THE SECOND RATIONAL HOMOLOGY OF THE TORELLI GROUP

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Not everything is a lesson. Sometimes you just fail.

For my parents

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SUMMARY

The Torelli group of S_g, a closed, orientable, connected surface of genus g, is the subgroup of the mapping class group $\mathrm{Mod}(S_g)$ that acts trivially on $H_1(S_g;\mathbb{Z})$. The Torelli group is denoted $\mathcal{I}_g$. We show that the second rational homology of the Torelli group is finitely generated for all closed, oriented surfaces of sufficiently high genus. This represents the most significant progress towards determining for which g, if any, $\mathcal{I}_g$ is finitely presentable.

CHAPTER 1

INTRODUCTION

Let S_g^b be a compact, orientable surface of genus g with b boundary components. The *mapping class group* $\mathrm{Mod}(S_g)$ is $\pi_0(\mathrm{Diff}^+(S_g))$. The action of $\mathrm{Mod}(S_g)$ on $H_1(S_g;\mathbb{Z})$ induces a representation $\mathrm{Mod}(S_g) \to \mathrm{Sp}(2g,\mathbb{Z})$ called the *symplectic representation*, where $\mathrm{Sp}(2g,\mathbb{Z})$ is the group of invertible linear transformations of $H_1(S_g;\mathbb{Z})$ that respect the algebraic intersection form $\langle \cdot, \cdot \rangle$. The kernel of this representation is called the *Torelli group* and is denoted $\mathcal{I}_g$. There is a short exact sequence

$$1 \to \mathcal{I}_g \to \mathrm{Mod}(S_g) \to \mathrm{Sp}(2g,\mathbb{Z}) \to 1.$$

By the work of McCullough–Miller [29], Mess [30] and Johnson [20], $\mathcal{I}_g$ finitely generated if and only if $g \neq 2$. Birman posed the following question [6, Problem 29]:

Question 1. *Is $\mathcal{I}_g$ finitely presented for any sufficiently large g?*

Similar questions were asked by Mess [25, Page 90] and Morita [32, Problem 2.1].

Obstructions to finite presentability. If G is a finitely generated group with $H_2(G;\mathbb{Q})$ infinite dimensional, then G is not finitely presentable. The main result of this thesis is Theorem A.

Theorem A. *Let $g \geq 33$. The vector space $H_2(\mathcal{I}_g;\mathbb{Q})$ is finite dimensional.*

In particular, Theorem A means that for $g \geq 33$, we cannot use $H_2(\mathcal{I}_g;\mathbb{Q})$ to obstruct the finite presentability of $\mathcal{I}_g$. Theorem A partially answers the following question of Bestvina [11, Page 5]. Margalit also asked a similar question [28, Question 5.12]).

Question 2. *For which choices of k, g, and commutative Noetherian ring R is $H_k(\mathcal{I}_g;R)$ finitely generated as an R–module?*

Prior partial answers to Question 2. Mess showed that $\mathcal{I}_2$ is an infinitely generated free group [30], which implies that $H_1(\mathcal{I}_2)$ is infinitely generated and $H_i(\mathcal{I}_2) = 0$ for $i \geq 2$. Johnson [19] constructed a map

$$\tau_g : \mathcal{I}_g \wedge^3 H_1(S_g;\mathbb{Q})/H_1(S_g;\mathbb{Q}).$$

now called the Johnson homomorphism, and showed that $(\tau_g)_*$ induces an isomorphism in first rational homology [22]. Akita showed that $H_*(\mathcal{I}_g;\mathbb{Z})$ is infinitely generated as an abelian group for $g \geq 7$ [1]. Hain showed that $H_3(\mathcal{I}_3;\mathbb{Z})$ is infinitely generated [15]. Bestvina, Bux, and Margalit showed that the cohomological dimension of $\mathcal{I}_g$ is $3g-5$ [4, Theorem A], which implies that $H_k(\mathcal{I}_g;\mathbb{Z}) = 0$ for $k \geq 3g-4$. Bestvina, Bux, and Margalit also proved in this same paper that $H_{3g-5}(\mathcal{I}_g;\mathbb{Z})$ is infinitely generated for $g \geq 2$ [4, Theorem C]. Gaifullin later showed that $H_k(\mathcal{I}_g;\mathbb{Z})$ is infinitely generated for $2g-3 \leq k \leq 3g-5$ [13], confirming a conjecture of Bestvina–Bux–Margalit [4].

The proof of Theorem A proceeds in three steps.

1.1 Step 1 of the proof of Theorem A: The complex of homologous curves

For the remainder of this thesis, a *curve* on S_g will be a homotopy class of oriented essential embedded circles $S^1 \to S_g$. A *multicurve* M will be a set of curves in S_g with pairwise disjoint representatives. A curve $c \subseteq S_g$ is *nonseparating* if c has a nonseparating representative. Likewise, a multicurve $M \subseteq S_g$ is *nonseparating* if every curve in M can be simultaneously represented by embedded S^1's whose union is nonseparating in S_g. We let $\mathcal{C}(S_g)$ denote the *curve complex* of S_g. The k–cells of this complex are multicurves $M \subseteq S_g$ containing $k+1$ curves.

Complex of homologous curves. If $c \subseteq S_g$ is a curve, we let $[c] \in H_1(S_g;\mathbb{Z})$ denote the homology class represented by c. Let $\vec{x} \in H_1(S_g;\mathbb{Z})$ be a primitive homology class. The *complex of homologous curves* $\mathcal{C}_{\vec{x}}(S_g)$, defined by Putman [33], is the subcomplex of $\mathcal{C}(S_g)$

generated by curves c with $[c] = \vec{x}$. Putman used the work of Johnson [21] to show that when $g \geq 3$, the complex $C_{\vec{x}}(S_g)$ is connected [36]. The first step in the proof of Theorem A is the following result.

Theorem B. *Let $g \geq 2$. The integral homology $\widetilde{H}_k(\mathcal{C}_{\vec{x}}(S_g); \mathbb{Z})$ vanishes for $k \leq g - 3$.*

If we set $\vec{x} = 0$, then the resulting complex is called the *complex of separating curves*, and is denoted $\mathcal{C}_{\text{sep}}(S_g)$. Looijenga has proven that $\mathcal{C}_{\text{sep}}(S_g)$ is $(g-3)$–connected [27, Theorem 1.1], which implies that $\widetilde{H}_k(\mathcal{C}_{\text{sep}}(S_g); \mathbb{Z}) = 0$ for $k \leq g - 3$.

Prior uses of $\mathcal{C}_{\vec{x}}(S_g)$ to study the Torelli group. The Torelli group $\mathcal{I}_g$ acts naturally on $C_{\vec{x}}(S_g)$ for any choice of primitive nonzero $\vec{x} \in H_1(S_g; \mathbb{Z})$. The complex $C_{\vec{x}}(S_g)$ has been used by Hatcher and Margalit to give a new proof that $\mathcal{I}_g$ is generated by bounding pair maps [18]. Gaster, Greene and Vlamis also connected colorings of $C_{\vec{x}}(S_g)$ with the Chillingworth homomorphism [14].

1.1.1 The strategy of the proof of Theorem B

Let $g \geq 2$ and let $\vec{x} \in H_1(S_g; \mathbb{Z})$ be a nonzero primitive homology class. Bestvina, Bux and Margalit defined a complex called the complex of minimizing cycles, denoted $\mathcal{B}_{\vec{x}}(S_g)$ [4]. We will make use of the following two properties of the complex of minimizing cycles:

- $C_{\vec{x}}(S_g)$ is a subcomplex of $\mathcal{B}_{\vec{x}}(S_g)$, and
- $\mathcal{B}_{\vec{x}}(S_g)$ is contractible [4, Theorem E].

Hatcher and Margalit [18] use PL–Morse theory to prove that $H_1(\mathcal{B}_{\vec{x}}(S_g), C_{\vec{x}}(S_g); \mathbb{Z}) = 0$ when $g \geq 3$. Along with the contractibility of $\mathcal{B}_{\vec{x}}(S_g)$, $H_1(\mathcal{B}_{\vec{x}}(S_g), C_{\vec{x}}(S_g); \mathbb{Z}) = 0$ implies that $C_{\vec{x}}(S_g)$ is connected when $g \geq 3$. In Section 3.3.3, we will use Hatcher and Margalit's PL–Morse function to prove $H_k(\mathcal{B}_{\vec{x}}(S_g)/C_{\vec{x}}(S_g); \mathbb{Z}) = 0$ for $k \leq g - 2$. This and the long exact sequence in homology complete the proof of Theorem B.

1.1.2 The complex of splitting curves

As part of the proof that $H_k(\mathcal{B}_{\vec{x}}(S_g)/\mathcal{C}_{\vec{x}}(S_g);\mathbb{Z}) = 0$ for $k \leq g-2$, we will also prove a result about the *complex of splitting curves*. Putman [34] defined the notion of a *partitioned surface*, which is a pair $\Sigma = (S, P)$ where S is a compact, oriented surface and P is a partition of the set of boundary components of S. The *complex of separating curves* $\mathcal{C}_{\text{sep}}(\Sigma)$ is the full subcomplex of the curve complex $\mathcal{C}(S)$ generated by curves δ such that each $p \in P$ is contained entirely in one connected component of $S \setminus \delta$. Suppose now that $|P| = 2$, and the blocks of P are labeled B_+ and B_-. The *complex of splitting curves* $\mathcal{C}_{\text{split}}(\Sigma)$ is the full subcomplex of $\mathcal{C}_{\text{sep}}(\Sigma)$ generated by curves δ such that each connected component of $S \setminus \delta$ contains a block in P. Looijenga [27, Theorem 1.5] has shown that for such Σ, the complex $\mathcal{C}_{\text{sep}}(\Sigma)$ is $(g-2)$–connected. We will prove in Proposition 3.1.1 that $\mathcal{C}_{\text{split}}(\Sigma)$ is at least $(g - 3 + \mathbb{1}_{|B_+|\geq 2} + \mathbb{1}_{|B_-|\geq 2})$–acyclic. For the remainder of this thesis, we will use $\mathbb{1}$ to denote the indicator function.

1.1.3 Comparing $\mathcal{C}_{\text{split}}(\Sigma)$ to $\mathcal{C}_{\text{sep}}(\Sigma)$

Note that for $\Sigma = (S, \{B_+, B_-\})$ with $g(S) \geq 1$, we have $\mathcal{C}_{\text{split}}(\Sigma) \neq \mathcal{C}_{\text{sep}}(\Sigma)$. To see this, observe that the curve δ in Figure 1.1 is not a vertex of $\mathcal{C}_{\text{split}}(\Sigma)$, but is a vertex of $\mathcal{C}_{\text{sep}}(\Sigma)$.

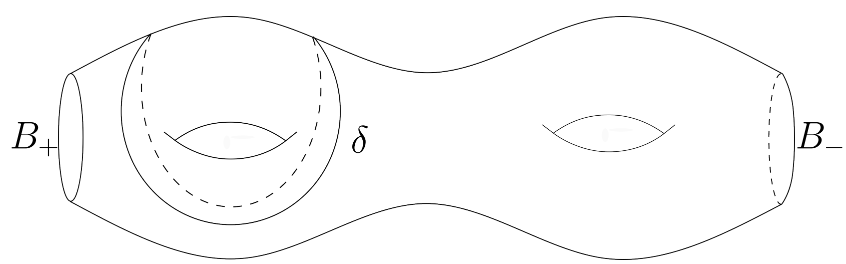

Figure 1.1: The curve δ is a vertex of $\mathcal{C}_{\text{sep}}(\Sigma)$ but not of $\mathcal{C}_{\text{split}}(\Sigma)$

Looijenga's theorem [27, Theorem 1.5] yields a stronger property for $\mathcal{C}_{\text{sep}}(\Sigma)$ than Proposition 3.1.1 does for $\mathcal{C}_{\text{split}}(\Sigma)$ when either $|B_+|$ or $|B_-|$ (or both) are equal to 1. However, if $|B_+|, |B_-| \geq 2$, Looijenga's theorem says that $\mathcal{C}_{\text{sep}}(\Sigma)$ is $(g-2)$–connected, while Proposition 3.1.1 says $\mathcal{C}_{\text{split}}(\Sigma)$ is $(g-1)$–acyclic. This suggests the following possible method of proof for showing that $\mathcal{C}_{\text{sep}}(\Sigma)$ is $(g-1)$–connected when $|B_+|, |B_-| \geq 2$.

Step 1. Upgrade Proposition 3.1.1 to show that $\mathcal{C}_{\text{split}}(\Sigma)$ is $(g-1)$–connected.

Step 2. Show that $\pi_k(\mathcal{C}_{\text{sep}}(\Sigma), \mathcal{C}_{\text{split}}(\Sigma)) = 0$ for $k \leq g-1$.

Then $\mathcal{C}_{\text{sep}}(\Sigma)$ would be $(g-1)$–connected via the long exact sequence in relative homotopy groups. However, $(g-1)$ may not necessarily be the best possible bound on the connectivity of $\mathcal{C}_{\text{sep}}(\Sigma)$. Looijenga asked the following question.

Question 3 (Looijenga [27, pg 4])**.** *For $\Sigma = (S, P)$ a partitioned surface, let $s(P)$ denote the number of blocks of P with at least two elements. Assume that $g(S) \geq 1$. Is it true that $\mathcal{C}_{\text{sep}}(\Sigma)$ is $(g - 4 + |P| + s(P))$–connected?*

If the answer to Question 3 is affirmative, then we might expect the connectivity of $\mathcal{C}_{\text{sep}}(\Sigma)$ to be strictly higher than the connectivity of $\mathcal{C}_{\text{split}}(\Sigma)$. Indeed, for $\Sigma = (S, \{B_+, B_-\})$ with $|B_+|, |B_-| \geq 2$, we see that $g - 4 + |P| + s(P) = g$, while Proposition 3.1.1 only says that $\mathcal{C}_{\text{split}}(\Sigma)$ is $(g-1)$–acyclic. This suggests that the sketch that $\mathcal{C}_{\text{sep}}(\Sigma)$ is $(g-1)$–connected for $|B_+|, |B_-| \geq 2$ might not yield a sharp bound on the connectivity of $\mathcal{C}_{\text{sep}}(\Sigma)$.

1.2 Step 2 of the proof of Theorem A: A criterion for $\mathrm{Sp}(2g, \mathbb{Z})$–representations to be finite dimensional

We now describe the main technical tool used in the proof of Theorem A.

Notation. Throughout this thesis, Stab will be taken to mean the pointwise stabilizer, as opposed to the setwise stabilizer. If V is a G–module and $f \in G$, let $V^f = \mathrm{Stab}_f\, V$.

Proposition 4.1.1. *Let $g \geq 1$, and let $N \subseteq S_g$ be a nonseparating multicurve such that $|N| < g$. Let $G \subseteq \mathrm{Sp}(2g, \mathbb{Z})$ be the image of the map $\mathrm{Stab}_{\mathrm{Mod}(S_g)}(N) \to \mathrm{Sp}(2g, \mathbb{Z})$. Let V be a G–representation over $\mathbb{Q}$. Suppose that there is a constant $0 \leq d \leq g - |N|$ such that the following hold:*

1. *For any multicurve $M \subseteq S_g$ such that:*

- *$|M| \geq d$,*
- *M is disjoint from N, and*
- *$M \sqcup N$ is nonseparating,*

the cokernel of the map $\bigoplus_{c \in M} V^{T_c} \to V$ is finite dimensional.

2. *For any multicurve $M \subseteq S_g$ such that:*
   - *$|M| < d$,*
   - *M is disjoint from N, and*
   - *$M \sqcup N$ is nonseparating,*

   the coinvariants module $V_{\mathrm{Stab}_G(M)}$ is finite dimensional.

Then V is finite dimensional.

In practice, we will often show that the first hypothesis is satisfied by showing that the natural map $\bigoplus_{c \in M} V^{T_c} \to V$ is surjective. We will prove Proposition 4.1.1 in Section 4.1.

1.3 Step 3 of the proof of Theorem A: Finite dimensionality of cokernels

The bulk of the work of proving Theorem A will be proving the following result.

Theorem C. *Let $g \geq 33$. Let $a \subseteq S_g$ be a nonseparating simple closed curve. The cokernel of the pushforward map $\iota_* : H_2(\mathrm{Stab}_{\mathcal{I}_g}(a); \mathbb{Q}) \to H_2(\mathcal{I}_g; \mathbb{Q})$ is finite dimensional.*

We will use Theorem C along with Proposition 4.1.1 to prove Theorem A. Setting $d = 1$, Theorem C will be used to verify the first hypothesis of Proposition 4.1.1, while the second hypothesis will be verified in Lemma 7.1.1.

The proof of Theorem C proceeds by considering the action of $\mathcal{I}_g$ on the complex of homologous curves, and then studying the equivariant homology spectral sequence for this group action.

