RESEARCH STATEMENT

MEG DOUCETTE

1. Introduction

I work in smooth dynamics with a focus on partially hyperbolic dynamics. I am particularly interested in questions of classification and rigidity for partially hyperbolic diffeomorphisms.

Partially hyperbolic diffeomorphisms are a rich and widely studied class of dynamical systems. As a generalization of Anosov diffeomorphisms, they display a mix of the rigid, chaotic behavior of Anosov systems along with more flexible behavior, which allows for interesting phenomena. My research focuses on fibered partially hyperbolic systems, which are a class of partially hyperbolic diffeomorphisms where the interplay between the rigid chaotic behavior and more flexible behavior can be more easily observed and quantified than in general partially hyperbolic diffeomorphisms.

Classification and rigidity results are an extremely important tool in understanding partially hyperbolic systems because they give a way to understand the dynamical behavior of general partially hyperbolic systems by understanding simpler models, for example smooth ones or "linear" ones. A focus of my research is to expand classification and rigidity results that exist for partially hyperbolic diffeomorphisms in dimension three to fibered partially hyperbolic diffeomorphisms in higher dimensions. I am particularly interested in combining tools from dynamics with tools from Lie theory, topology, and geometry to study these systems.

1.1. **Outline.** In Section 2, I give some basic definitions and give a broader context for my results. In Section 3, I discuss the two main results of my thesis. Section 3.1 describes a rigidity result that gives smooth models for certain fibered partially hyperbolic systems. Section 3.2 discusses homotopy rigidity of fibered partially hyperbolic diffeomorphisms of Heisenberg nilmanifolds. Finally, in Section 4, I discuss plans for future research that builds upon my past and current work.

2. Background

A diffeomorphism $f : M \to M$ of a closed Riemannian manifold is said to be *partially hyperbolic* if there is a continuous Df-invariant splitting $TM = E^s \oplus E^c \oplus E^u$, constants $0 < \lambda < \hat{\gamma} < 1 < \gamma < \mu$, and constant $C > 0$ such that for all $x \in M$ and all unit vectors $v^s \in E^s(x)$, $v^c \in E^c(x)$, and $v^u \in E^u(x)$, we have that for all $n \in \mathbb{N}$,

$$\|d_x f^n v^s\| \leq C\lambda^n, \qquad \frac{1}{C}\hat{\gamma}^n \leq \|d_x f^n v^c\| \leq C\gamma^n, \qquad \frac{1}{C}\mu^n \leq \|d_x f^n v^s\|.$$

In other words, Df is uniformly contracting in the direction of the stable bundle E^s, is uniformly expanding in the direction of the unstable bundle E^u, and is less contracting and/or expanding in the direction of the center bundle E^c than it is of the other two bundles.

Partially hyperbolic diffeomorphisms are a generalization of *Anosov* diffeomorphisms, which are partially hyperbolic diffeomorphisms with $E^c = \{0\}$. Up to finite covers, all known Anosov diffeomorphisms lie on nilmanifolds, which are manifolds of the form N/Γ, where N is a simply-connected nilpotent Lie group and $N < \Gamma$ is a uniform lattice.

Anosov diffeomorphisms have been extensively studied and the chaotic behavior they display exhibits an extraordinary degree of stability and rigidity. For example, Franks and Manning classified Anosov diffeomorphisms of tori and nilmanifolds by proving that any Anosov diffeomorphism of a torus or nilmanifold is topologically conjugate to a linear model [6], [12].

Partially hyperbolic diffeomorphisms are important because they combine the chaotic behavior of Anosov diffeomorphisms in the directions of the stable and unstable bundles E^s and E^u, respectively, with a more flexible range of behaviors in the direction of the center bundle E^c. Remarkably little is known about partially hyperbolic diffeomorphisms, making the study of them an exciting and dynamic field.

My research is focused on a class of partially hyperbolic systems called *fibered partially hyperbolic diffeomorphisms*. These are partially hyperbolic diffeomorphisms that have an integrable center bundle E^c, tangent to a continuous invariant fibration by compact submanifolds. The fibration here allows us to directly observe the interplay between the distinct behavior in the stable/unstable directions and in the transverse center direction.

Fibered partially hyperbolic systems form a rich class of dynamical systems. Beyond the relatively simple examples of skew products on trivial bundles, fibered systems appear as automorphisms of nilmanifolds and play a role in the construction of exotic partially hyperbolic systems (e.g. [7]) and in several rigidity contexts [1], [2], [13]. They also have featured in the proofs of several classification results for partially hyperbolic diffeomorphisms.

