M_0 = M$ and $M_c = \{*\}$, such that $M_{s-1}$ is a smooth $T_s$-bundle over $M_s$. This can be done according to the following theorem:

**Theorem B.7.** *Let $M = N/\Gamma$ be a nilmanifold. Let $\{e\} = N_0 \leq ... \leq N_c = N$ be a central series for $N$ such that $N_s$ is closed and connected. Let $\Gamma_s = N_s \cap \Gamma$. Then, we can define the following:*

- *the compact, connected, abelian Lie groups (i.e. a tori), $T_s = N_s/(\Gamma_s N_{s-1})$, for $1 \leq s \leq c$, and*
- *the manifolds $M_s = N/(\Gamma N_s)$ for $0 \leq s \leq c$, where $M_0 = M$ and $M_c = \{*\}$.*

*Then, we can define a free, smooth, proper action of $T_s$ on $M_{s-1}$ given by left multiplication, which gives us the following principal $T_s$-bundle:*

$$\begin{array}{ccc} T_s & \longrightarrow & M_{s-1} \\ & & \Big\downarrow{\scriptstyle \pi_s} \\ & & M_s \end{array}$$

*where $\pi_s : M_{s-1} \to M_s$ is the natural projection map, $n\Gamma N_{s-1} \mapsto n\Gamma N_s$.*

This tells us that $M$ looks like a twisted product of tori. In other words, locally, $M$ looks like a finite product of tori, $T_1 \times ... \times T_c$. Globally, $M$ will differ from $T_1 \times ... \times T_c$ by twists in how the fibers are glued together.

Our goal in writing a nilmanifold as a sequence of torus bundles was to generalize properties of tori to nilmanifolds via induction. We'll now give an alternative (equivalent) ways of writing the bundles in Theorem B.7 which makes this task a bit easier.

**Corollary B.8.** *Let $M = N/\Gamma$ be a nilmanifold, and let $N_s$ and $\Gamma_s$ for $0 \leq s \leq c$ be as in Theorem B.7. Define*

- *the compact, connected, abelian Lie groups (i.e. tori), $T'_s = (N_s/N_{s-1})/(\Gamma_s N_{s-1}/N_{s-1}) = (N_s/N_{s-1})/(\Gamma_s/\Gamma_{s-1})$ for $1 \leq s \leq c$, and*
- *the manifolds $M'_s = (N/N_s)/(\Gamma N_s/N_s) = (N/N_s)/(\Gamma/\Gamma_s)$ for $0 \leq s \leq c$. Note that $M'_0 = M$ and $M'_c = \{*\}$*

*Then, $T'_s$ acts on $M'_{s-1}$ via left multiplication, which gives us the following principal $T'_s$-bundle:*

$$\begin{array}{ccc} T'_s & \longrightarrow & M'_{s-1} \\ & & \Big\downarrow{\scriptstyle \pi'_s} \\ & & M'_s \end{array}$$

*where $\pi'_s : M'_{s-1} \to M'_s$ is the natural projection map, given by sending the equivalence class of $n$ in $M'_{s-1}$ to the equivalence class of $n$ in $M'_s$.*

We've shown that if $M = N/\Gamma$ is a nilmanifold, and $N$ is $c$-step nilpotent, then $M$ is a principal torus bundle over a nilmanifold $M_1 = (N/N_1)/(\Gamma/\Gamma_1)$. Since $N/N_1$ is simply-connected and $(c-1)$-step nilpotent, this gives a way to generalize properties of tori to nilmanifolds via induction on $c$.

B.2.1. *Application: the Lefschetz number of an endomorphism of a nilmanifold.* As stated at the beginning of the section, the fact that a nilmanifold can be written as a finite sequence of torus extensions allows us to generalize many results from tori to nilmanifolds. In this subsection, we'll use this strategy to generalize the following result from tori to nilmanifolds.