The equivariant homology spectral sequences. We consider the equivariant homology spectral sequence $\mathbb{E}^*_{*,*}$ for the action of $\mathcal{I}_g$ on $\mathcal{C}_{[a]}(S_g)$ (see [8, Section VII] for an expository account of this spectral sequence). Let $a \subseteq S_g$ be a nonseparating curve and let $X_g = \mathcal{C}_{[a]}(S_g)/\mathcal{I}_g$. We will use the notation X_g throughout this thesis to refer to this quotient. If σ is a cell of X_g, let $(\mathcal{I}_g)_\sigma$ denote the stabilizer $\mathrm{Stab}_{\mathcal{I}_g}(\widehat{\sigma})$ for some arbitrary lift of σ to $\mathcal{C}_{[a]}(S_g)$. It is known that if x and y are nonseparating homologous curves in S_g, then there is an $f \in \mathcal{I}_g$ such that $fx = y$ (see, e.g., [34, Lemma 6.2]). Hence the set X_g^0 of vertices of X_g is a singleton. It follows that page 1 of $\mathbb{E}^*_{*,*}$ is as in Figure 1.2.

$$
\begin{array}{c|cccc}
2 & H_2((\mathcal{I}_g)_a) \longleftarrow & \bigoplus_{e\in X^{(1)}} H_2((\mathcal{I}_g)_e) & & \\
1 & H_1((\mathcal{I}_g)_a) \longleftarrow & \bigoplus_{e\in X^{(1)}} H_1((\mathcal{I}_g)_e) \leftarrow & \bigoplus_{\sigma\in X^{(2)}} H_1((\mathcal{I}_g)_\sigma) & \\
0 & & \bigoplus_{e\in X^{(1)}} H_0((\mathcal{I}_g)_e) \leftarrow & \bigoplus_{\sigma\in X^{(2)}} H_0((\mathcal{I}_g)_\sigma) \leftarrow & \bigoplus_{\rho\in X^{(3)}} H_0((\mathcal{I}_g)_\rho) \\
\hline
 & 0 & 1 & 2 & 3
\end{array}
$$

Figure 1.2: Page 1 of $\mathbb{E}^*_{*,*}$ for the action of $\mathcal{I}_g$ on $\mathcal{C}_{[a]}(S_g)$. All coefficients are in $\mathbb{Q}$.

Since $\mathcal{C}_{[a]}(S_g)$ is 2–acyclic for $g \geq 5$ by Theorem B, the spectral sequence $\mathbb{E}^*_{*,*}$ converges to $H_2(\mathcal{I}_g;\mathbb{Q})$ [8, Section VII]. Examining Figure 1.2, we see that the cokernel of the map

$$\iota_* : H_2(\mathrm{Stab}_{\mathcal{I}_g}(a);\mathbb{Q}) \to H_2(\mathcal{I}_g;\mathbb{Q})$$

is isomorphic to the direct sum

$$\mathbb{E}^\infty_{1,1} \oplus \mathbb{E}^\infty_{2,0}.$$

Hence the cokernel of ι_* is isomorphic to a subquotient of

$$\mathbb{E}^2_{1,1} \oplus \mathbb{E}^2_{2,0}.$$

The proof of Theorem C will proceed in two steps:

I. the vector space $\mathbb{E}^2_{2,0}$ is finite dimensional, and

II. the vector space $\mathbb{E}^2_{1,1}$ is finite dimensional.

By definition, $\mathbb{E}^2_{2,0}$ is canonically identified with $H_2(X_g;\mathbb{Q})$. Step I is recorded as the following proposition.

Proposition 1.3.1. *Let $g \geq 33$ and $a \subseteq S_g$ be a nonseparating simple closed curve. Let $\vec{x} = [a]$. Let $X_g = \mathcal{C}_{\vec{x}}(S_g)/\mathcal{I}_g$. The vector space $H_2(X_g;\mathbb{Q})$ is finite dimensional.*

Proposition 1.3.1 will be proven in Sections 4.2–5.3. Step II of the proof of Theorem C is recorded as the following result.

Proposition 1.3.2. *Let $g \geq 33$ and let $a \subseteq S_g$ be a nonseparating simple closed curve. Let $\vec{x}[a]$. Let $\mathbb{E}^*_{*,*}$ denote the equivariant homology spectral sequence in rational coefficients for the action of $\mathcal{I}_g$ on $\mathcal{C}_{\vec{x}}(S_g)$. The vector space $\mathbb{E}^2_{1,1}$ is finite dimensional.*

Proposition 1.3.2 will be proven in Sections 6.1–6.2. We now summarize the purpose of each section.

1.4 The outline of the thesis

This thesis is organized into the following chunks.

- Sections 2.1–2.3, consisting of general connectivity and acyclicity results.
- Section 3.1, where we prove that the complex of splitting curves is sufficiently acyclic, Proof of Proposition 3.1.1.
- Sections 3.2 and 3.3, where we show that the complex of homologous curves is $(g-3)$–acyclic. This result is recorded as Theorem B.
- Section 4.1, where we Proposition 4.1.1. This is the main tool we will use to verify finite–dimensionality.
- Sections 4.2 – 5.3, where we verify that $H_2(\mathcal{C}_{\vec{x}}(S_g)/\mathcal{I}_g;\mathbb{Q})$ is finite dimensional.

- Sections 6.1 and 6.2, where we verify that the vector space $\mathbb{E}^2_{1,1}(\mathcal{C}_{\vec{x}}(S_g), \mathcal{I}_g; \mathbb{Q})$ is finite dimensional. Additionally, Section 6.2 contains the proof of Theorem C.

- Section 7.1, which contains the proof of Theorem A.

We now explain the organization of some of the chunks in more detail.

Section 2.1. We discuss some general facts about connectivity and acyclicity of simplicial complexes. We then prove Lemma 2.1.3, which says that the relative homology of a pair of simplicial complexes $A \subseteq B$ can be computed using PL–Morse theory. Specifically, suppose that there is a function $W : B^{(0)} \to \mathbb{Z}_{\geq 0}$ such that $W^{-1}(\{0\}) = A^{(0)}$. We will show that if the function W satisfies certain local acyclicity properties, then the relative homology $H_k(B, A; \mathbb{Z})$ vanishes in a range depending on W.

Section 2.2. We prove Proposition 2.2.1. This is a packaging of some standard results about the $\check{\mathrm{C}}\mathrm{ech}$-to-singular spectral sequence. We will assume that we have some simplicial complex A and a simplicial cover $\mathcal{U}$ of A with $\mathcal{U}$ indexed by the vertices of another simplicial complex B. We will show that if $\widetilde{H}_k(B; \mathbb{Z})$ vanishes in a range and the elements of the cover $\mathcal{U}$ also satisfy some acyclicity properties, then $\widetilde{H}_k(A; \mathbb{Z})$ also vanishes in a range.

Section 2.3. We prove Lemma 2.3.1. This is a result based on the work of Brendle, Broaddus and Putman [7] that allows us to compute the acyclicity of certain subcomplexes of the curve complex of surfaces with boundary.

Section 3.1: The proof of Proposition 3.1.1. We will use Lemma 2.1.3 to prove that a variant of the arc complex on surfaces with certain decoration on the boundary are acyclic in a range. We then use Proposition 2.2.1 to prove that a more general version of the complex of splitting curves is acyclic in a range. This will imply Proposition 3.1.1.

Section 3.2. We will revisit some of the ideas from Section 2.1 in a slightly different context. In particular, the results in Section 2.1 apply only to simplicial complexes. However, the complex of minimizing cycles $\mathcal{B}_{\vec{x}}(S_g)$ is not a simplicial complex. We will resolve this issue in Lemma 3.2.6 by showing that Lemma 2.1.3 can be applied to CW–complexes equipped with some convex structure.

Section 3.3: The proof of Theorem B. We will apply a variant of Lemma 2.1.3. The required local acyclicity properties will be verified by inductively applying Proposition 2.3.1. The base case of this argument uses Proposition 3.1.1.

Sections 4.2 – 5.3, which prove Proposition 1.3.1. Recall that this result said that for $g \geq 33$ and for $a \subseteq S_g$, the vector space $H_2(X_g; \mathbb{Q})$ is finite dimensional. Here $X_g = \mathcal{C}_{\vec{x}}(S_g)/\mathcal{I}_g$, with $\vec{x} = [a]$. Our goal is to prove Proposition 1.3.1 by applying Proposition 4.1.1 with $G = \mathrm{Stab}_{\mathrm{Sp}(2g,\mathbb{Z})}(\vec{x})$, $d = 1$, and $V = H_2(X_g; \mathbb{Q})$. In particular we will prove the following:

1. For any nonseparating curve c disjoint from and not homologous to a, the cokernel of the map $H_2(X_g; \mathbb{Q})^{T_{[c]}} \to H_2(X_g; \mathbb{Q})$ is finite dimensional. Here, $T_{[c]}$ denotes the transvection along $[c]$, defined in Section 4.1.

2. The coinvariants module $H_2(X_g; \mathbb{Q})_G$ is finite dimensional.

The latter property is verified in the proof of Proposition 1.3.1, while verifying the former property is the bulk of the work of Sections 4.2– 5.3. The verification of the first hypothesis proceeds in the following steps:

1. Construct a subspace spanned by fundamental classes of tori $T \subseteq X_g$ called Bestvina–Margalit tori. These tori are constructed in Section 4.2, and the subspace spanned by their fundamental classes is denoted $\mathrm{BM}_2(X_g; \mathbb{Q})$.

2. Show that for any nonseparating $c \subseteq S_g$ disjoint from and not homologous to a, the cokernel $\mathrm{BM}_2(X_g; \mathbb{Q})^{T_{[c]}} \to \mathrm{BM}_2(X_g; \mathbb{Q})$ is spanned by classes in $\mathrm{BM}_2(X_g; \mathbb{Q})$. This is the bulk of the work of Section 4.2, and is recored as Lemma 5.3.1.

3. Use Proposition 4.1.1 with $G = \mathrm{Stab}_{\mathrm{Sp}(2g,\mathbb{Z})}(\vec{x}, [c])$, $d = 9$, and V the cokernel of the map $H_2(X_g; \mathbb{Q})^{T_{[c]}} \to H_2(X_g; \mathbb{Q})$. This is carried out in two substeps, which each verify a hypothesis of Proposition 4.1.1.

(a) The subspace $\mathrm{BM}_2(X_g;\mathbb{Q})$ is the image under the map $H_2(\mathcal{I}_g;\mathbb{Q}) \to H_2(X_g;\mathbb{Q})$ of a certain subspace $H_2^{\mathrm{ab,bp}}(\mathcal{I}_g;\mathbb{Q}) \subseteq H_2(\mathcal{I}_g;\mathbb{Q})$ generated by abelian cycles consisting of bounding pair maps. The main work of Section 5.1 is Proposition 5.1.1, which says that for a nonseparating multicurve $M \subseteq S_g$ containing at least 9 curves, the map

$$\bigoplus_{d \in M} \left(H_2^{\mathrm{ab,bp}}(\mathcal{I}_g;\mathbb{Q})\right)^{T_d} \to H_2^{\mathrm{ab,bp}}(\mathcal{I}_g;\mathbb{Q})$$

is surjective. Since $V = H_2(X_g;\mathbb{Q})/H_2(X_g;\mathbb{Q})^{T_{[c]}}$ is a quotient of $\mathrm{BM}_2(X_g;\mathbb{Q})$ by Lemma 5.3.1 and $\mathrm{BM}_2(X_g;\mathbb{Q})$ is a quotient of $H_2^{\mathrm{ab,bp}}(\mathcal{I}_g;\mathbb{Q})$, the map

$$\rho : \bigoplus_{d \in M} V^{T_{[d]}} \to V$$

is surjective. In particular, this implies that $\mathrm{cok}(\rho)$ is finite dimensional, so hypothesis (1) of Proposition 4.1.1 is satisfied for $G = \mathrm{Stab}_{\mathrm{Sp}(2g,\mathbb{Z})}(\vec{x},[c])$, $V = H_2(X_g;\mathbb{Q})/H_2(X_g;\mathbb{Q})^{T_{[c]}}$ and $d = 9$.

(b) We now show that for M a multicurve with $|M| \leq 8$ such that M is disjoint from a and c and $a \sqcup c \sqcup M$ is nonseparating, the coinvariants module

$$V_{\mathrm{Stab}_{[M]} G}$$

is finite dimensional, where $[M]$ denotes the set of homology classes represented by elements of M. Since V is a quotient of $\mathrm{BM}_2(X_g;\mathbb{Q})$ by Lemma 5.3.1, it suffices to show that $\mathrm{BM}_2(X_g;\mathbb{Q})_{\mathrm{Stab}_{[M]} G}$ is finite dimensional. This is the content of Lemma 5.2.1 and is the main work of Section 5.2.

Statements (a) and (b) are the hypotheses of Proposition 4.1.1 for the group $G = \mathrm{Stab}_{\mathrm{Sp}(2g,\mathbb{Z})}(\vec{x},[c])$, $d = 9$, and V the cokernel of the map $H_2(X_g;\mathbb{Q})^{T_{[c]}} \to H_2(X_g;\mathbb{Q})$.

Hence V is finite dimensional by Proposition 4.1.1.

Hence for any primitive c disjoint from and not homologous to a, the cokernel of the map $H_2(X_g;\mathbb{Q})^{T_{[c]}} \to H_2(X_g;\mathbb{Q})$ is finite dimensional. If $G = \mathrm{Stab}_{\mathrm{Sp}(2g,\mathbb{Z})}(\vec{x})$ then it is not too difficult to show that $H_2(X_g;\mathbb{Q})_G$ is finite dimensional. Hence by applying Proposition 4.1.1 with $G = \mathrm{Stab}_{\mathrm{Sp}(2g,\mathbb{Z})}(\vec{x})$, $d = 1$, and $V = H_2(X_g;\mathbb{Q})$, we conclude that $H_2(X_g;\mathbb{Q})$ is finite dimensional, which is the statement of Proposition 1.3.1.

Sections 6.1 and 6.2, which prove Proposition 1.3.2. The approach is to apply Proposition 4.1.1 with $G = \mathrm{Stab}_{\mathrm{Sp}(2g,\mathbb{Z})}(\vec{x})$, $d = 8$, and $V = \mathbb{E}^2_{1,1}$. We verify each hypothesis of Proposition 4.1.1 in turn.

1. Hypothesis (1) is stated as Lemma 6.2.1, and is the main content of Section 6.2.1.
2. Hypothesis (2) is stated as Lemma 6.1.1 and is the main content of Section 6.2.

Given these two results, we prove Proposition 1.3.2 by applying Proposition 4.1.1. Additionally, Section 6.2 contains the proof of Theorem C.

CHAPTER 2

PRELIMINARY WORK FOR THE PROOF OF THEOREM B

2.1 Connectivity and PL–Morse theory

In this section, we explain some basic algebraic topology facts. We then explain some of the basic ideas of PL-Morse theory, namely:

- PL–Morse functions and
- descending links.

For more background, see Bestvina's survey [3]. We begin with Section 2.1.1, which includes some basic definitions and results about connectivity and acyclicity. In Section 2.1.2, we prove Lemma 2.1.2, which is a slight reformulation of Bestvina's results. We then use Lemma 2.1.2 to prove Lemma 2.1.3, which is a basic application of PL-Morse theory that allows us to compute the relative acyclicity and connectivity of certain pairs of complexes. The latter result is the main PL–Morse theory result used throughout the paper.

2.1.1 Some terminology and algebraic topology facts

Let X be a topological space with a basepoint $x \in X$ and let $Y \subseteq X$ be a subspace with $x \in Y$. Let $n \geq 0$ be a non–negative integer. We say that X is *n–connected* if $\pi_k(X, x) = 0$ for every $k \leq n$. We say that X is *n–acyclic* if $\widetilde{H}_k(X; \mathbb{Z}) = 0$ for every $k \leq n$. We say that the pair (X, Y) is *relatively n–connected* if $\pi_k(X, Y) = 0$ for every $k \leq n$. We say that the pair (X, Y) is *relatively n–acyclic* if $H_k(X, Y) = 0$ for every $k \leq n$.

Notation. We will use $c(X)$ and $\mathfrak{a}(X)$ to denote the connectivity and acyclicity respectively of the space X.

We require the following fact from algebraic topology (see [27, Lemma 2.1]).

Lemma 2.1.1. *Let $X_1, \dots, X_n$ be a collection of topological spaces.*

(a) If each X_i is k_i–connected, then the join

$$X_1 * \ldots * X_n$$

is $(-2 + \sum_{i=1}^{n} (k_i + 2))$–connected.

(b) If each X_i is k_i–acyclic, then the join

$$X_1 * \ldots * X_n$$

is $(-2 + \sum_{i=1}^{n} (k_i + 2))$–acyclic.

2.1.2 PL–Morse theory

We now discuss the basics of PL–Morse theory and prove Lemma 2.1.3.

PL-Morse functions. Let X be a simplicial complex and Y a subcomplex of X. A *PL-Morse function* on X is a function $W : X^{(0)} \to \mathbb{Z}_{\geq 0}$. We define the *min-set* of W to be $M(W) = W^{-1}(\{0\})$.

Remark. In Bestvina's formulation of PL-Morse theory, it is assumed that two vertices in X with the same weight and positive weight are not adjacent. For our purposes it is not necessary to assume this, so it is not part of our definition.

Descending links. Let X be a simplicial complex equipped with a PL–Morse function W. Let $\sigma \subseteq X$ be a cell, and let $\operatorname{lk}(\sigma)$ denote the link of σ in X. If σ is a cell of X such that W is positive and constant on the vertices of σ, then we say that σ is a *W–constant cell.* If σ is a W–constant cell of X, then the descending link $d_W(\sigma)$ is the subcomplex of $\operatorname{lk}(\sigma)$ generated by vertices $w \in X$ such that $W(w) < W(v)$ for all vertices v of σ. Similarly, the descending star $s_W(\sigma)$ is the join $\sigma * d_W(\sigma)$.

Connectivity and acyclicity of PL-Morse functions. Let W be a PL-Morse function on a simplicial complex X. Suppose there is a positive integer n such that for every positive weight W–constant k-cell σ, the descending link $d_W(\sigma)$ is $(n-k)$–connected. In this case, we will say that W is an *n–connected PL-Morse function*. Similarly, if there is a positive integer n such that for W–constant k–cell σ, the descending link $d_W(\sigma)$ is $(n-k)$–acyclic, we say that W is an *n–acyclic PL-Morse function*.