A natural notion of equivalence for partially hyperbolic dynamical systems is leaf conjugacy. Two partially hyperbolic diffeomorphisms $f : M \to M$ and $g : M' \to M'$ are said to be *leaf conjugate* if there exists a homeomorphism $h : M \to M'$, a f-invariant foliation W^c_f tangent to the center bundle of f, and a g-invariant foliation W^c_g tangent to the center direction of g such that h maps center leaves of f to center leaves of g (i.e. $h(W^c_f(x)) = W^c_g(h(x))$ and $h(f(W^c_f(x))) = g(h(W^c_f(x))))$.

The classification of partially hyperbolic diffeomorphisms up to leaf conjugacy (and also of fibered partially hyperbolic systems) is almost completely open in dimensions greater than three. There are a number of results in dimension three. For example, Hammerlindl and Potrie showed that every partially hyperbolic diffeomorphisms on three-dimensional nilmanifolds and tori is leaf conjugate to the "linear" model in its homotopy class [8], [10]. They also provided a partial classification up to leaf conjugacy of partially hyperbolic diffeomorphisms on 3-manifolds whose fundamental group is solvable and has exponential growth [11]. All of these results rely heavily on the topology of three-manifolds and tools from [3], [4], which only apply in dimension three. As a result, the methods used in dimension three do not generalize well (if at all) to higher dimension.

While there are a lot of results about classification of partially hyperbolic diffeomorphisms in dimension three, almost nothing is known in higher dimension. The results that exist typically require very strong assumptions. For example, Hammerlindl proved that a partially hyperbolic diffeomorphism of the torus $f : \mathbb{T}^d \to \mathbb{T}^d$ $(d > 3)$ is leaf conjugate to the linear automorphism of $\mathbb{T}^d$ in its homotopy class under the assumption that the center foliation for f is one-dimensional and the stable and unstable foliations for f are quasi-isometric [8]. Sandfeldt has shown that under similar assumptions of one-dimensional center and quasi-isometry of stable and unstable foliations, a partially hyperbolic diffeomorphism of a nilmanifold modeled on the $(2n+1)$-dimensional Heisenberg group is leaf conjugate to a nilmanifold automorphism [15].

In order to even conjecture a classification result, one has to find candidates for the "simpler" system. This limits how general a classification result can be. For example, any diffeomorphism on a nilmanifold (e.g. a torus) gives rise to a "linear" model. In other words, by considering the action of a diffeomorphism on the fundamental group of a nilmanifold or torus, one can find a toral/nilmanifold automorphism in the same homotopy class as the original diffeomorphism. This observation was used by Hammerlindl, Potrie, and Sandfeldt [8], [10], [15].

Even though these results are about the more general class of partially hyperbolic diffeomorphism, fibered partially hyperbolic systems do play a role in their proofs– in constructing the leaf conjugacy, both results show that the original partially hyperbolic systems are fibered. This shows one way in which classification results for fibered partially hyperbolic systems could be of use in understanding the more general class of partially hyperbolic systems.

3. Main Results

A central goal of my research is to classify fibered partially hyperbolic diffeomorphisms up to leaf conjugacy. One thing that makes this challenging is that the fibration of such a system is typically only continuous, and its fibers are C^1 in general. While fibered partially hyperbolic systems are not generally smooth, when they are, the induced map on the base of the fibration, which represents the stable and unstable directions, is Anosov. This allows us to use tools and results about Anosov diffeomorphisms to study smooth fibered partially hyperbolic systems. It also suggests a natural subclass of fibered partially hyperbolic systems to consider.

As noted earlier, Anosov diffeomorphisms are classified on nilmanifolds, but a classification on arbitrary manifolds of does not exist. While all known examples of Anosov diffeomorphisms exist on nilmanifolds, it is unknown if nilmanifolds are the only manifolds that can support Anosov diffeomorphisms. In my research, I consider fibered partially hyperbolic systems over nilmanifolds. This allows for use and modification of existing results about Anosov diffeomorphisms on nilmanifolds. It also allows us to construct examples of fibered partially hyperbolic systems using existing examples of Anosov diffeomorphisms on nilmanifolds.

3.1. **Smooth models for fibered partially hyperbolic systems.** In [5], I showed that under certain conditions, every fibered partially hyperbolic system over a nilmanifold is leaf conjugate to a smooth model that is isometric on the fibers and descends to a hyperbolic nilmanifold automorphism on the base.