**Proposition B.9** ([13])**.** *Let $A : T^n \to T^n$ be an automorphism of an $n$-torus. Then the Lefschetz number of $A$ is $L(A) = \det(\mathrm{Id} - A) = \prod_{i=1}^{n}(1-\lambda_i)$, where $\lambda_1, ..., \lambda_n$ are the eigenvalues of $A$.*

The goal of this subsection is to prove the following generalization by following the argument in [15].

**Theorem B.10** ([16])**.** *Let $M = N/\Gamma$ be a nilmanifold, and let $A : N/\Gamma \to N/\Gamma$ be a nilmanifold automorphism. Let $\lambda_1, ..., \lambda_n$ be the eigenvalues (counted with multiplicity) of $D_{e\Gamma}A$. Then, the Lefschetz number of $A$ is $L(A) = \prod_{i=1}^{n}(1-\lambda_i)$.*

This generalization was integral to Manning's extension of Franks' proof of Theorem 2.2 to nilmanifolds. It will similarly necessary to our generalization of Theorem 4.1 in Section 4. We now proceed to its proof.

*Proof.* This proof is taken from [15]. Suppose that $N$ is $c$-step nilpotent. We'll prove Theorem B.10 by induction on $c$. If $c = 1$, then $N$ is abelian. Since all subgroups of an abelian group are normal, we get that $M = N/\Gamma$ is a compact, connected, abelian Lie group, in other words, a torus. The conclusion of Theorem B.10 thus holds. Now, we suppose that the conclusion of the theorem holds when $N$ is $c$-step nilpotent for some $c \geq 1$. Suppose $N$ is $(c+1)$-step nilpotent.

In the last section, we showed that $M = N/\Gamma$ can be written as a principal torus bundle over a nilmanifold, $M' = N'/\Gamma'$, where $N'$ is $c$-step nilpotent. We now recall how to do that. Let $N_0 \leq N_1 \leq ... \leq N_{c+1}$ be the upper central series for $N$. The upper central series for $\Gamma$ is $\Gamma_0 \leq \Gamma_1 \leq ... \leq \Gamma_{c+1}$ where $\Gamma_s = N_s \cap \Gamma$. [9] As in Corollary B.8, we define $T_1 = N_1/\Gamma_1$ and $M_1 = (N/N_1)/(\Gamma/\Gamma_1)$. Then, $M$ is the principal $T_1$-bundle,

$$\begin{array}{ccc} T_1 & \longrightarrow & M \\ & & \big\downarrow \pi \\ & & M_1 \end{array}$$

where $\pi : M \to M_1$ takes $n\Gamma \in M$ to the equivalence class of $n$ in $M_1$. This means that $M$ is a "twisted product" of the torus, $T_1$ and the nilmanifold $M_1$.

The strategy of the proof will be to "break down" the nilmanifold automorphism, $A : M \to M$, into a toral automorphism $A_1 : T_1 \to T_1$ and a nilmanifold automorphism $B : M_1 \to M_1$. We can do this because the induced map $A_* : \Gamma \to \Gamma$ on the fundamental group $\pi_1(M, e\Gamma) \cong \Gamma$ preserves the upper central series of $\Gamma$. Proposition B.9 along with the induction hypothesis will then tell us that the Lefschetz number of $A_1 \times B : T_1 \times M_1 \to T_1 \times M_1$ will be given by the expression in Theorem B.10. Since $A$ will differ from $A_1 \times B$ by a twist in the fiber, we'll then conclude by showing that this twist doesn't affect the Lefschetz number. This is an immediate consequence of the following lemma.

**Lemma B.11** ([15])**.** *Let $\pi : (X, *) \to (B, *)$ is a fiber bundle with fiber $F = \pi^{-1}(*)$ such that $F$ or $B$ is compact and such that the fundamental group $\pi_1(B, *)$ acts trivially on the homology of $F$. Then, if $(\psi, \chi)$ is a bundle map and $\omega = \psi|_F$, then the Lefschetz numbers $L(\psi) = L(\chi \times \omega)$ are equal.*

□

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