We have the following general result about PL–Morse functions due to Bestvina [3]. We will assume that all simplicial complexes are finite dimensional and countable.

Lemma 2.1.2. *Let X be a finite–dimensional, countable simplicial complex equipped with a PL-Morse function W. Let $Y = M(W)$.*

(a) If W is n–connected, then the pair (X, Y) is relatively $(n+1)$–connected.

(b) If W is n–acyclic, then the pair (X, Y) is relatively $(n+1)$–acyclic.

Proof. We will begin by constructing a double–indexed filtration of X. If σ is a k-cell, we let $W(\sigma) = \max\{W(v) : v \in \sigma_0\}$. For integers $k \geq 0$ and $m \geq 1$, we set

$$X_{k,m} = W^{-1}\left([0,m)\right) \bigcup \{\sigma : W(\sigma) \leq m, \dim(\sigma) \leq k\}.$$

We will use the notation $W_{\infty,m}$ to mean the full subcomplex of X generated by vertices of weight $\leq m$. The set

$$\{X_{k,m}\}_{k \in \mathbb{Z}_{\geq 0}, m \in \mathbb{Z}_{\geq 1}}$$

is a filtration of X with $X_{k,m} \subseteq X_{k',m'}$ if either $m' > m$ or $m' = m$ and $k' \geq k$. We will prove part (a) of the lemma. The proof of (b) follows from a similar argument.

The proof of (a). Since the filtration

$$X_{0,0} \subseteq X_{0,1} \subseteq \ldots$$

is well-founded, it suffices to prove that for every $k \leq n+1$, the following hold:

1. The pair $(X_{k,m}, X_{k-1,m})$ is relatively $(n+1)$–connected for every $k > 0$ and $m > 0$.

2. The pair $(X_{0,m}, X_{\infty,m-1})$ is relatively $(n+1)$–connected for every $m > 0$.

Since (1) and (2) follow by similar reasoning, we only prove (1).

The proof of (1). Fix integers $k \geq 1$ and $m \geq 1$. Let $\mathcal{T}^{k,m}$ be the set of W–constant k-cells of weight m. The complex $X_{k,m}$ is constructed from $X_{k-1,m}$ by attaching, for each $\sigma \in \mathcal{T}^{k,m}$, the complex $\sigma * d_W(\sigma)$ to $\partial\sigma * d_W(\sigma)$ in the natural way. over $d_W(\sigma)$. See Figure 2.1 for an example of this filtration. By hypothesis, $d_W(\sigma)$ is $(n-k)$–connected.

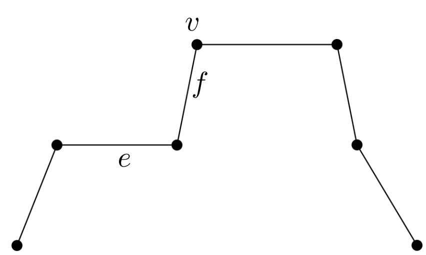

Figure 2.1: The filtration $X_{k,m}$. The edge $e \subseteq X^{1,1}$, the vertex $v \in X^{0,2}$, the edge $f \subseteq X_{0,2}$.

By Lemma 2.1.1, the join $\partial\sigma * d_W(\sigma)$ is $((k-2)+(n-k)+2)$–connected and hence n–connected. There is a countable filtration of $X_{k,m}$ given by arbitrarily indexing the cells of $\mathcal{T}^{k,m}$ by the natural numbers $\mathbb{N}$, and then attaching them one at a time to $X_{k-1,m}$. We will notated the jth term of this filtration by $X^j_{k-1,m}$. If τ is the cell added between $X^{j-1}_{k-1,m}$ and $X^j_{k-1,m}$, then the pair

$$\left(X^{j-1}_{k-1,m} \sqcup_{\partial\tau * d_W(\tau)} \tau * d_W(\tau), X^{j-1}_{k-1,m}\right)$$

is relatively $(n+1)$–connected. Therefore $(X^j_{k-1,m}, X^{j-1}_{k-1,m})$ is relatively $(n+1)$–connected for every $j \geq 1$, so the pair $(X_{k,m}, X_{k-1,m})$ is relatively $(n+1)$–connected. □

We have the following consequence of Lemma 2.1.2.

Lemma 2.1.3. *Let X be a finite–dimensional, countable simplicial complex with a PL-Morse function W and let $Y = M(W)$. Let $n \geq 0$ be an integer.*

(a) If W is n–connected and Y is $(n+1)$–connected, then X is $(n+1)$–connected.

(b) If W is n–connected and X is n–connected, then Y is n–connected.

(c) If W is n–acyclic and Y is $(n+1)$–acyclic, then X is $(n+1)$–acyclic.

(d) If W is n–acyclic and X is n–acyclic, then Y is n–acyclic.

Proof of Lemma 2.1.3. We prove (a). The others follows by a similar argument. It is enough to prove that for $i \leq n$, we have $\pi_i(Y) = 0$. Fix some $i \leq n$. We have the long exact sequence in relative homotopy associated to the pair (X, Y), a part of which is given by

$$\ldots \to \pi_{i+1}(X, Y) \to \pi_i(Y) \to \pi_i(X) \to \ldots$$

Then $\pi_{i+1}(X, Y) = 0$ by Lemma 2.1.2 and $\pi_i(X) = 0$ by hypothesis, so $\pi_i(Y) = 0$. □

2.2 Covers indexed by simplicial complexes

The main output of this section is Proposition 2.2.1, which is a technical result about the acyclicity of complexes covered by simplicial subcomplexes that are indexed by the cells of another complex. Proposition 2.2.1 is a homological version of a result of Mirzaii and van der Kallen [31]. Proposition 2.2.1 can also be viewed as a specific application of the homotopy colimit spectral sequence [10].

Bi-cellular covers. Let X and Y be two simplicial complexes. Let L_X be a function taking a vertex $y \in Y^{(0)}$ to a subcomplex of X. If σ is a k-cell of Y with vertices $y_0, \ldots, y_k$, we denote

$$L_X(\sigma) = L_X(y_0) \cap \ldots \cap L_X(y_k).$$

If the set $\{L_X(y)\}_{y \in Y^{(0)}}$ covers X, we will call such an L_X a *Y-indexed cover of X*. If L_X is a Y-indexed cover of X, then there is an associated X-indexed cover of Y given by setting $L_Y(x)$ to be the full subcomplex of Y generated by vertices y such that $x \in L_X(y)$ for x a vertex of X. We say that a Y-indexed cover of X is *cellularly n–acyclic* if for each k-cell σ of Y, the complex $L_X(\sigma)$ is $(n-k)$–acyclic. We say L_X is *bi-cellularly n–acyclic* if L_X and L_Y are both cellularly n–acyclic.

Proposition 2.2.1. *Let X and Y be finite–dimensional, countable simplicial complexes. Let L_X be a Y-indexed cover of X. Suppose there is an integer n such that the following hold:*

- *L_X is bi-cellularly n–acyclic and*
- *Y is n–acyclic.*

Then X is n–acyclic.

We will now introduce the main algebraic object that we use to prove Proposition 2.2.1.

Bi–cellular spectral sequence. Let X and Y be simplicial complexes, and let L_X be a Y–indexed cover of X. We have a bi-graded double complex called the *bi-cellular complex* given by

$$C_{p,q} = \bigoplus_{\sigma \in Y^{(p)}} C_q(L_X(\sigma))$$

where C_* denotes the complex of integral chains on a simplicial complex. Let $\mathbb{E}^{*,\leftarrow}_{*,*}$ and $\mathbb{E}^{*,\downarrow}_{*,*}$ denote the leftward and downward spectral sequences associated to $C_{*,*}$ associated to L_X and L_Y. We refer to these respectively as the *leftward* and *downward bi-cellular spectral sequences*.

The leftward versus downward strategy. Let $C_{p,q}$ be a double complex. There are two spectral sequences associated to $C_{p,q}$, which are the leftward and downward spectral sequences. We denote these $\mathbb{E}^{*,\leftarrow}_{*,*}$ and $\mathbb{E}^{*,\downarrow}_{*,*}$. These two spectral sequences are each constructed from

$C_{p,q}$ by two different filtrations of the total complex C_{p+q}. The key point is that both $\mathbb{E}^{*,\leftarrow}_{*,*}$ and $\mathbb{E}^{*,\downarrow}_{*,*}$ converge to filtrations of the total homology of $C_{p,q}$. Denote the total homology by $H_{p+q}(C_{*,*})$. Suppose that we want to compute some of the groups $\mathbb{E}^{2,\leftarrow}_{p,q}$. The strategy is as follows:

1. Show that $\mathbb{E}^{*,\downarrow}_{p,q}$ converges to 0 in a range $0 < p+q < n$. This implies that $H_{p+q}(C_{*,*})$ converges to 0 for $0 < p+q < n$.

2. Use the fact that $\mathbb{E}^{*,\leftarrow}_{p,q}$ must also converge to 0 for $0 < p+q < n$ to say something about the groups $\mathbb{E}^{2,\downarrow}_{p,q}$.

This is a standard technique, used for example to show that the G–equivariant homology of a contractible CW–complex X converges to the group homology of G when G acts on X without rotations [8, Section VII].

Proof of Proposition 2.2.1. We will apply the leftward versus downward strategy discussed above. In particular, we will show that the downward bi-cellular spectral sequence converges to $\mathbb{Z}$ for $p+q=0$ and converges to 0 for $0 < p+q \leq n$, and that the leftward bi-cellular spectral sequence converges to $H_{p+q}(X;\mathbb{Z})$ for $0 \leq p+q \leq n$. This completes the proof since the leftward and downward sequence both converge to $H_{p+q}(C_{*,*})$.

The downward sequence. On page 1 of $\mathbb{E}^{*,\downarrow}_{*,*}$, we have

$$\mathbb{E}^{1,\downarrow}_{p,q} = \bigoplus_{\sigma \in Y^{(p)}} H_q(L_X(\sigma);\mathbb{Z}).$$

By hypothesis, for $0 \leq p+q \leq n$ and $q > 0$ we have

$$\mathbb{E}^{1,\downarrow}_{p,q} = 0.$$

Then for $q = 0$ in this range we have

$$\mathbb{E}^{1,\downarrow}_{p,*} = C_p(Y).$$

Hence $\mathbb{E}^{*,\downarrow}_{p,q}$ converges to $H_{p+q}(Y;\mathbb{Z})$, which by hypothesis is $\mathbb{Z}$ when $p+q=0$ and 0 for $0<p+q\leq n$.

The leftward sequence. By construction, there is a canonical isomorphism

$$C_{*,*} \cong \bigoplus_{\sigma\in X^{(q)}} C_p(L_Y(\sigma)).$$

Then by the same argument as the downward case, $\mathbb{E}^{*,\leftarrow}_{*,*}$ converges to $H_{p+q}(X;\mathbb{Z})$ for $p+q\leq n$. Hence $H_0(X;\mathbb{Z})\cong\mathbb{Z}$ and $H_{p+q}(X;\mathbb{Z})=0$ for $0<p+q<n$. □

2.3 Brendle–Broaddus–Putman flow

Our goal in this section is to prove Proposition 2.3.1, which is a technical result about how adding boundary components to a surface increases the acyclicity of certain subcomplexes of the curve complex. Let $S=S_g^b$ be a surface with $b\geq 1$. Let p_0 be a boundary component of S. Let S' be a surface and let $\iota : S\to S'$ be an embedding such that $S'\setminus\iota(S)$ is a disk bounded by $\iota(p_0)$. Let $\mathcal{C}(S)$ denote the curve complex of S. Let $\mathcal{C}'(S)$ be the subcomplex of $\mathcal{C}(S)$ generated by curves δ such $\iota(\delta)$ is essential. Let $K(S)\subseteq\mathcal{C}(S)$ be the subcomplex generated by vertices δ such that $\iota(\delta)$ is inessential. There is a pushforward map

$$\iota_* : \mathcal{C}'(S)\to\mathcal{C}(S').$$

Assuming that S has one boundary component, Kent, Leininger and Schleimer proved that ι_* is a homotopy equivalence [24, Theorem 7.2]. Brendle, Broaddus and Putman [4] use a flow argument to show that the map ι_* is a homotopy equivalence assuming that $b\geq 2$. This is a similar technique to methods used by Hatcher [17], Looijenga [27], and Bell–Margalit [2]. We describe technique of Brendle–Broaddus–Putman here.

Hatcher flow. Associated to each vertex $\delta\in K(S)$, there is an oriented arc α connecting a boundary component p_1 to p_0 as in Figure 2.2. Let $\mathrm{lk}(\delta)$ denote the link of δ in $\mathcal{C}(S)$. If

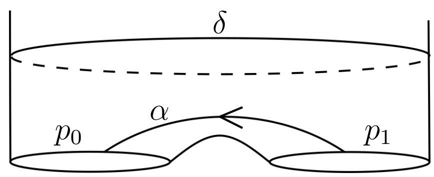

Figure 2.2: The arc α associated to the curve $\delta \in K$. α is oriented from p_1 to p_0.

$\delta \in K$, then $\mathrm{lk}(\delta) \subseteq \mathcal{C}'(S)$. Then $\mathcal{C}'(S)$ is homotopy equivalent to $\mathrm{lk}(\delta)$. The homotopy equivalence is given iteratively surgering β along the arc α. An example of this surgery is given in Figure 2.3. The full definition is given below.

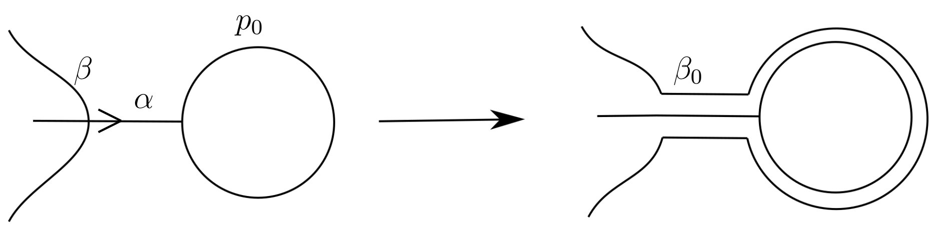

Figure 2.3: Surgering on boundary components [7].

Brendle, Broaddus and Putman show that there is a homotopy equivalence

$$K(S) * \mathcal{C}(S') \simeq \mathcal{C}(S).$$

We will say that a complex $X \subseteq C(S)$ is *Brendle–Broaddus–Putman-compatible* with a boundary component $p \in \pi_0(\partial S)$ if for every $\delta \in K(S) \cap X$ and every $\beta \in X^{(0)} \setminus (K(S) \cap X)$, the surgery of β along an arc α as in Figure 2.2 is also in X.

Remark. Since curves are defined only up to homotopy, this definition of surgery is not *a priori* well–defined. We will more precisely state the surgery construction and prove that it is well defined in Lemma 2.3.2.

The main goal of the section is to prove the following result.

Proposition 2.3.1. *Let $S = S_g^b$ with $b \geq 2$ and $\chi(S) \leq -1$. Let p_0 be a boundary component of X. Let $X \subseteq \mathcal{C}(S)$ be Brendle–Broaddus–Putman compatible relative to p_0. Let $\iota : S \to S'$ be the inclusion map from S to the surface S' with p_0 filled in by a* *disk. Let $\mathcal{C}'(S)$ denote the subcomplex of the curve complex generated by curves δ such that δ remains essential when p_0 is filled in with a disk, and let $K(S)$ denote the remaining vertices of $\mathcal{C}'(S)$. Then*

(a) For every $\delta \in K(S)$, the inclusion $X \cap \mathcal{C}'(S) \hookrightarrow \mathrm{lk}(\delta)$ is a homotopy equivalence.

(b) $(X \cap K(S)) * (X \cap \mathcal{C}'(S)) \simeq X$.

Before we prove this proposition, we will precisely define surgery along arcs.

Surgery along arcs. We now explicitly construct the surgery of a curve β along an arc δ. Let $S = S_g^b$ be a compact oriented surface with $b \geq 2$. Let p_0, p_1 be two boundary components of S, and let $\mathcal{C}(S)$, $\mathcal{C}'(S)$ and $K(S)$ be as above. Let $\delta \in K(S)$ be a curve and let $\beta \in \mathcal{C}'(S)$ be another curve. We will define the *surgery of β along δ*, which we denote $\mathrm{surg}_\delta(\beta)$. Let $\widehat{\delta}$ be a smooth representative of δ, and let $\widehat{\alpha}$ be an oriented arc connecting p_1 to p_0 such that α is disjoint from $\widehat{\delta}$. Choose a representative $\widehat{\beta}$ of β such that the intersection number $\left|\widehat{\beta} \cap \widehat{\alpha}\right|$ is minimal. If $\widehat{\beta}$ is disjoint from $\widehat{\alpha}$, then we say $\mathrm{Surg}_\delta(\beta) = \beta$. Otherwise, let q be the point of intersection of $\widehat{\beta}$ with $\widehat{\alpha}$ that is closest along $\widehat{\alpha}$ to the boundary component p_0. Let U be an annulus with $p_0 \subseteq \partial U$. Let $\widehat{\alpha}_{q,p_0}$ denote the embedded interval given by restricting $\widehat{\alpha}$ to the sub–interval connecting q to p_0. Let V be an embedded copy of $I \times I$ such that $I \times \frac{1}{2} = \alpha_{q,p_0}$. Define

$$\mathrm{Surg}_\delta(\beta) = \text{ the isotopy class of } \widehat{\beta}\Delta(\partial(I \cup U))$$

where Δ denotes symmetric difference. We have the following result.