Theorem 1 ([5], Theorem A). *Let $f : M \to M$ be a fibered partially hyperbolic system with quotient a nilmanifold B and fiber F (where F is a closed manifold). Suppose that the structure group of the F-bundle M is $G \subset \mathrm{Diff}^1(F)$ and that there exists a Riemannian metric on F and a subgroup I of $\mathrm{Isom}(F) \cap G$ such that the inclusion $I \hookrightarrow G$ is a homotopy equivalence.*

Then f is leaf conjugate to a C^∞-fibered partially hyperbolic system $g : \widehat{M} \to \widehat{M}$ (in which both g and the fibration are C^∞) such that

1. *The projection of the leaf conjugacy to B is a map homotopic to the identity;*
2. *the F-bundles M and $\widehat{M}$ are isomorphic;*
3. *the structure group of $\widehat{M}$ is $\mathrm{Isom}(F)$; and*
4. *the projection of g to B is a hyperbolic nilmanifold automorphism.*

The general application of Theorem 1 raises a homotopical question about the fiber F: for which manifolds F and which subgroups G of $\mathrm{Diff}^\infty(F)$ is there a subgroup $H \subset \mathrm{Isom}(F) \cap G$ such that the inclusion $H \hookrightarrow G$ is a homotopy equivalence? There are numerous examples of manifolds for which this question can be answered. For example, for the spheres S^1, S^2, and S^3 and for hyperbolic 3-manifolds, the inclusion of their isometry group in their diffeomorphism group is a homotopy equivalence; this means that Theorem 1 applies to any fibered partially hyperbolic system with base a nilmanifold and fiber one of these manifolds. Thus, for example, for $k \leq 3$ Theorem 1 gives a complete classification of all fibered partially hyperbolic systems with S^k fibers.

The proof of Theorem 1 only requires the assumption on the structure group G of the bundle at the very end. Without this assumption, the argument gives that the initial fibered partially hyperbolic $f : M \to M$ is leaf conjugate to an extension over a hyperbolic nilmanifold automorphism. In other words, without the assumption, we can construct a conjugacy between the map induced by f on the base B and a hyperbolic nilmanifold automorphism $A : B \to B$, and we can also construct a smooth F-bundle $\widehat{M}$ over B that is isomorphic to the original F-bundle M. Without the assumption on the structure group G, we can lift A to a homeomorphism of $\hat{M}$ that is leaf-conjugate to the original f, but we cannot guarantee that the lift is smooth or partially hyperbolic. I would like to explore other situations where A can be lifted to a fibered partially hyperbolic systems in my future research. I would like to use this setup to further explore the following question.

Guiding Question 1. *Let $f : M \to M$ be a fibered partially hyperbolic system with quotient a nilmanifold B and fiber F (where F is a closed manifold). When can we find a C^∞-fibered partially hyperbolic system $g : \widehat{M} \to \widehat{M}$ that is leaf conjugate to f and satisfies conclusions* (1), (2), *and* (4) *of Theorem 1?*

In the setup from [5], this question boils down to the question of lifting $A : B \to B$ to a partially hyperbolic diffeomorphism $g : \widehat{M} \to \widehat{M}$. In addition to showing that Question 1 can be answered affirmatively in the setting of Theorem 1, in [5], I also answered Question 1 when the original bundle M is trivial:

Theorem 2 ([5], Proposition B). *Let $f : M \to M$ be a fibered partially hyperbolic system with quotient a nilmanifold B and fiber F (where F is a closed manifold). Suppose that the F-bundle M is trivial (i.e. that the F-bundle M is isomorphic to $B \times F$).*

Then f is leaf conjugate to a C^∞ fibered partially hyperbolic system $g : \widehat{M} \to \widehat{M}$ such that 1., 2, and 4. from Theorem 1 hold.

Theorem 2 follows from the setup described above by lifting $A : B \to B$ by making the lift the identity in the fiber direction (which is possible because the bundle $\widehat{M}$ is trivial).