Lemma 2.3.2. *Let $S, S', p_0, p_1, \delta, \beta$ be as above. The curve $\mathrm{Surg}_\delta(\beta)$ is well–defined.*

Proof. In the construction of $\mathrm{Surg}_\delta(\beta)$, there are six choices made, namely $\widehat{\delta}$, $\widehat{\alpha}$, $\widehat{\beta}$, U, and I. Only the choice of $\widehat{\alpha}$ is still a choice up to isotopy, since two arcs $\widehat{\alpha}$ and $\widehat{\alpha}'$ need not be isotopic. Since we are working up to isotopy, we may assume that $\widehat{\alpha}$ and $\widehat{\alpha}'$ have the same endpoints. Let α and α' denote the isotopy classes of the arcs $\widehat{\alpha}$ and $\widehat{\alpha'}$. There are integers $m, n \in \mathbb{Z}$ such that $T_{p_0}^m T_{p_1}^n \alpha = \alpha'$, where T_{p_i} denotes the Dehn twist along a curve cobounding an annulus with p_i. But then if γ denotes the surgery of β along α up to isotopy and γ' for β along α' up to isotopy, we see that $T_{p_0}^m T_{p_1}^n \gamma = \gamma'$. But T_{p_0} and T_{p_1} act trivially on $\mathcal{C}(S)$, so $\mathrm{Surg}_\delta(\beta)$ is well–defined. □

We now describe our extension of the construction of Brendle–Broaddus–Putman [7]. Let $S = S_g^b$ with $b \geq 2$. Let $X \subseteq \mathcal{C}(S)$ be a subcomplex of the curve complex. Let p_0 be a boundary component of S. Let S', $\mathcal{C}'(S)$ and $K(S)$ be as above. We say that X is *Brendle–Broaddus–Putman compatible relative to* p_0 if for every curve $\delta \in X \cap K(S)$ and every $\beta \in X \cap \mathcal{C}'(S)$, we have $\mathrm{Surg}_\delta(\beta) \in X$. We say that X is *Brendle–Broaddus–Putman compatible* if it is Brendle–Broaddus–Putman compatible relative to every boundary component of S. We have following result.

Our argument is essentially the same as the argument of Brendle, Broaddus and Putman [7, pg 13-16]. We require the following lemma from their paper [7, Lemma 4.1].

Lemma 2.3.3. *Let X be a simplicial complex, let I be a discrete set of points, and let $Y \subseteq I * X$ be a subcomplex such that $I, X \subseteq Y$. Suppose that, for any $i \in I$, the inclusion $\mathrm{lk}_Y(i) \hookrightarrow X$ is a homotopy equivalence. Then the inclusion $Y \hookrightarrow I * X$ is a homotopy equivalence.*

Proof of Proposition 2.3.1. Part *(b)* of the lemma follows from part *(a)* and Lemma 2.3.3, so it suffices to prove part *(a)*. Let $\delta \in K(S)$. Since $X \cap \mathcal{C}'(S)$ and $X \cap \mathrm{lk}(\delta)$ both have the homotopy type of CW–complexes, it suffices to show that the inclusion map

$$\kappa : X \cap \mathrm{lk}(\delta) \to X \cap \mathcal{C}'(S)$$

induces an isomorphism on homotopy groups. Equivalently, it suffices to show that $\pi_k(X \cap \mathcal{C}'(S), X \cap \mathrm{lk}(\delta)) = 0$ for all $k \geq 0$.

Choose a hyperbolic metric on S with geodesic boundary. Then for each $\beta \in X \cap \mathcal{C}'(S)$, let $\widehat{\beta}$ be the geodesic representative. Let $\widehat{\alpha}$ be an oriented geodesic arc connecting p_1

to p_0 and disjoint from δ. Since these are geodesics representatives, any pair of chosen representatives intersects minimally [12]. By perturbing the $\widehat{\beta}$ slightly, we may additionally assume that:

- each $\widehat{\beta}$ intersects $\widehat{\alpha}$ in a different point than any other $\widehat{\beta}$, and
- $\widehat{\beta}$ minimally intersects $\widehat{\beta'}$ for any $\beta, \beta' \in \mathcal{C}(S)$.

Let $W(\beta)$ denote the distance along α between p_1 and q, where q is the closest point of intersection of β and α with p_0. Let $\psi : S^k \to X \cap \mathcal{C}'(S)$ be a simplicial representative of a class in $\pi_k(X \cap \mathcal{C}'(S), X \cap \mathrm{lk}(\delta))$, where S^k is a simplicial decomposition of a k–sphere. Let $\beta \in \psi(S^k)$ be a vertex with $W(\beta)$ maximal. If $W(\beta) = 0$ then we are done, so assume otherwise. Note that $d_W(\beta)$ is a cone with cone point $\mathrm{Surg}_\delta(\beta)$. Hence ψ is homotopic to a map $\psi' : S^k \to X \cap \mathcal{C}'(S)$ such that the maximal intersection number of any vertex in $\psi'(S^k)$ is not larger than for $\psi(S^k)$. Since iteratively applying Surg_δ to a curve β eventually stabilizes in a curve $\overline{\beta}$ with $W(\overline{\beta})) = 0$, the above process eventually terminates in ψ with both $\psi \simeq \psi''$ and $\psi'' : S^k \to X \cap \mathrm{lk}(\delta)$, so the lemma holds. □

CHAPTER 3

THE PROOF OF THEOREM B

3.1 The complex of splitting curves

Our main goal in this section is to prove Proposition 3.1.1, which bounds the acyclicity of the complex of splitting curves. We begin with a sequence of definitions, and then we state Proposition 3.1.1. In Section 3.1.1, we will prove Lemma 3.1.2, which is an acyclicity result about a complex called the nonseparating arc complex. We conclude with Section 3.1.2, where we prove Proposition 3.1.1.

Cutting curves on surfaces. Let S be a surface and M a multicurve on S. The notation $S \searrow M$ denotes Farb and Margalit's notion of cutting curves on surfaces [12].

Partitioned surface. Following Putman [34], a *partitioned surface* $\Sigma = (S, P)$ is a pair consisting of a compact, connected, oriented surface S and a partition P of the set of boundary components of S. A *block of a partition* is one set in the partition. There is a poset $\mathfrak{TSur}$ of partitioned surfaces where $(S, P) \leq (S', P')$ if there is an embedding $\iota : S \to S'$ such that:

- for each block $B \in P'$, there is a connected component $S_B \subseteq S' \searrow \iota(S)$ with $B \subseteq S_B$, and
- for each connected component $\widehat{S} \subseteq S' \searrow \iota(S)$, we have $\partial\widehat{S} \cap \iota(S) \in P$.

Complex of separating curves. The *complex of separating curves* $\mathcal{C}_{\text{sep}}(\Sigma)$ is the full subcomplex of $\mathcal{C}(S)$ generated by separating curves δ such that each block $B \in P$ is contained entirely in one connected component of $S \searrow \delta$. A theorem of Looijenga [27, Theorem 1.3] tells us that this complex is $(g-2)$–connected when $|\pi_0(\partial S)| \geq 2$.

Vertex complement. Following Hatcher and Margalit [18], a *vertex complement* is a partitioned surface $\Sigma = (S, P)$ such that:

- $|P| \geq 2$, and
- two blocks of P come equipped with labels. One is labeled B_+, the other is B_-.

We will denote a vertex complement by

$$(S, P, B_+, B_-).$$

The terminology "vertex complement" refers to Σ sometimes being the complement of a vertex of a certain complex called the complex of minimizing cycles, which we discuss in Section 3.3. We will say that Σ is a *vertex complement on a surface* S if the underlying surface of Σ is S. The set of vertex complements is a poset which we will denote $\mathfrak{VerCom}$. We have $\Sigma \leq \Sigma'$ if:

- $\Sigma \leq \Sigma'$ in $\mathfrak{TSur}$, and
- there is an inclusion $\iota : S \to S'$ that realizes $\Sigma \leq \Sigma'$ in $\mathfrak{TSur}$ such that there are two connected components of $S' \setminus \iota(S)$ called S_+ and S_- that satisfy $\partial S_+ = B_+ \cup B'_+$ and $\partial S_- = B_- \cup B'_-$.

We will denote the genus of the underlying surface S in $\Sigma = (S, P, B_+, B_-)$ by $g(\Sigma)$.

The complex of splitting curves. Let $\Sigma = (S_g^b, P, B_+, B_-)$ be a vertex complement. The *complex of splitting cycles* $\mathcal{C}_{\text{split}}(S, P)$ is the full subcomplex of $\mathcal{C}_{\text{sep}}(S, P)$ generated by curves δ such that B_+ and B_- are contained in separate connected components of $S \setminus \delta$.

The remainder of this section will be devoted to the proof of the following proposition.

Proposition 3.1.1. *Let $\Sigma = (S_g^b, P, B_+, B_-)$ be a vertex complement. The acyclicity of the complex $\mathcal{C}_{\text{split}}(\Sigma)$ satisfies*

$$\mathfrak{a}(\mathcal{C}_{\text{split}}(\Sigma)) \geq g - 3 + \mathbb{1}_{|B_+| \geq 2} + \mathbb{1}_{|B_-| \geq 2}.$$

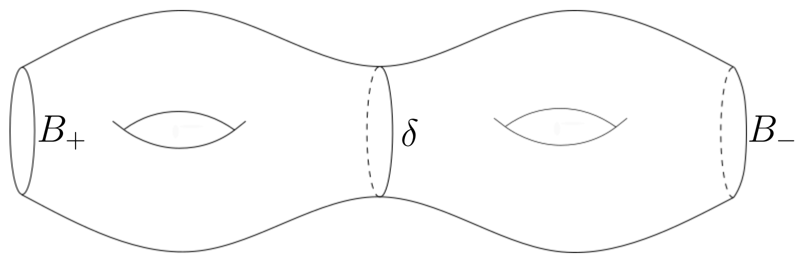

Figure 3.1: A splitting curve.

3.1.1 The nonseparating arc complex

The goal of this section is to prove Lemma 3.1.2, which is a result about the acyclicity of a certain subcomplex of the arc complex called the nonseparating arc complex. We begin by defining a poset of compact surfaces with marked compact intervals on the boundary of the surface. We then prove Lemma 3.1.3, which is an auxiliary result about the contractibility of another subcomplex of the arc complex. We conclude by leveraging Lemma 3.1.3 and Proposition 2.2.1 to prove Lemma 3.1.2.

Marked vertex complements. The poset $\mathfrak{MarkSur}$ has elements consisting of vertex complements $\Sigma = (S, P, B_+, B_-)$ equipped with a set of compact intervals Q on the boundary components in B_+ such that:

- $|Q| \geq 2$, and
- each $p \in B_+$ contains an interval of Q.

If $p \in B_+$ is a boundary component, we let $Q(p)$ denote the set of intervals contained in p. We say that $(\Sigma, Q) \leq (\Sigma', Q')$ if $\Sigma \leq \Sigma'$ in $\mathfrak{VerCom}$.

The nonseparating arc complex. Let $(\Sigma, Q) \in \mathfrak{MarkSur}$. The *nonseparating arc complex* $\mathcal{A}_{\text{nosep}}(\Sigma, Q)$ is the subcomplex of the arc complex [40] of S consisting of cells σ satisfying:

1. $S \setminus \sigma$ is connected, and
2. every arc of σ has endpoints in distinct intervals of Q.

Note that the arc complex is a complex of arcs up to isotopy. We will allow isotopies that move endpoints as long as the endpoints of the arcs remain in the intervals in Q. The goal of Section 3.1.1 is to prove the following lemma.

Lemma 3.1.2. *Let $S = S_g^b$ be a compact surface and let $(\Sigma, Q) \in \mathfrak{MarkSur}$ with underlying surface S. The complex $\mathcal{A}_{\text{nosep}}(\Sigma, Q)$ is $(g + |B_+| - 3)$–acyclic.*

The proof proceeds by applying Proposition 2.2.1 to the complex $\mathcal{A}_{\text{nosep}}(\Sigma, Q)$ and the *complex of Q–arcs* $\mathcal{A}(\Sigma, Q)$, which is the complex of arcs with endpoints in distinct intervals in Q. For $\mathcal{A}(\Sigma, Q)$, we will allow isotopies of arcs to change the endpoints of α, but the endpoints must remain inside the intervals of Q throughout the isotopy.

The outline of Section 3.1.1. We will first discuss the process of cutting open an arc α in $(\Sigma, Q) \in \mathfrak{VerCom}$. We will use the method of "Hatcher flow" [17] to prove Lemma 3.1.3 that $\mathcal{A}(\Sigma, Q)$ is contractible, and then use Proposition 2.2.1 to prove Lemma 3.1.2.

Cutting open arcs in (Σ, Q). Let $(\Sigma, Q) \in \mathfrak{MarkSur}$ and let $\alpha \in \mathcal{A}(\Sigma, Q)$ be a vertex. We will define $(\Sigma, Q) \searrow \alpha \in \mathfrak{MarkSur}$. Let $\Sigma = (S, P, B_+, B_-)$. The object $\Sigma \searrow \alpha \in \mathfrak{VerCom}$ is given by the unique vertex complement structure on $S \searrow \alpha$ such that:

- $\Sigma \searrow \alpha < \Sigma$, and
- the inclusion $\iota : \Sigma \searrow \alpha \to \Sigma$ that realizes $\Sigma \searrow \alpha < \Sigma$ is the inclusion $S \searrow \alpha \to S$.

We will denote $\Sigma \searrow \alpha = (S \searrow \alpha, P^\alpha, B_+^\alpha, B_-^\alpha)$. Let $(\Sigma \searrow \alpha, Q_\alpha) = (\Sigma, Q) \searrow \alpha$. The set of intervals Q_α is defined as follows. Let $\iota : S \searrow \alpha \to S$ denote the natural inclusion. If $p \in B_+^\alpha$ is a boundary component with $\iota(p)$ isotopic to a boundary component $p' \in B_+$, then we will define Q_α to have $|Q_\alpha(p)| = |Q_\alpha(p')|$. If $\iota(p)$ is not isotopic to a boundary component of S, we have two cases.

- Case 1: there is an embedded $S_{0,3} \hookrightarrow S$ with $\partial S_{0,3} = \iota(p) \cup p_0 \cup p_1$ with $p_0, p_1 \subseteq \partial S$. In this case, α has endpoints in p_0 and p_1. Let $f : p \to p_0 \cup \alpha \cup p_1$ be a smooth map that extends to a smooth map $F : S \searrow \alpha \to S$ which is homotopic to ι. We define

$$Q_\alpha(p) = \pi_0\left(f^{-1}(\alpha) \bigcup_{I \in Q(p_0)\cup Q(p_1)} f^{-1}(I)\right).$$

- Case 2: there is no such $S_{0,3} \subseteq S$. In this case, the endpoints of α are in the same boundary component $q \in B_+$. There is a unique boundary component $p' \in B_+^\alpha$ such that $\iota(p'), \iota(p)$ and q cobound an embedded $S_{0,3} \subseteq S$. Let $f : p \cup p' \to q \cup \alpha$ be a smooth map that extends to a smooth map $F : S \searrow \alpha \to S$ such that F is homotopic to ι. We define
$$Q_\alpha(p) = \pi_0\left(\bigcup_{I \in Q(q)} f^{-1}(I)\right).$$

The motivation behind this definition is that for σ a cell of $\mathcal{A}_{\text{nosep}}(\Sigma, Q)$, we have

$$\text{lk}_{\mathcal{A}_{\text{nosep}}(\Sigma,Q)}(\sigma) = \mathcal{A}_{\text{nosep}}((\Sigma, Q) \searrow \sigma)$$

and similarly for $\sigma \subseteq \mathcal{A}(\Sigma, Q)$.

We will prove the following lemma about the complex of Q–arcs $\mathcal{A}(\Sigma, Q)$. This proof follows a method of Hatcher [17], which is sometimes referred to as Hatcher flow. This strategy is also used in the proof of Lemma 2.3.1.

Lemma 3.1.3. *Let* $(\Sigma, Q) \in \mathfrak{MarkSur}$. *The complex* $\mathcal{A}(\Sigma, Q)$ *is contractible.*

Proof. Since the join of contractible spaces is contractible, we may assume that the underlying surface of Σ is connected. Fix an arc $\beta \in \mathcal{A}(\Sigma, Q)$. Let $q_0, q_1 \in Q$ be the endpoints of β and orient β from q_0 to q_1. We will define a surgery function Surg_β on $\mathcal{A}(\Sigma, Q)$.

We begin by defining $\text{Surg}_\beta(\alpha)$ for α a vertex of $\mathcal{A}(\Sigma, Q)$. Assume that α is isotoped to be in minimal position with β. If α is disjoint from β, then $\text{Surg}_\alpha(\beta) = \alpha$. Otherwise, suppose that p is the point of intersection of α and β closest along β to the point q_1. Cut α at the point p and paste in the subsegment of β connecting p to q_1, and then homotope slightly to get 2 arcs α' and α'' with α' disjoint from α and β, and α'' disjoint from α and minimally intersecting β. Refer to Figure 3.2 for a picture of the construction.