In the proofs of both Theorem 1 and Theorem 2 in [5], the method of lifting $A : B \to B$ to a partially hyperbolic system $g : \widehat{M} \to \widehat{M}$ did not use the fact that $f : M \to M$ was partially hyperbolic at all– the existence of f was solely used to guarantee that there was no topological obstruction to lifting A (and the fact that f was partially hyperbolic was completely irrelevant to doing this). Once we had that A could be lifted to a homeomorphism, we relied solely on the topological and geometric structure of the fiber bundle M (and $\widehat{M}$) to get the lift to be partially hyperbolic with center tangent to the fibers. Asking for such a lift to exist without using the partial hyperbolicity of f in a general setting amounts to asking the following question:

Guiding Question 2. *Given a smooth F-bundle M over a nilmanifold (or torus) B and a hyperbolic automorphism $A : B \to B$, if A can be lifted to a homeomorphism $\hat{g} : M \to M$, can A be lifted to a partially hyperbolic diffeomorphism $g : M \to M$ with center direction tangent to the fibers F?*

Such a lift g would be required to be smooth, partially hyperbolic, and to have E^c tangent to the fibers. These requirements can create restrictions on the topology of the bundle M. It therefore seems unreasonable to expect an affirmative answer to Question 2 without further restrictions on the bundle. In fact, it does not even seem obvious that A must lift to a diffeomorphism in the general case described in Question 2.

Thus, the fact that we didn't use that f was partially hyperbolic at all to lift A in [5] suggests a major area for improvement. I see two obvious strategies (which can certainly be combined) for how to use the partial hyperbolicity of f to lift A to a fibered partially hyperbolic system: (1) Use the existence of f to create restrictions on the topology and geometry of the fiber bundle, and then use those restrictions to get the desired lift g of A; (2) Directly use the existence of f to construct the lift g of A.

3.2. **Homotopy rigidity of fibered partially hyperbolic diffeomorphisms of Heisenberg nilmanifolds.** Currently I am working on extending the results and methods of Hammerlindl and Potrie in [9], [10] to higher dimensional Heisenberg nilmanifolds. Hammerlindl and Potrie's methods rely heavily on the results of [3], [4], which are extremely rooted in the specifics of three-manifold topology and geometry. As part of a work in progress, under slightly different assumptions, I have established results similar to those of Sandfeldt in [15] using the techniques of [10]. The assumptions used are quite strong, but they are designed to enable limited versions of the 3-manifold results used by Hammerlindl and Potrie to hold in the necessary higher dimensional situations.

Disregarding the difficulties that arise here from the differences in topology in 3-manifolds and higher dimensional manifolds, Hammerlindl and Potrie's strategy fundamentally will not work for partially hyperbolic diffeomorphisms of Heisenberg nilmanifolds with E^c of dimension greater than one. This is because they use the way E^c interacts with the algebraic center of the Heisenberg group, which is one-dimensional. They use this to show that the original partially hyperbolic diffeomorphism is fibered with one dimensional fiber. This makes this approach unsuited to classifying partially hyperbolic diffeomorphisms of Heisenberg nilmanifolds with E^c of dimension greater than one.

4. Future Directions for Research

In general, the fibered partially hyperbolic systems $g : \widehat{M} \to \widehat{M}$ found in Theorem 1 and in Theorem 2 are not in the same homotopy class as the original system $f : M \to M$. While f and g induce the same action on the base (in terms of homotopy), they do not generally induce the same action on the fiber (in terms of homotopy).

I want to know whether the smooth model g can be taken to be in the same homotopy class as f. More generally, I would like to find answers to the following guiding question in the case of fibered partially hyperbolic systems over nilmanifolds.

Guiding Question 3. *When is a fibered partially hyperbolic system $f : M \to M$ leaf conjugate to a smooth model that is in the same homotopy class as the original system?*

This question is of interest even when the original bundle M is smooth. For example, even when the original bundle M is smoothly trivial, it is not obvious how to lift A to a fibered partially hyperbolic diffeomorphism that doesn't change the action on fibers.

One idea here is to consider cases where M has additional structure. One way to do this is to suppose that M is a nilmanifold. It's easy to see that if M and B are nilmanifolds, then the fiber F must be topologically a nilmanifold using the long exact sequence of homotopy groups for the fiber bundle. This allows us to find an algebraic model for f and also for its action on F as seen in [8] and [10]. Inspired by the approach in these papers, I want to answer the following question:

Question 4. *Let $f : M \to M$ be a fibered partially hyperbolic system over a nilmanifold, and assume M is a nilmanifold. Under what circumstances is f leaf-conjugate to the unique automorphism of M in its homotopy class?*

The lack of results in dimension greater than 3 makes any answer to this question extremely interesting. Additionally, any obstructions or barriers to constructing such a leaf-conjugacy would indicate potential avenues for creating exotic or pathological fibered partially hyperbolic systems.

The question of whether the linearization of a partially hyperbolic diffeomorphism on a nilmanifold (or torus) is hyperbolic is answered affirmatively in dimension 3, but is open in higher dimensions [3],[14]. Thus, finding circumstances where we can answer Question 4 is not sufficient to provide an answer to Questions 1 and 3. To do this, we need to answer an additional question.

Question 5. *Let $f : M \to M$ be a fibered partially hyperbolic system over a nilmanifold, and assume M is a nilmanifold. When is the "linearization" of f partially hyperbolic?*

Answering these questions in the context that M is a nilmanifold and g is a nilmanifold automorphism poses the question

Question 6. *Suppose that $f : M \to M$ is a fibered partially hyperbolic system over a nilmanifold B and that M is a nilmanifold. How does assuming that f is a nilmanifold automorphism restrict the structure of the bundle $M \to B$?*

The algebraic structure of a nilmanifold significantly limits the forms the "linearization" of a diffeomorphism can take (e.g. see section 2 of [9]). Because the fiber bundle in a fibered partially hyperbolic system commutes with the diffeomorphism, when the diffeomorphism in the system is an automorphism, the bundle will have to respect the algebraic structure of the nilmanifold, at least to some extent (again this can be seen in [9]). This allows use of tools from Lie theory and algebraic topology to be more directly used to study the dynamics of these systems.

Because the bundles M and $\widehat{M}$ in Question 1 are isomorphic, requiring that the fibered partially hyperbolic system $g : \widehat{M} \to \widehat{M}$ be the "linearization" of $f : M \to M$ restricts the forms f and M can take. This makes Question 4 much more tractable.

References

[1] Avila, A., Viana, M., and Wilkinson, A. (2022). Absolute continuity, Lyapunov exponents, and rigidity II: systems with compact center leaves. *Ergodic Theory and Dynamical Systems*, 42(2):437-490.

[2] Damjanovic, D., Wilkinson, A., and Xu, D. (2021). Pathology and asymmetry: Centralizer rigidity for partially hyperbolic diffeomorphisms. *Duke Mathematical Journal*, 170(17):3815-3890.

[3] Brin M., Burago D. and Ivanov S. (2009). Dynamical coherence of partially hyperbolic diffeomorphisms of the 3-torus. *Journal of Modern Dynamics*, 3(1):1-11.

[4] Burago D. and Ivanov S. (2008). Partially hyperbolic diffeomorphisms of 3-manifolds with abelian fundamental groups. *Journal of Modern Dynamics*, 2(4):541-580.

[5] Doucette M. (2023). Smooth Models for Certain Fibered Partially Hyperbolic Systems. To appear in *Ergodic Theory and Dynamical Systems*. https://arxiv.org/abs/2208.07286.

[6] Franks, J. (1969). Anosov diffeomorphisms on tori. *Transactions of the American Mathematical Society*, 145:117-124.

[7] Gogolev, A., Ontaneda, P., and Rodriguez Hertz, F. (2015). New partially hyperbolic dynamical systems I. *Acta Mathematica*, 215(2):363-393.

[8] Hammerlindl, A. (2013). Leaf conjugacies on the torus. *Ergodic Theory and Dynamical Systems*, 33(3):896-933. Publisher: Cambridge University Press.

[9] Hammerlindl, A. (2013). Partial hyperbolicity on 3-dimensional nilmanifolds. *Discrete and Continuous Dynamical Systems*, 33(8):3641-3669.

[10] Hammerlindl, A. and Potrie, R. (2014). Pointwise partial hyperbolicity in 3-dimensional nilmanifolds. *Journal of the London Mathematical Society*, 89(3):853-875.

[11] Hammerlindl, A. and Potrie, R. (2015). Classification of partially hyperbolic diffeomorphisms on 3-manifolds with solvable fundamental group. *Journal of Topology*, 8(3):842-870.

[12] Manning, A. (1974). There are No New Anosov Diffeomorphisms on Tori. *American Journal of Mathematics*, 96(3):422-429.

[13] Nitica, V. and Torok, A. (2001). Local rigidity of certain partially hyperbolic actions of product type. *Ergodic Theory and Dynamical Systems*, 21(4):1213-1237.

[14] Rodriguez Hertz F., Rodriguez Hertz M.A., and Ures R. (2008). Partial hyperbolicity and ergodicity in dimension 3. *Journal of Modern Dynamics*, 2(2):187-208.

[15] Sandfeldt S. (2023) Global Rigidity of higher rank partially hyperbolic actions on Heisenberg nilmanifolds. preprint.