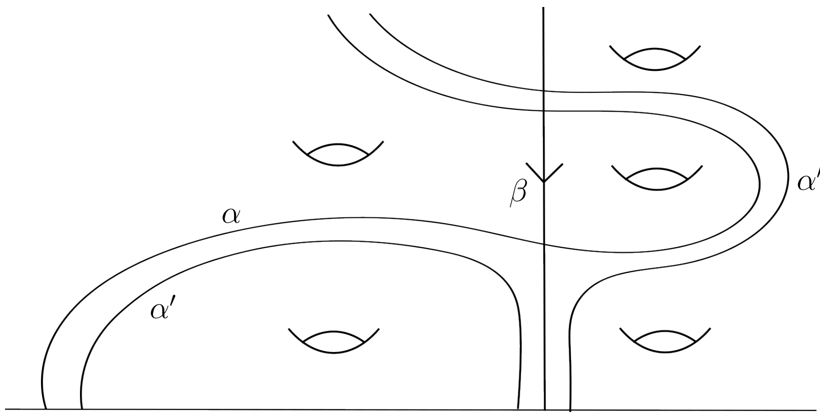

Figure 3.2: Surgery on arcs.

Now, one of α' or α'' must have endpoints in distinct intervals in Q since both α and β both have endpoints in distinct intervals of Q. Hence, the function Surg_β is defined on vertices by:

$$\mathrm{Surg}_\beta(\alpha) = \begin{cases} \alpha' & \text{if } \alpha'' \text{ is not a vertex of } \mathcal{A}(\Sigma, Q) \\ \alpha'' & \text{if } \alpha' \text{ is not a vertex of } \mathcal{A}(\Sigma, Q) \\ \frac{1}{2}\alpha' + \frac{1}{2}\alpha'' & \text{otherwise} \end{cases}$$

We now extend Surg_β to a homotopy $\mathcal{A}'(\Sigma, Q) \to \mathrm{st}(\beta)$ using the same strategy as in the proof of Lemma 2.3.3. Choose geodesic representatives for every arc, and perturb them so that no two arcs α, α' intersect β at the same point. Then there is a weight function given by the distance along β from this point of intersection to q_1. The descending link of this weight function is a cone with cone point $\mathrm{Surg}_\beta(\alpha)$, so $\mathcal{A}'(\Sigma, Q) \simeq \mathrm{st}(\beta)$. Since $\mathrm{st}(\beta)$ is contractible, the proof is complete. □

We are now almost ready to prove Lemma 3.1.2, which we recall says that $\mathcal{A}_{\text{nosep}}(\Sigma, Q)$ is $(g + |B_+| - 3)$–acyclic. We first must extend the notion of cutting open curves and arcs on surfaces to the poset $\mathfrak{MarkSur}$.

Proof of Lemma 3.1.2. We will induct on the poset $\mathfrak{MarkSur}$.

Base cases. The base cases are given by elements $(\Sigma, Q) \in \mathfrak{MarkSur}$ such that $g + |B_+| - 3 \leq -1$, or equivalently that $g + |B_+| \leq 2$. Since Q has at least two intervals, $|B_+|$ is non–empty. Hence the two cases to consider are:

- $S = S_1^b$ with $|B_+| = 1$, and
- $S = S_0^b$ with $|B_+| = 2$.

In both cases, for any choice of Q such that $(\Sigma, Q) \in \mathfrak{MarkSur}$, the complex $\mathcal{A}_{\text{nosep}}(\Sigma, Q)$ is non–empty.

Inductive step. Let $\Sigma = (S, P, B_+, B_-) \in \mathfrak{VerCom}$ and $Q \subseteq \partial\Sigma$ with $(\Sigma, Q) \in \mathfrak{MarkSur}$ such that for every $(\Sigma', Q') \in \mathfrak{MarkSur}$ with $(\Sigma', Q') < (\Sigma, Q)$, the lemma holds for (Σ', Q'). We will then show that the lemma holds for (Σ, Q) as well.

We will apply Proposition 2.2.1 with the complexes $\mathcal{A}_{\text{nosep}}(\Sigma, Q)$ and $\mathcal{A}(\Sigma, Q)$. The set $L_{\mathcal{A}(\Sigma,Q)}(\sigma)$ is the simplicial star in $\mathcal{A}(\Sigma, Q)$ of the cell $\sigma \subseteq \mathcal{A}_{\text{nosep}}(\Sigma, Q) \subseteq \mathcal{A}(\Sigma, Q)$. For any k–cell σ of $\mathcal{A}_{\text{nosep}}(\Sigma, Q)$, the subcomplex $L_{\mathcal{A}(\Sigma,Q)}(\sigma)$ is contractible. Since $\mathcal{A}(\Sigma, Q)$ is contractible by Lemma 3.1.3, it suffices to show that for any k–cell τ of $\mathcal{A}(\Sigma, Q)$, we have

$$\mathfrak{a}(L_{\mathcal{A}_{\text{nosep}}(\Sigma,Q)}(\tau)) = \dim(\mathcal{A}_{\text{nosep}}(\Sigma, Q)) - k - 1$$

In this case, $L_{\mathcal{A}(\Sigma,Q)}$ is bi-cellularly $(g(\Sigma) + |B_+| - 3)$–acyclic. Since $\mathcal{A}(\Sigma, Q)$ is contractible, Proposition 2.2.1 implies that $\mathcal{A}_{\text{nosep}}(\Sigma, Q)$ is $(g(\Sigma) + |B_+| - 3)$–acyclic, as desired.

Observe that if α is an arc on Σ with endpoints in distinct intervals in Q, then cutting Σ along α does one of three things:

1. decreases $|B_+|$ by 1 if α joins two distinct boundary components,
2. decreases g by 1 and increases $|B_+|$ by 1 if α has endpoints in the same boundary component, or

3. splits Σ into two surfaces Σ' and Σ'' with

$$g(\Sigma') + g(\Sigma'') = g(\Sigma) \text{ and } |B'_+| + |B''_+| = |B_+| + 1.$$

We will refer to arcs in the first collection as Type (1) arcs, and similarly for Type (2) and Type (3). Let $\sigma \subseteq \mathcal{A}(\Sigma, \mathbb{Q})$ be a k–cell. Let $(\Sigma_1, Q_1), \ldots, (\Sigma_m, Q_m) \in \mathfrak{MarkSur}$ be the connected components of $(\Sigma, Q) \searrow \sigma$. If $m = 1$, then $\sigma \subseteq \mathcal{A}_{\text{nosep}}(\Sigma, Q)$, so $L_{\mathcal{A}_{\text{nosep}}(\Sigma,Q)}(\sigma)$ is contractible. Otherwise, let $\tau \subseteq \sigma$ be a cell of maximal dimension with the property that $(\Sigma, Q) \searrow \tau$ is connected. Let $\tau' \subseteq \sigma$ be the unique sub-cell with $\tau * \tau' = \sigma$. Let $(\Sigma, Q) \searrow \tau = (\Sigma_\tau, Q_\tau)$, with $\Sigma_\tau = (S \searrow \tau, P^\tau, B^\tau_+, B^\tau_-)$. Note that $\dim(\tau) = k + 1 - m$. Since every arc in τ must be either type (1) or type (2), we have

$$g(\Sigma_\tau) + |B^\tau_+| \geq g(\Sigma) + |B_+| - (k - m + 2).$$

Then by construction, every arc of τ' is of Type (3) when restricted to (Σ_τ, Q_τ). Therefore

$$\sum_{j=1}^{m} g(\Sigma_j) + |B^j_+| = g(\Sigma_\tau) + |B^\tau_+| + m - 1.$$

Putting this together and applying Lemma 2.1.1 (our fact about joins) along with the inductive hypothesis, we see that

$$\begin{aligned}
\mathfrak{a}(L_{\mathcal{A}_{\text{nosep}}(\Sigma,Q)}(\sigma) &= -2 + \sum_{j=1}^{m} \big(g(\Sigma_j) + |B^+_j| - 3 + 2\big) \\
&= -2 + \sum_{j=1}^{m} \big(g(\Sigma_j) + |B^+_j|\big) - m \\
&= -2 + g(\Sigma_\tau) + |B^\tau_+| + m - 1 - m \\
&= +g(\Sigma_\tau) + |B^\tau_+| - 3 \\
&\geq g(\Sigma) + |B_+| - k + m - 5.
\end{aligned}$$

We have assumed that $m \geq 2$, so this last expression is bounded below by $g(\Sigma) + |B_+| - k - 3$. Hence $L_{\mathcal{A}(\Sigma,Q)}$ is bi–cellularly $(g(\Sigma) + |B_+| - 3)$–acyclic, so the proof is complete by applying Proposition 2.2.1. □

3.1.2 The proof of Proposition 3.1.1

We now show prove Proposition 3.1.1, which we recall says

$$\mathfrak{a}\left(\mathcal{C}_{\text{split}}(\Sigma)\right) \geq g(\Sigma) + \mathbb{1}_{|B_+|\geq 2} + \mathbb{1}_{|B_-|\geq 2} - 3.$$

We will begin by describing the outline of the proof of Proposition 3.1.1. We will then carry out some of the steps in the outline in a pair of lemmas. We will conclude Section 3.1.2, and Section 3.1, by proving Proposition 3.1.1.

The setup of the proof of Proposition 3.1.1. As in the proof of Lemma 3.1.2, the proof follows by induction on the poset $\mathfrak{VerCom}$. Let $\Sigma = (S, P, B_+, B_-)$ be a vertex complement such that Proposition 3.1.1 holds for all $\mathcal{T} < \Sigma$. Suppose without loss of generality that $|B_+| \geq |B_-|$. Choose a set Q of compact subintervals of $\partial\Sigma$ with $|Q|$ minimal such that $(S, Q) \in \mathfrak{MarkSur}$. Let $\mathcal{A} = \mathcal{A}_{\text{nosep}}(\Sigma, Q)$. If σ is a cell of $\mathcal{C}_{\text{split}}(\Sigma)$, let $L_{\mathcal{A}}(\sigma)$ be the full subcomplex of $\mathcal{A}$ generated by arcs disjoint from σ. Similarly, for cells $\tau \in \mathcal{A}$, let $L_{\mathcal{C}}(\tau)$ denote the subcomplex of $\mathcal{C}_{\text{split}}(\Sigma)$ generated by cells σ with σ disjoint from τ. Let $C_{p,q}$ be the bi-cellular complex

$$\bigoplus_{\sigma \in \mathcal{A}^{(p)}} C_q(L_C(\sigma)).$$

Let $\mathbb{E}^{*,\downarrow}_{*,*}$ and $\mathbb{E}^{*,\leftarrow}_{*,*}$ be the downward and leftward bi-cellular spectral sequences respectively. The goal is to prove the following two facts:

1. the downward sequence converges to $\mathbb{Z}$ for $p + q = 0$ and converges to 0 for $0 < p + q \leq g(\Sigma) - 3 + \mathbb{1}_{|B_+|\geq 2} + \mathbb{1}_{|B_-|\geq 2}$, and
2. the leftward sequence converges to $H_{p+q}(\mathcal{C}_{\text{split}}(\Sigma))$ for $0 \leq p + q \leq g(\Sigma) - 3 +$

$\mathbb{1}_{|B_+|\geq 2} + \mathbb{1}_{|B_-|\geq 2}$.

Given these two facts, $\widetilde{H}_{p+q}(\mathcal{C}_{\text{split}}(\Sigma)) = 0$ for $0 \leq p+q < \dim(C_{\text{split}}(\Sigma))$. Fact (1) follows from essentially the same argument as in the proof of Proposition 2.2.1 and will be handled in the proof of Proposition 3.1.1. The bulk of Section 3.1.2 will be devoted to proving Fact (2), which we will record as the following result.

Lemma 3.1.4. *Let $\Sigma \in \mathfrak{VerCom}$ such that for every $\mathcal{T} < \Sigma$, the complex $\mathcal{C}_{\text{split}}(\mathcal{T})$ satisfies the conclusion of Proposition 3.1.1. Let $\mathbb{E}^{1,\leftarrow}_{p,q}$ be the leftward spectral sequence discussed above. Then*

$$\mathbb{E}^{1,\leftarrow}_{p,q} \Rightarrow H_{p+q}(\mathcal{C}_{\text{split}}(\Sigma); \mathbb{Z})$$

for $p+q \leq g(\Sigma) - 3 + \mathbb{1}_{|B_+|\geq 2} + \mathbb{1}_{|B_-|\geq 2}$.

We need the following auxiliary lemma.

Lemma 3.1.5. *Let $\Sigma \in \mathfrak{VerCom}$ be as in Lemma 3.1.4. Let $p, q \geq 0$ be integers with $p > 0$ and $p+q < \dim(\mathcal{C}_{\text{split}}(\Sigma))$. Let $\delta \in \mathcal{C}_{\text{split}}(\Sigma)$. Label the connected components of $\Sigma \setminus \delta$ by Σ_+ and Σ_- where $B_+ \subseteq \Sigma_+$ and $B_- \subseteq \Sigma_-$. Suppose that $g(\Sigma_+) + |B_+| - 3 < p$. Then $\mathcal{C}_{\text{split}}(\Sigma_-)$ is at least $(q-1)$–acyclic.*

Note that the assumption on δ in Lemma 3.1.5 says that if $(\Sigma, Q) \in \mathfrak{MarkSur}$, then Lemma 3.1.2 does not tell us that the homology group $H_p(L_{\mathcal{A}}(\delta); \mathbb{Z})$ vanishes. The content of Lemma 3.1.5 is that for these δ, the acyclicity of $C_{\text{split}}(\Sigma_-)$ is high enough to allow us to carry out something resembling PL–Morse theory inside the spectral sequence $\mathbb{E}^{1,\leftarrow}_{p,q}$.

Proof of Lemma 3.1.5. Let $p' = g(\Sigma_+) + |B_+| - 3$. Observe that $\dim(\mathcal{C}_{\text{split}}(\Sigma)) = g(\Sigma) + \mathbb{1}_{|B_+|\geq 2} + \mathbb{1}_{|B_-|\geq 2} - 3$. Then $p' < p$ by hypothesis, so

$$g(\Sigma) - 3 + \mathbb{1}_{|B_+|\geq 2} + \mathbb{1}_{|B_-|\geq 2} - p' \geq q + 1.$$

We also have $g(\Sigma_+) + g(\Sigma_-) = g(\Sigma)$. Therefore, we have

$$\begin{aligned} g(\Sigma_-) - 2 + \mathbb{1}_{|B_-|\geq 2} &= g(\Sigma) - g(\Sigma_+) + \mathbb{1}_{|B_-|\geq 2} - 2 \\ &\geq g(\Sigma) - p' - 5 + |B_+| + \mathbb{1}_{|B_-|\geq 2} \\ &\geq q - 1 + |B_+| - \mathbb{1}_{|B_+|\geq 2} \\ &\geq q. \end{aligned}$$

In particular, we have $g(\Sigma_-) - 3 + \mathbb{1}_{|B_-|\geq 2} \geq q - 1$. Then $\Sigma_- < \Sigma$, so by hypothesis we have $\mathfrak{a}(\mathcal{C}_{\text{split}}(\Sigma_-)) \geq q - 1$. □

We are now ready to prove Lemma 3.1.4. Recall that this said that

$$\mathbb{E}^{1,\leftarrow}_{p,q} \Rightarrow H_{p+q}(\mathcal{C}_{\text{split}}(\Sigma); \mathbb{Z})$$

for $p + q \leq g(\Sigma) - 3 + \mathbb{1}_{|B_+|\geq 2} + \mathbb{1}_{|B_-|\geq 2}$.

Proof of Lemma 3.1.4. On page 1, the sequence $\mathbb{E}^{1,\leftarrow}_{p,q}$ is given by

$$\mathbb{E}^{1,\leftarrow}_{p,q} = \bigoplus_{\tau \in \mathcal{C}_{\text{split}}(\Sigma)^{(q)}} H_p(L_{\mathcal{A}}(\tau)).$$

Observe that in the column $p = 0$, the chain complex $\mathbb{E}^{1,\leftarrow}_{0,q}$ is identified with $C_*(\mathcal{C}_{\text{split}}(\Sigma))$, since Lemma 3.1.2 implies that each cell $\tau \in \mathcal{C}_{\text{split}}(\Sigma)$ with $\dim(\tau) \leq g(\Sigma) - 3 + \mathbb{1}_{|B_+|} + \mathbb{1}_{|B_-|}$ has $L_{\mathcal{A}}(\tau)$ connected. Indeed, if the connected component Σ' of $\Sigma \setminus \tau$ containing B_+ has $g(\Sigma') \geq 1$, then Lemma 3.1.2 implies $\mathcal{A}_{\text{nosep}}(\Sigma', Q)$ is connected. Otherwise, $\mathcal{A}_{\text{nosep}}(\Sigma, Q) \cong \mathcal{A}(\Sigma, Q)$, the latter of which is contractible by Lemma 3.1.3. Hence it is enough to show that $\mathbb{E}^{2,\leftarrow}_{p,q} = 0$ for $p > 0$ and $0 < p + q \leq g(\Sigma) - 3 + \mathbb{1}_{|B_+|\geq 2} + \mathbb{1}_{|B_-|\geq 2}$.

Pick a pair p, q with $p > 0$ and $p + q \leq g(\Sigma) - 3 + \mathbb{1}_{|B_+|\geq 2} + \mathbb{1}_{|B_-|\geq 2}$. We will show that $\mathbb{E}^{2,\leftarrow}_{p,q} = 0$. For a vertex $\delta \in \mathcal{C}_{\text{split}}(\Sigma)$, let $W(\delta) = g(\Sigma_+) + |B_+| - 3$, where Σ_+ is the connected component of $\Sigma \setminus \gamma$ that contains B_+. Let Σ_- be the connected component of the vertex complement on the connected component of $\Sigma \searrow \delta$ that contains B_-. An example of δ, Σ_+ and Σ_- can be seen in Figure 3.3.

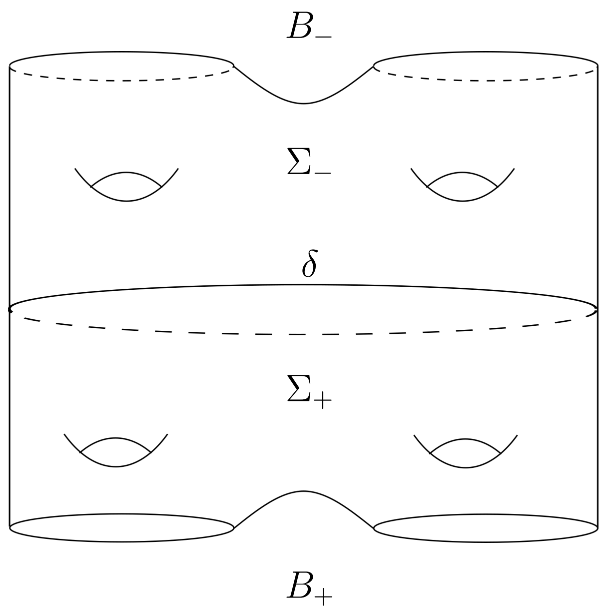

Figure 3.3: A curve δ with minimal $\dim(L_{\mathcal{A}}(\delta))$.

Let $\mathbb{E}^{1,\leftarrow}_{p,*}$ denote the chain complex in the pth column of the spectral sequence $\mathbb{E}^{1,\leftarrow}_{*,*}$. We will show that $\mathbb{E}^{2,\leftarrow}_{p,q} = H_q\left(\mathbb{E}^{1,\leftarrow}_{p,*};\mathbb{Z}\right) = 0$. There is a filtration X_k of $\mathcal{C}_{\text{split}}(\Sigma)$ given by $X_k = \bigcup_{\tau \in F_k} \tau$, where

$$F_k = \left\{\tau \subseteq \mathcal{C}_{\text{split}}(\Sigma) : \min_{\delta \in \tau^{(0)}} W(\delta) \geq p - k\right\}$$

This filtration starts at $k = 0$ and runs to $k = p$. Each X_k induces a subcomplex of $\mathbb{E}^{1,\leftarrow}_{p,*}$ by

$$\mathbb{E}^{1,\leftarrow}_{p,q}(X_k) = \bigoplus_{\tau \in \mathcal{C}_{\text{split}}(\Sigma)^{(q)} : \tau \subseteq X_k} H_p(L_{\mathcal{A}}(\tau)).$$

By construction, these $\mathbb{E}^{1,\leftarrow}_{p,*}(X_k)$ are a filtration of $\mathbb{E}^{1,\leftarrow}_{p,*}$. It therefore suffices to prove the following two facts:

- $H_q(\mathbb{E}^{1,\leftarrow}_{p,*}(X_1)) = 0$, and
- $H_q(\mathbb{E}^{1,\leftarrow}_{p,*}(X_k), \mathbb{E}^{1,\leftarrow}_{p,*}(X_{k-1})) = 0$ for $1 \leq k \leq p$.

We will prove each of these in turn.

$H_q(\mathbb{E}^{1,\leftarrow}_{p,*}(X_0);\mathbb{Z}) = 0$. Each $\tau \in X_0$ has $\mathfrak{a}(L_{\mathcal{A}}(\tau)) \geq p$ by Lemma 3.1.2. Hence $\mathbb{E}^{1,\leftarrow}_{p,*}(X_0)$ is identically 0.

$H_q(\mathbb{E}^{1,\leftarrow}_{p,*}(X_k), \mathbb{E}^{1,\leftarrow}_{p,*}(X_{k-1});\mathbb{Z}) = 0$ *for* $1 \leq k \leq p$. Let

$$\Delta_k = \{\delta \in \mathcal{C}_{\text{split}}(\Sigma) : W(\delta) = p - k\}.$$

Note that Δ_k is a discrete set, i.e., no two vertices in Δ_k are adjacent. Assume that Δ_k is ordered arbitrarily, which induces a filtration Y_k^j of X_k, where Y_k^j is given by attaching to X_{k-1} the first j elements in the order on Δ_k. Each Y_k^j induces a subcomplex of $\mathbb{E}^{1,\leftarrow}_{p,*}(X_k)$, which we denote $\mathbb{E}^{1,\leftarrow}_{p,*}(Y_k^j)$. We have the following claim.

Claim: For any $j \geq 1$, the relative homology $H_q(\mathbb{E}^{1,\leftarrow}_{p,*}(Y_k^j), \mathbb{E}^{1,\leftarrow}_{p,*}(Y_k^{j-1}))$ vanishes.

Proof of claim. Let δ be the unique vertex in $Y_k^j \setminus Y_k^{j-1}$. If $\delta \in \mathcal{C}_{\text{split}}(\Sigma)$ is a vertex, let $a_W(\delta)$ denote the subcomplex of $\mathcal{C}_{\text{split}}(\Sigma)$ generated by all $\delta' \in \text{lk}(\delta)$ with $W(\delta') > W(\delta)$. Note that if $\tau \subseteq a_W(\delta)$ is a cell, the inclusion map $L_{\mathcal{A}}(\tau * \delta) \hookrightarrow L_{\mathcal{A}}(\delta)$ is an isomorphism. Hence, there is short exact sequence of chain complexes

$$0 \to C_*(a_W(\delta)) \otimes H_p(L_{\mathcal{A}}(\delta);\mathbb{Z}) \to \mathbb{E}^{1,\leftarrow}_{p,*}(Y_k^{j-1}) \oplus C_*\left(\delta * a_W(\delta)\right) \otimes H_p(L_{\mathcal{A}}(\delta);\mathbb{Z}) \to \mathbb{E}^{1,\leftarrow}_{p,*}(Y_k^j) \to 0.$$

Now, by Lemma 3.1.5, we have $\widetilde{H}_k(a_W(\delta)) = 0$ for $k \leq q-1$, and thus we have

- $H_k(C_*(a_W(\delta)) \otimes H_p(L_{\mathcal{A}}(\delta);\mathbb{Z}) = 0$ for $1 \leq k \leq q-1$, and
- $H_0(C_*(a_W(\delta)) \otimes H_p(L_{\mathcal{A}}(\delta);\mathbb{Z})) = H_p(L_{\mathcal{A}}(\delta);\mathbb{Z})$.

The complex $\delta * a_W(\delta)$ is contractible, so we have

- $H_k(C_*\left(\delta * a_W(\delta)\right) \otimes H_p(L_{\mathcal{A}}(\delta);\mathbb{Z}) = 0$ for $1 \leq k$, and
- $H_0(C_*\left(\delta * a_W(\delta)\right) \otimes H_p(L_{\mathcal{A}}(\delta);\mathbb{Z}) = H_p(L_{cA}(\delta);\mathbb{Z})$.

Therefore the pushforward map

$$H_k(\mathbb{E}^{1,\leftarrow}_{p,*}(Y^j_k)) \to H_k(\mathbb{E}^{1,\leftarrow}_{p,*}(Y^{j-1}_k);\mathbb{Z}))$$

is surjective for $k = q$ and bijective for $k = q-1$, so the claim holds.

Given the claim, we now have $H_q(\mathbb{E}^{1,\leftarrow}_{p,*}(X_k), \mathbb{E}^{1,\leftarrow}_{p,*}(X_{k-1})) = 0$ for $1 \leq k \leq p$. Since $H_q(\mathbb{E}^{1,\leftarrow}_{p,*}(X_0)) = 0$, we have $H_q(\mathbb{E}^{1,\leftarrow}_{p,*}(X_p)) = H_q(\mathbb{E}^{1,\leftarrow}_{p,*}) = 0$, as desired. □

We are now ready to show that $\mathfrak{a}(\mathcal{C}_{\text{split}}(\Sigma)) \geq g(\Sigma) - 3 + \mathbb{1}_{|B_+|\geq 2} + \mathbb{1}_{|B_-|\geq 2}$.

Proof of Proposition 3.1.1. We will induct on the poset $\mathfrak{VerCom}$. Namely, if Σ is a vertex complement, we will assume that Proposition 3.1.1 holds for all $\mathcal{T} \in \mathfrak{VerCom}$ with $\mathcal{T} < \Sigma$, and show that it holds for Σ as well.

Base cases. Our base cases are given by any choice of g, $|B_+|$ and $|B_-|$ such that

$$g - 3 + \mathbb{1}_{|B_+|\geq 2} + \mathbb{1}_{|B_-|\geq 2} \geq -1.$$

This is equivalent to

$$g + \mathbb{1}_{|B_+|\geq 2} + \mathbb{1}_{|B_-|\geq 2} \geq 2.$$

All such choices of $g, |B_+|, |B_-|$ satisfying this condition are as follows:

- $g \geq 2$,
- $g \geq 1, |B_+| \geq 2$ or $|B_-| \geq 2$, or
- $|B_+|, |B_-| \geq 2$.

In all these cases $C_{\text{split}}(\Sigma)$ is non–empty by inspection, so the result holds.

Induction on $\mathfrak{VerCom}$. Let $\Sigma = (S, P, B_+, B_-)$ be a vertex complement such that the proposition holds for all $\mathcal{T} < \Sigma$. Suppose without loss of generality that $|B_+| \geq |B_-|$.

Let $C_{p,q}$, $\mathbb{E}^{*,\leftarrow}_{p,q}$ and $\mathbb{E}^{*,\downarrow}_{p,q}$ be as discussed in the beginning of Section 3.1.2. It suffices to prove that the downward sequence converges to $\mathbb{Z}$ for $p + q = 0$ and converges to 0 for $0 < p + q < \dim(\mathcal{C}_{\text{split}}(\Sigma))$, and that the leftward sequence converges to $H_{p+q}(\mathcal{C}_{\text{split}}(\Sigma))$ for $0 < p + q < \dim(\mathcal{C}_{\text{split}}(\Sigma))$.

The downward spectral sequence converges to 0 *for* $0 < p+q < \dim(\mathcal{C}_{\text{split}}(\Sigma))$. On page 1, the downward spectral sequence $\mathbb{E}^{*,\downarrow}_{*,*}$ is given by

$$\mathbb{E}^{1,\downarrow}_{p,q} = \bigoplus_{\sigma \in \mathcal{A}^{(p)}} H_q(L_C(\sigma)).$$

If $|B_+| = 2$ and σ is a vertex, then $L_{\mathcal{C}}(\sigma)$ is contractible. For all other situations, the inductive hypothesis says that the groups $\widetilde{H}_q(L_{\mathcal{C}}(\sigma))$ are trivial for σ a p-cell and $0 \leq q \leq g(\Sigma) - 3 + \mathbb{1}_{|B_+| \geq 2} + \mathbb{1}_{|B_-| \geq 2} - p$. Therefore we have

$$\mathbb{E}^{*,\downarrow}_{p,q} \Rightarrow H_{p+q}(\mathcal{A}_{\text{nosep}}(\Sigma, Q))$$

for $p + q \leq g(\Sigma) - 3 + \mathbb{1}_{|B_+| \geq 2} + \mathbb{1}_{|B_-| \geq 2}$. By Lemma 3.1.2, $\mathbb{E}^{*,\downarrow}_{p,q} \Rightarrow 0$ for $0 < p + q \leq g(\Sigma) - 3 + \mathbb{1}_{|B_+| \geq 2} + \mathbb{1}_{|B_-| \geq 2}$ and $\mathbb{Z}$ for $p + q = 0$.

The leftward spectral sequence converges to $H_{p+q}(\mathcal{C}_{\text{split}}(\Sigma))$. This is the content of Lemma 3.1.4. Since both $\mathbb{E}^{1,\leftarrow}_{p,q}$ and $\mathbb{E}^{1,\downarrow}_{p,q}$ converge to the total homology of $C_{p,q}$, we have $\widetilde{H}_{p+q}(\mathcal{C}_{\text{split}}(\Sigma)) = 0$ for $p + q \leq g(\Sigma) - 3 + \mathbb{1}_{|B_+| \geq 2} + \mathbb{1}_{|B_-| \geq 2}$, as desired. $\square$

3.2 PL–Morse theory for cell complexes

Recall from the introduction that the proof of Theorem B proceeds by using PL–Morse theory on the complex of minimizing cycles $\mathcal{B}_{\vec{x}}(S_g)$ [4]. As defined, $\mathcal{B}_{\vec{x}}(S_g)$ is not a simplicial complex and as such the results of Section 2.1 do not directly apply.

Remark. Hatcher showed that $\mathcal{B}_{\vec{x}}(S_g)$ could be turned into a simplicial complex by triangulating cells [16].

We will address this issue here, by explaining how to use PL–Morse theory on certain types of non–simplicial complexes. The main goal is to prove Lemma 3.2.6, which is a version of Lemma 2.1.3 that applies for CW–complexes equipped with some additional linear structure.

Locally linear cell complex. Let $\mathbb{R}^{\mathbb{N}} = \bigoplus_{n\in\mathbb{N}} \mathbb{R}$. Let $S_1, S_2 \subseteq \mathbb{R}^{\mathbb{N}}$ be two subsets. The convex hull of S_1, denoted $\mathrm{Hull}(S_1)$, is the set

$$\{t_0 s_0 + \ldots + t_n s_n : n \in \mathbb{Z}_{\geq 0}, s_0, \ldots, s_n \in S, \sum_{i=0}^{n} t_i = 1, t_i \geq 0 \text{ for all } 0 \leq i \leq n\}.$$

The convex join of S_1 and S_2, denoted $\mathrm{ConvJoin}(S_1, S_2)$, is the set

$$\{t_1 s_1 + t_2 s_2 : s_i \in S_i, t_1 + t_2 = 1, t_i \geq 0\}.$$

For notational convenience, we set $\mathrm{ConvJoin}(S, \emptyset) = S$.

Locally linear cell complex. A finite-dimensional cell complex X is *locally linear* if there is an inclusion $\iota : X \hookrightarrow \mathbb{R}^{\mathbb{N}}$ such that $\iota(\sigma)$ is the convex hull of its vertices and $\iota(\partial\sigma)$ is the union of the faces of σ for every cell $\sigma \subseteq X$. We conflate X with its image under this map. Let $W : X^{(0)} \to \mathbb{N}$ be a PL–Morse funtion. We say that W is a *linear PL–Morse function* if W is the restriction of some linear function $\mathbb{R}^{\mathbb{N}} \to \mathbb{R}$, which by abuse of notation we will refer to by W as well. Linear PL–Morse functions have the following property.

Lemma 3.2.1. *Let σ be a cell of a locally linear cell complex X and let W be a linear PL–Morse function on X. The set of all points $x \in \sigma$ with $W(x)$ of maximal weight among the points of σ is a face of σ, and is the convex hull of the vertices of σ of maximal weight.*

We will say that a function $W : X^{(0)} \to \mathbb{N}$ is a *quasi–linear PL–Morse function* if, for every cell $\sigma \subseteq X$, the convex hull of the set of all vertices $v \in \sigma^{(0)}$ with $W(v)$ maximal over all vertices of $\sigma^{(0)}$ is a face of σ. We say that a cell $\sigma \subseteq X$ is *W–constant* if $W(v) = W(w)$ for every $v, w \in \sigma^{(0)}$. We say that W is *sharp* if the only W–constant cells of X are vertices.

Descending links in locally linear cell complexes. Let X be a locally linear cell complex and let W be a quasi–linear PL–Morse function on X. Let $\sigma \subseteq X$ be a W–constant cell. We will define three different versions of the descending link $d_W(\sigma)$, and then show that all threare homotopy equivalent. Let $\mathcal{R}_\sigma$ denote the set of all cells $\rho \subseteq X$ such that

- σ is a face of ρ, and
- $W(r) \leq W(\sigma)$ for all $r \in \rho^{(0)}$, with equality if and only if $r \in \sigma$.

The variants of the descending link are as follows.

- *The facial descending link* $d_W^{\text{face}}(\sigma)$. Let $\rho \in \mathcal{R}_\sigma$ be a cell and let $\mathcal{T}(\rho)$ be the set of all faces $\tau \subseteq \rho$ such that $\tau \cap \sigma = \emptyset$. We define

$$d_W^{\text{face}}(\sigma) = \bigcup_{\rho \in \mathcal{R}_\sigma} \mathcal{T}(\rho).$$

- *The adjacent descending link* $d_W^{\text{adj}}(\sigma)$. For all $\rho \in \mathcal{R}_\sigma$, define $V(\rho)$ to be the set of all vertices $v \in \rho^{(0)}$ with $v \notin \sigma$ such that vw is an edge of X for some $w \in \sigma$. We define

$$d_W^{\text{adj}}(\sigma) = \bigcup_{\rho \in \mathcal{R}_\sigma} \text{Hull}(V(\rho)).$$

- *The total descending link* $d_W^{\text{tot}}(\sigma)$. This is given by

$$d_W^{\text{tot}}(\sigma) = \text{ConvJoin}(d_W^{\text{face}}(\sigma), d_W^{\text{adj}}(\sigma)).$$

Note that there are two inclusions

$$d_W^{\text{face}}(\sigma) \hookrightarrow d_W^{\text{tot}}(\sigma) \hookleftarrow d_W^{\text{adj}}(\sigma).$$

We have the following lemma that describes the relationships between the three types of descending links.

Lemma 3.2.2. *Let W be a quasi–linear PL–Morse function on a locally linear cell complex X. Let $\sigma \subseteq X$ be a W–constant cell. Then each inclusion*

$$d_W^{\text{face}}(\sigma) \hookrightarrow d_W^{\text{tot}}(\sigma) \hookleftarrow d_W^{\text{adj}}(\sigma)$$

induces an isomorphism in homology.

Proof. Let $\mathcal{R}_\sigma$ be as above. Let $\mathcal{R}_\sigma^{\max}$ denote the set of maximal cells in $\mathcal{R}_\sigma$, i.e., the set of all ρ such that there is no $\rho' \in \mathcal{R}_\sigma$ with $\rho \subsetneq \rho'$. For each $\dagger \in \{\text{face}, \text{tot}, \text{adj}\}$ and for each $\rho \in \mathcal{R}^{\max}$ let

$$U_\rho^\dagger = \rho \cap d_W^\dagger(\sigma).$$

For $\dagger \in \{\text{tot}, \text{adj}\}$, $U_\rho^\dagger$ is contractible for any $\rho \in \mathcal{R}^{\max}$ since convex hulls of nonempty sets are contractible. Since W is quasi–linear, U_ρ^{face} is contractible for every $\rho \in \mathcal{R}^{\max}$ as well, since $\partial\rho$ is homotopy equivalent to a sphere, and $U_\rho^\dagger$ is homotopy equivalent to $\partial\rho \setminus \sigma$, which is homotopy equivalent to a sphere with a point removed, and hence is contractible. Furthermore, let $\rho_0, \ldots, \rho_k \in \mathcal{R}_\sigma^{\max}$. Suppose that $U_{\rho_0}^\dagger \cap \ldots \cap U_{\rho_k}^\dagger \neq \emptyset$. Then for any choice of $\dagger' \in \{\text{tot}, \text{face}, \text{adj}\}$,

$$U_{\rho_0}^{\dagger'} \cap \ldots \cap U_{\rho_k}^{\dagger'} \simeq *.$$

Indeed, since $\rho_0, \ldots, \rho_k$ are cells of a locally linear complex X, their intersection $\rho = \rho_0 \cap \ldots \cap \rho_k$ must be a cell of X. Then by assumption, there must be a vertex $v \in \rho$ with $v \notin \sigma$. Hence the complex $V(\rho)$ in the definition of d_W^{adj} must be non–empty, and thus $\text{Hull}(V(\rho)) = \rho \cap d_W^{\text{adj}}(\sigma)$ is contractible. Likewise, $\mathcal{T}(\rho)$ as in the definition of d_W^{face} must be non–empty, since $V(\rho) \subseteq \mathcal{T}(\rho)$, so $d_W^{\text{face}}(\sigma) \cap \rho$ is non–empty. The cell σ is a face of ρ because W is quasi–linear. Therefore, the subcomplex $d_W^{\text{face}}(\sigma) \cap \rho$ is given by taking ρ and removing a contractible subspace glued over a contractible subspace, so we conclude that $d_W^{\text{face}}(\sigma) \cap \rho$ is contractible. Finally, if $d_W^{\text{face}}(\sigma) \cap \rho$ and $\rho \cap d_W^{\text{adj}}(\sigma)$ are both contractible, then $\rho \cap d_W^{\text{tot}}(\sigma)$ is contractible as well.

Now, let $\dagger \in \{\mathrm{adj}, \mathrm{face}, \mathrm{tot}\}$. Let $\mathcal{N}(U_\rho^\dagger)$ denote the nerve of the set $\{U_\rho^\dagger\}_{\rho \in \mathcal{R}_\sigma^{\max}}$. The nerve lemma implies that we have isomorphisms in homology

$$H_*(\mathcal{N}(U_\rho^\dagger); \mathbb{Z}) \cong H_*(d_W^\dagger(\sigma); \mathbb{Z}).$$

It now remains to show that for a choice of $\dagger = \{\mathrm{adj}, \mathrm{face}\}$, the induced map of chain complexes

$$C_*(\mathcal{N}(U_\rho^\dagger); \mathbb{Z}) \to C_*(\mathcal{N}(U_\rho^{\mathrm{tot}}); \mathbb{Z})$$

induces an isomorphism in homology. In fact, we will show that the above map induces an isomorphism of chain complexes. It is clear that the induced map is an injection for either choice of $\dagger$, so it remains to prove that the natural map is a surjection.

Let $\rho_0, \ldots, \rho_n \in \mathcal{R}_\sigma^{\max}$ be a collection of cells with $U_{\rho_0}^{\mathrm{tot}} \cap \ldots \cap U_{\rho_n}^{\mathrm{tot}} \neq \emptyset$. Let $\rho = \rho_0 \cap \ldots \cap \rho_n$. This ρ is a cell of X and $\sigma \neq \rho$, so there is a vertex $\tau \in X$ with $\tau \cap \sigma = \emptyset$ and $\tau \subseteq U_\rho^{\mathrm{tot}}$. Now, τ must contain a vertex adjacent to σ, since ρ is a cell. Therefore U_ρ^{adj} and U_ρ^{face} are both nonempty, so the induced map of chain complexes is surjective. □

We will need one more auxiliary result.

Lemma 3.2.3. *Let W be a linear PL–Morse function on a locally linear complex X, let σ be a W–constant cell, and let $k = \dim(\sigma)$. If $d_W^{\mathrm{tot}}(\sigma)$ is n–acyclic for some n, then $\mathrm{ConvJoin}(\partial\sigma, d_W^{\mathrm{adj}}(\sigma))$ is $(n+k)$–acyclic.*

Proof. By Lemma 3.2.2, $d_W^{\mathrm{adj}}(\sigma)$ is n–acyclic. We will show that $\mathrm{ConvJoin}(\partial\sigma, d_W^{\mathrm{adj}})$ has the same homology groups as $\partial\sigma * d_W^{\mathrm{adj}}$. Applying Lemma 2.1.1 completes the proof.

Recall the set $\mathcal{R}_\sigma$ and $\mathcal{R}_\sigma^{\max}$ from Lemma 3.2.2. For each $\rho \in \mathcal{R}_\sigma^{\max}$, let $U_\rho = \mathrm{ConvJoin}(\partial\sigma, \rho \cap d_W^{\mathrm{adj}})$. Then the set

$$\mathcal{U} = \{U_\rho\}_{\rho \in \mathcal{R}_\sigma^{\max}}$$

is a cellular cover of $\mathrm{ConvJoin}(\partial\sigma, d_W^{\mathrm{adj}})$. Furthermore, for any $\rho_1, \ldots, \rho_n \in \mathcal{R}_\sigma^{\max}$, there is a natural map

$$\iota : \partial\sigma * (\rho_1 \cap \ldots \cap \rho_n \cap d_W^{\mathrm{adj}}) \hookrightarrow U_{\rho_1} \cap \ldots \cap U_{\rho_n}.$$

Since the set $\left\{\partial\sigma * \left(\rho \cap d_W^{\mathrm{adj}}\right)\right\}_{\rho \in \mathcal{R}_\sigma^{\max}}$ is a cover of $\partial\sigma * d_W^{\mathrm{adj}}(\sigma)$ by contractible sets, it suffices to show that ι is a homotopy equivalence for any $\rho_1, \ldots, \rho_n \in \mathcal{R}_\sigma^{\max}$.

Let $\rho_1, \ldots, \rho_n \in \mathcal{R}_\sigma^{\max}$. Let $x \in U_{\rho_1} \cap \ldots \cap U_{\rho_n}$. Since each U_{ρ_i} is contained in ρ_i, we have $x \in \rho_1 \cap \ldots \cap \rho_n$. Hence $U_{\rho_1} \cap \ldots \cap U_{\rho_n}$ contains the minimal dimensional cell τ with $x \in \tau$ and $\tau \subseteq \rho_1 \cap \ldots \cap \rho_n$. Furthermore, if τ is such cell and $\tau' \subseteq \tau$ is a maximal dimensional subcell with $\tau \cap \sigma = \emptyset$, we have $\partial\sigma * \tau' \subseteq U_{\rho_1} \cap \ldots \cap U_{\rho_n}$. Then we must have $\tau' \subseteq d_W^{\mathrm{adj}}(\sigma)$ by our assumption on x. Therefore, $U_{\rho_1} \cap \ldots \cap U_{\rho_n} = \mathrm{ConvJoin}(\partial\sigma, \rho_1 \cap \ldots \cap \rho_n)$. Thus ι is a homotopy equivalence, since the source and target of ι are both either empty or contractible, and so the natural map of chain complexes

$$C_*(\mathcal{N}(\partial\sigma * (\rho_1 \cap d_W^{\mathrm{adj}}(\sigma)); \mathbb{Z}) \to C_*(\mathcal{N}(U_\rho); \mathbb{Z})$$

is an isomorphism of chain complexes. Since each intersection is contractible or empty, an application of the nerve lemma followed by Lemma 2.1.1 completes the proof. □

We will say that a linear PL–Morse function W on a locally linear complex X is n*–acyclic* if $d_W^{\mathrm{tot}}(\sigma)$ is $(n - \dim(\sigma))$–acyclic for every W–constant cell $\sigma \subseteq X$. The *min–set* of W, denoted $M(W)$, is the union of all cells $\sigma \subseteq X$ with

$$\max\{W(v) : v \in \sigma^0\} = \min\{W(w) : w \in X^0\}.$$

We now have the following lemma, which is similar to Lemma 2.1.2. Recall that Lemma 2.1.2 allowed us to use PL–Morse theory for simplicial complexes. This is similar, except it allows us to use PL–Morse theory for locally linear cell complexes.

Lemma 3.2.4. *Let X be a finite dimensional locally linear cell complex and let W be a linear PL–Morse function. Suppose that W is n–acyclic. Then the pair $(X, M(X))$ is*

$(n+1)$–*acyclic.*

Proof. The proof proceeds by the same strategy as Lemma 2.1.2. Let

$$X_{m,k} = W^{-1}([0,m)) \bigcup \{\sigma \text{ a cell of } X : \max\{W|_\sigma\} \leq m, \dim(\sigma) \leq k\}.$$

If $m > 0$, then $X_{m,k}$ is built out of $X_{m-1,k}$ iteratively by attaching, for each cell $\sigma \subseteq X_{m,k}$ with $\sigma \not\subseteq X_{m-1,k}$, the following spaces in order:

- $d_W^{\text{tot}}(\sigma)$ over $d_W^{\text{face}}(\sigma)$, then
- $\text{ConvJoin}(\partial\sigma, d_W^{adj}(\sigma))$ over $X_{m-1,k}$, and
- a convex neighborhood of the cell σ.

By Lemma 3.2.2, the inclusion $d_W^{\text{face}}(\sigma) \hookrightarrow d_W^{\text{tot}}(\sigma)$ is a homotopy equivalence, so the first attachment does not change homotopy type. The second is given by gluing in contractible subspaces over contractible subspaces, which also induces a homotopy equivalence. The third is attaching a contractible space over an n–acyclic space by Lemma 3.2.3, so $H_*(X_{m,k}) \cong H_*(X_{m-1,k})$ for $* \leq n+1$. □

We have the following lemma, which is similar to Lemma 3.2.4 except it deals with the case that W is sharp. This also follows from argument similar to the proof of Lemma 2.1.2.

Lemma 3.2.5. *Let X be a finite dimensional locally linear cell complex and let W be a sharp, quasi–linear PL–Morse function. Suppose that W is n–acyclic. Then the pair $(X, M(X))$ is $(n+1)$–acyclic.*

Proof. Let X_k denote the subcomplex of X generated by vertices v such that $W(v) \leq k$. Then since W is sharp, the complex X_k is constructed from X_{k-1} by attaching each $v \in X^{(0)}$ with $W(v) = k$ over the complex $d_W^{\text{face}}(v)$. By hypothesis $d_W^{\text{face}}(v)$ is n–acyclic, so the Mayer–Vietoris sequence implies that (X_k, X_{k-1}) is relatively $(n+1)$–acyclic. Since $X = \bigcup_{k \geq 0} X_k$, we have that $(X, M(W))$ is relatively $(n+1)$–acyclic, as desired. □

We now have the following lemma, which follows from an argument similar to the proof of Lemma 2.1.3.

Lemma 3.2.6. *Let X be an $(n+1)$–acyclic, finite-dimensional, locally linear cell complex and let W be an n–acyclic quasi–linear PL–Morse function on X. Assume that W is either sharp or linear. Then $M(W)$ is n–acyclic.*

Proof. By the long exact sequence in relative homology for the pair $(X, M(W))$ and the fact that $\widetilde{H}_k(X; \mathbb{Z}) = 0$ for $k \leq n+1$, the connecting homomorphism

$$\widetilde{H}_{k+1}(X, M(W); \mathbb{Z}) \to \widetilde{H}_k(M(W); \mathbb{Z})$$

is an isomorphism for $0 \leq k \leq n$. Hence applications of Lemmas 3.2.4 and 3.2.5 complete the proof. □

3.3 The complex of homologous curves

We now move on to the study of the complex of homologous curves $\mathcal{C}_{\vec{x}}(S_g)$. Our goal is to prove our main theorem (Theorem B), which says that $C_{\vec{x}}(S_g)$ is $(g-3)$–acyclic for $g \geq 2$.

Outline of the proof of Theorem B. In Section 3.3.1, we discuss the complex of minimizing cycles defined by Bestvina–Bux–Margalit [4]. We also discuss a PL-Morse function W(in the sense of Section 3.2) on the complex of minimizing cycles. This function is originally due to Hatcher and Margalit [18]. The min-set of this function W will be $\mathcal{C}_{\vec{x}}(S_g)$. We show in Lemma 3.3.1 that an auxiliary complex called the complex of draining cycles is highly acyclic. Instances of this complex will turn out to be the descending links of the PL–Morse function W. We complete the proof of Theorem B in Section 3.3.3 using the PL–Morse function W and the results of Section 3.2.

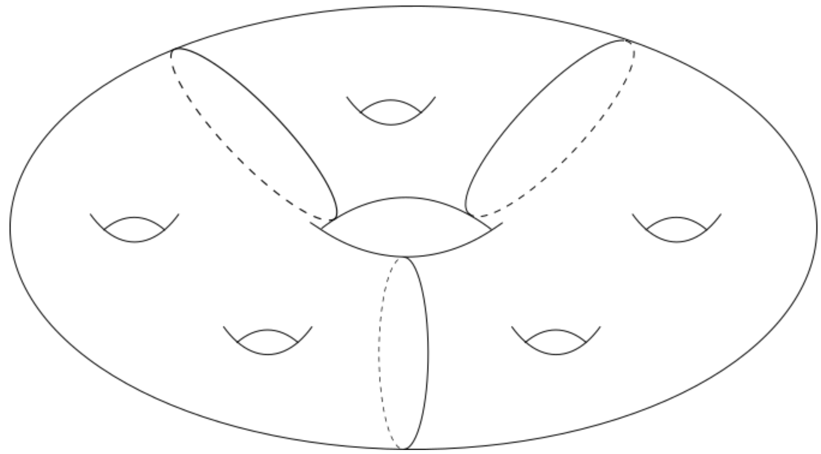

Figure 3.4: A 2-cell in $\mathcal{C}_{\vec{x}}(S_g)$.

3.3.1 The complex of minimizing cycles

We first define the complex of minimizing cycles, and then discuss Hatcher and Margalit's PL–Morse function on the complex of minimizing cycles.

Let $S = S_g$ and let $\vec{x}$ a primitive class in $H_1(S;\mathbb{Z})$. A *basic cycle* for $\vec{x}$ is a nonseparating oriented multicurve $M = a_1 \sqcup \ldots \sqcup a_k$ such that there is a collection of positive integers $\lambda_1, \ldots, \lambda_k$ with

$$\vec{x} = \sum_{i=1}^{k} \lambda_i [a_i].$$

We will say that a multicurve $M = a_1 \sqcup \ldots \sqcup a_m$ is a *cycle* if

1. each a_i is a member of a basic cycle $M' \subseteq M$, and
2. any non–trivial linear combination of the $[a_i]$ with nonnegative integer coefficients is nonzero.

Remark. Condition (2) is not present in Bestvina, Bux, and Margalit's original definition. Gaifullin demonstrated that condition (2) needed to be included in the definition [13]. All of Bestvina, Bux and Margalit's [4] results still hold, since they implicitly assumed that condition (2) followed from (1).

Let $\mathfrak{S}$ be the set of isotopy classes of nonseparating simple closed curves in S and let $\mathbb{R}^{\mathfrak{S}}$

be the real vector space spanned by $\mathfrak{S}$. If M is a cycle, denote by $P_M \subseteq \mathbb{R}^{\mathfrak{S}}$ the convex hull of the set of points

$$\{\lambda_i c_i : c_i \in M', M' \subseteq M \text{ is a basic cycle}\}.$$

Let $\mathcal{M}$ denote the set of all cycles in S_g. The *complex of minimizing cycles* $\mathcal{B}_{\vec{x}}(S_g)$ is the union

$$\bigcup_{M \in \mathcal{M}} P_M.$$

Observe that $\mathcal{B}_{\vec{x}}(S_g)$ is a CW-complex where the k–cells correspond to cycles $M \in \mathcal{M}$ with $\dim(P_M) = k$. Bestvina, Bux and Margalit proved that the complex $\mathcal{B}_{\vec{x}}(S)$ is contractible [4]. We will prove Theorem B by constructing a sharp, quasi-linear PL-Morse function W on $\mathcal{B}_{\vec{x}}(S_g)$ that is $(g-3)$–acyclic and has min-set equal to $\mathcal{C}_{\vec{x}}(S)$.

The PL-Morse function. We will use the PL-Morse function on $\mathcal{B}_{\vec{x}}(S_g)$ originally defined by Hatcher and Margalit [18]. Let v be a vertex in $\mathcal{B}_{\vec{x}}(S)$ represented by a basic cycle $M = a_0 \sqcup \ldots \sqcup a_m$. By definition, $\vec{x} = \sum_{i \leq m} \lambda_i [a_i]$ for some unique positive integers λ_i. We define

$$W(v) = \sum_{i \leq m} \lambda_i.$$

The min-set $M(W)$. There are no vertices $v \in \mathcal{B}_{\vec{x}}(S_g)$ with $W(v) = 0$. In fact, the lowest weight vertices are given by all v with $W(v) = 1$. The subcomplex of $\mathcal{B}_{\vec{x}}(S)$ generated by v with $W(v) = 1$ is precisely the complex of homologous curves $\mathcal{C}_{\vec{x}}(S)$, so $M(W)$ is the subcomplex of $\mathcal{B}_{\vec{x}}(S)$ generated by vertices v with $W(v) = 1$.

3.3.2 The complex of draining cycles

Our goal in this section is to prove Lemma 3.3.1, which says that a certain complex called the complex of draining cycles is highly acyclic. This complex is a generalization of the complex of splitting curves introduced in Section 3.1. It is also inspired by Hatcher–

Margalit's proof of the connectivity of $\mathcal{C}_{\vec{x}}(S_g)$ [18]. We will leverage Lemma 3.3.1 to prove that the PL-Morse function W is $(g-3)$–acyclic.

The complex of draining cycles. We begin by defining a certain class of labeled surfaces. Let $S = S_g^b$. We say that S is a *partial cobordism* if it:

- comes equipped with an orientation of some of the boundary components of S and
- a partition of the oriented boundary components into two disjoint sets B_+ and B_- such that the surface S' given by gluing a disk to each nonoriented boundary component of S is a cobordism from B_+ to B_-.

We will denote a partial cobordism by $\Sigma = (S, B_+, B_-, B_0)$, where $B_0 = \pi_0(\partial S) \setminus (B_+ \cup B_-)$. We will say that a partial cobordism is:

- *draining* if $|B_+| > |B_-|$,
- *balanced* if $|B_+| = |B_-|$, and
- *flooding* if $|B_+| < |B_-|$.

Let $S = S_g^b$ and let $\Sigma = (S, B_+, B_-, B_0)$ be a partial cobordism. A vertex of $\mathcal{C}_{\text{dr}}(\Sigma)$ is an oriented multicurve $M \subseteq S$ such that:

- $|\pi_0(S \searrow M)| = 2$,
- one connected component of $S \searrow M$ is a partial cobordism Σ_M from a subset of B_+ to a union of M and a subset of B_-, and
- the partial cobordism Σ_M is draining.

The partial cobordism Σ_M is unique. Two examples of vertices in this complex can be found in Figure 3.5. We now define the higher-dimensional cells in the complex $\mathcal{C}_{\text{dr}}(\Sigma)$. Let $\mathfrak{S}_\Sigma$ be set of oriented isotopy classes of essential simple closed curves on Σ. We say that an oriented multicurve $M \subseteq S$ is *representative* if:

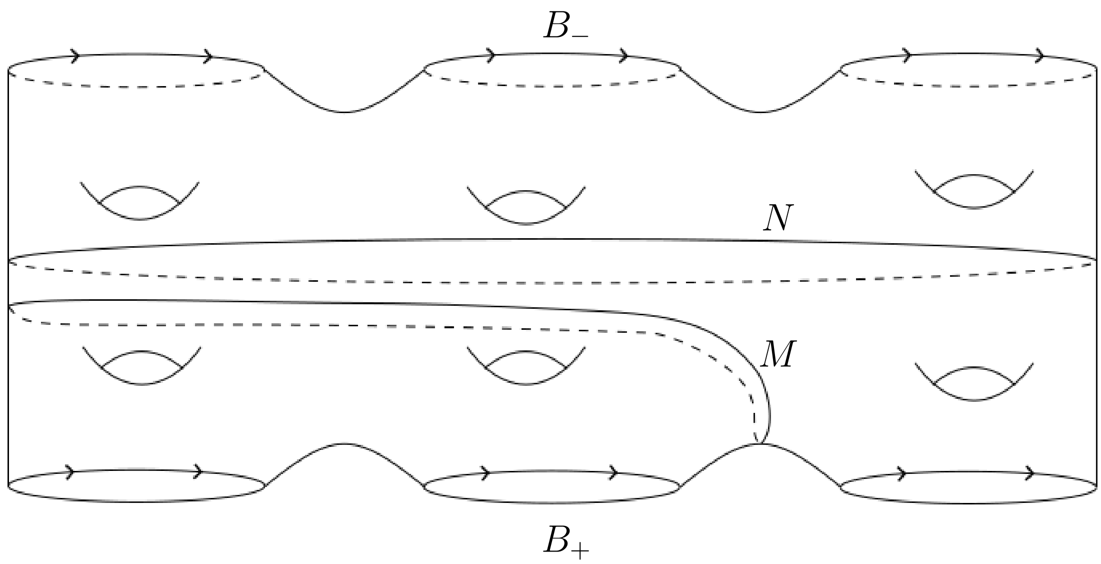

Figure 3.5: Two vertices M and N in the complex of draining cycles.

- each curve in M is contained in a vertex of $\mathcal{C}_{\mathrm{dr}}(\Sigma)$ supported on M,
- if $N \subseteq M$ is a multicurve such that $|\pi_0(S \searrow N)| = 2$ and at least one connected component Σ_N of $S \searrow N$ is a partial cobordism from a subset of B_+ to a union of N and a subset of B_-, then Σ_N is draining, and
- any non–trivial linear combination of the homology classes represented by curves in M with non–negative integer coefficients is nonzero in $H_1(S; \mathbb{Z})$, where S is the underlying surface of Σ.

Let $\mathfrak{S}_\Sigma$ denote the set of all oriented simple closed curves in S. Let $\mathcal{M}$ denote the set of representative multicurves in Σ. Each vertex $M \in \mathcal{M}$ corresponds to the point $a_1 + \ldots + a_k \in \mathbb{R}^{\mathcal{S}}$ with $M = a_1 \sqcup \ldots \sqcup a_k$. We will henceforth identify a vertex $M \in \mathcal{M}$ with this point in $\mathbb{R}^{\mathcal{S}}$. For an arbitrary $M \in \mathcal{M}$, let $P_M \subseteq \mathbb{R}^{\mathcal{S}_\Sigma}$ be the convex hull of the vertices supported in M. The complex $\mathcal{C}_{\mathrm{dr}}(\Sigma)$ is the union

$$\bigcup_{M \in \mathcal{M}} P_M.$$

Observe that $\mathcal{C}_{\mathrm{dr}}(\Sigma)$ is naturally a locally linear cell complex. We will prove the following.

Lemma 3.3.1. *Let* $\Sigma = (S_g^b, B_+, B_-, B_0)$ *be a partial cobordism with* $|B_+|, |B_-| \geq 2$ *and* $|B_+| \leq |B_-| + 1$*. The complex* $\mathcal{C}_{\text{dr}}(\Sigma)$ *is* $(g - 3 + |B_+|)$*–acyclic.*

In order to prove Lemma 3.3.1, we will need the following auxiliary result.

Lemma 3.3.2. *Let* $\Sigma = (S_g^b, B_+, B_-, B_0)$ *be a partial cobordism with* $|B_+|, |B_-| \geq 2$*,* $|B_0| \geq 1$*. Let* $b \in B_0$ *be a boundary component and let* Σ' *be the partial cobordism given by gluing a disk along* p*. The map* $\mathcal{C}_{\text{dr}}(\Sigma) \to \mathcal{C}_{\text{dr}}(\Sigma')$ *is a homotopy equivalence.*

Proof. Surgering a curve surrounding b onto any vertex M along any arc α yields a vertex M'. Hence the lemma follows by Proposition 2.3.1. □

We now prove Lemma 3.3.1. The proof proceeds by induction on the poset of partial cobordisms, denoted $\mathfrak{ParCob}$. The elements of $\mathfrak{ParCob}$ are partial cobordisms, and $\Sigma < \Sigma'$ if either $g(\Sigma) < g(\Sigma')$, or $g(\Sigma) = g(\Sigma')$ and $|B_+| < |B'_+|$

Proof of Lemma 3.3.1. We will induct on the poset of partial cobordism $\mathfrak{ParCob}$.

Base cases. The base cases are given by partial cobordisms with $|B_+| = 2$ and $|B_0| = 0$. In this case, $\mathcal{C}_{\text{dr}}(\Sigma)$ is identified with the complex of splitting curves. Lemma 3.1.1 says that when $|B_+| = 2$ and $|B_-| \geq 2$, the complex of splitting curves is $(g-1)$–acyclic, so the base case holds.

Induction on $\mathfrak{ParCob}$. Let $\Sigma = (S_g^b, B_+, B_-, B_0) \in \mathfrak{ParCob}$ with $|B_+| \geq 3$, $|B_-| \geq 2$ and $|B_+| \leq |B_-| + 1$. Assume by the inductive hypothesis that the lemma holds for all $\mathcal{T} < \Sigma$ satisfying the hypotheses of the lemma.

Lemma 3.3.2 says that we can fill in boundary components of B_0 with disks without changing the homotopy type of $\mathcal{C}_{\text{dr}}(\Sigma)$, so we may assume that $|B_0| = 0$. Let $\Sigma' = (S_g^b, B'_+, B_-, B_0)$ be the partial cobordism given by relabeling a boundary component $b \in B_+$ to be a boundary component of B_0 and let $\Sigma'' = (S_g^{b-1}, B'_+, B_-)$ be the partial cobordism given by filling in b with a disk. By Lemma 3.3.2, there is a homotopy equivalence

$$\mathcal{C}_{\text{dr}}(\Sigma') \simeq \mathcal{C}_{\text{dr}}(\Sigma'').$$

By the inductive hypothesis, $\mathcal{C}_{\mathrm{dr}}(\Sigma')$ is $(g - 4 + |B_+|)$–acyclic. Hence it suffices to show that $\mathfrak{a}(\mathcal{C}_{\mathrm{dr}}(\Sigma')) + 1 \leq \mathfrak{a}(\mathcal{C}_{\mathrm{dr}}(\Sigma))$. We will prove this statement in three steps:

1. We add to $\mathcal{C}_{\mathrm{dr}}(\Sigma')$ any vertices $M \in \mathcal{C}_{\mathrm{dr}}(\Sigma)$ that become inessential when b is capped with a disk, and show that this strictly increases acyclicity by 1.

2. We add vertices $M \in \mathcal{C}_{\mathrm{dr}}(\Sigma)$ such that M is a single simple closed curve, and show that this does not decrease acyclicity.

3. We add in the remaining vertices and show again that acyclicity does not decrease.

Adding curves that become inessential when b is filled in with a disk. Let K be the set of vertices in $\mathcal{C}_{\mathrm{dr}}(\Sigma)$ such that at least one curve becomes inessential when b is filled in with a disk. The set K is a discrete set, and every element of K is a single curve γ such that γ surrounds b and one other boundary component in B_+. Let $\mathcal{C}_{\mathrm{dr}}(\Sigma', K)$ denote the full subcomplex of $\mathcal{C}_{\mathrm{dr}}(\Sigma)$ generated by $\mathcal{C}_{\mathrm{dr}}(\Sigma')$ and K. An example of a curve in K can be found in Figure 3.6.

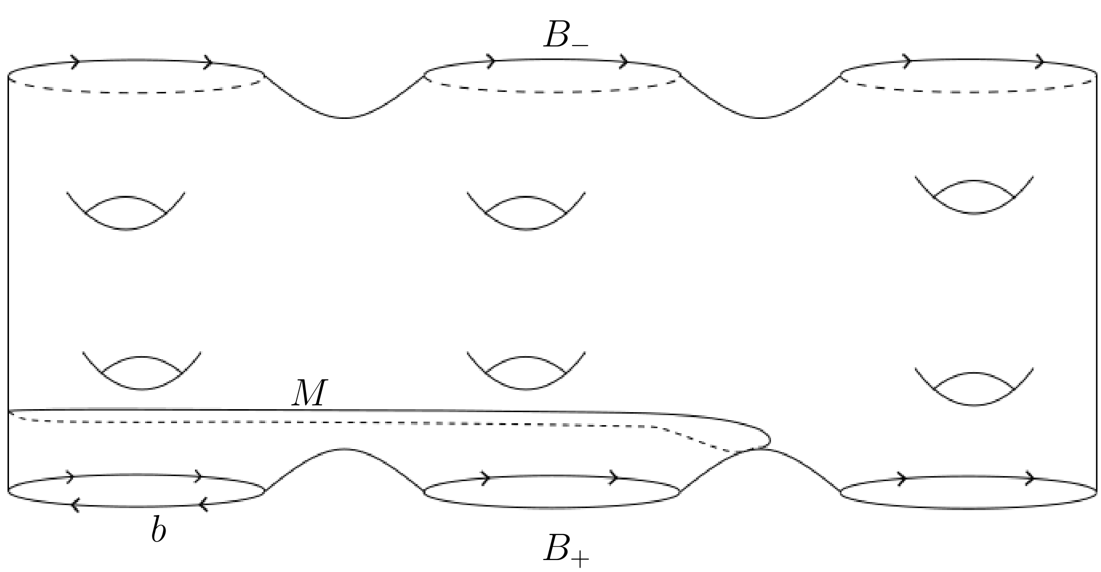

Figure 3.6: M is a vertex in in K

By Lemma 3.3.2 and Lemma 2.3.3, there is a homotopy equivalence

$$\mathcal{C}_{\mathrm{dr}}(\Sigma', K) \simeq \mathcal{C}_{\mathrm{dr}}(\Sigma') * K.$$

Hence $\mathfrak{a}(\mathcal{C}_{\mathrm{dr}}(\Sigma', K)) \geq (g - 3 + |B_+|)$ by Lemma 2.1.1 and the inductive hypothesis.

Adding in single curves. Let $C^1_{\mathrm{dr}}(\Sigma', K)$ denote the subcomplex of $\mathcal{C}_{\mathrm{dr}}(\Sigma)$ consisting of multicurves M such that every vertex $N \subseteq M$ either has $N \in \mathcal{C}_{\mathrm{dr}}(\Sigma', K)$ or N connected. Let M be a vertex of $C^1_{\mathrm{dr}}(\Sigma', K)$ and let Σ_M denote the associated cobordism from the definition of the complex of draining cycles. Let $\mathcal{T}_M$ denote the other connected component of $S^b_g \searrow M$, which is naturally a cobordism. An example of such a vertex M can be found in Figure 3.7.

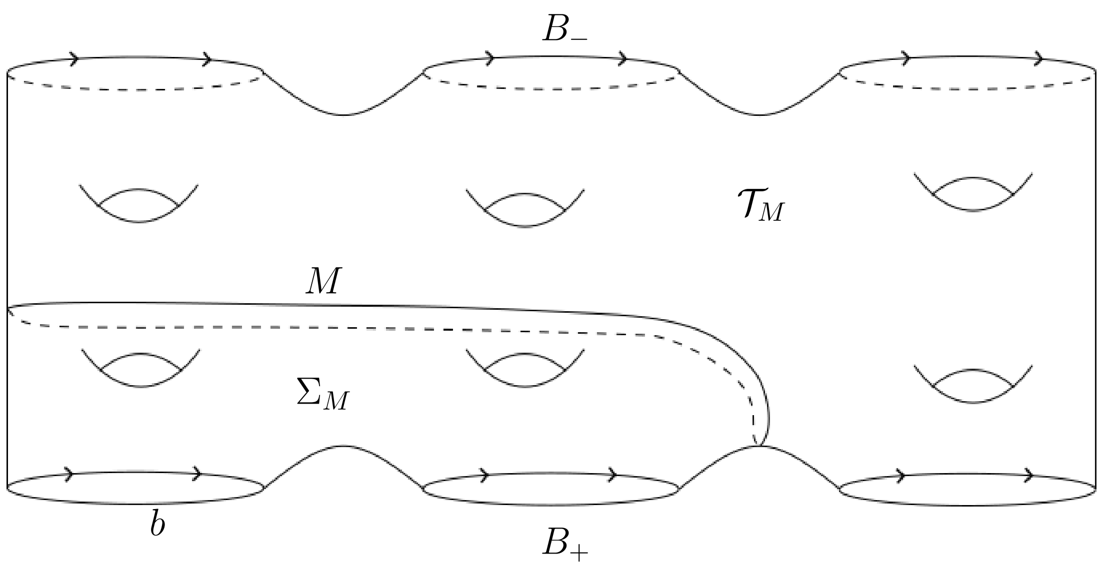

Figure 3.7: A curve $M \in C^1_{\mathrm{dr}}(\Sigma', K)$. The top boundary components are in B_-, the bottom components are in B_+.

For M a vertex of $\mathcal{C}^1_{\mathrm{dr}}(\Sigma', K)$, let

$$W(M) = \begin{cases} 0 & \text{if } M \in \mathcal{C}_{\mathrm{dr}}(\Sigma', K) \\ g(\Sigma_M) & \text{otherwise.} \end{cases}$$

We will show that W is a $(g-4+|B_+|)$–acyclic PL–Morse function. Since $\mathcal{C}_{\mathrm{dr}}(\Sigma', K)$ is at least $(g-3+|B_+|)$–acyclic and W is sharp (as in Section 3.2), Lemma 3.2.6 will imply that $\mathcal{C}^1_{\mathrm{dr}}(\Sigma', K)$ is $(g-3+|B_+|)$–acyclic.

Let $M \in \mathcal{C}^1_{\mathrm{dr}}(\Sigma', K)$ be a positive-weight vertex and let $N \in d^{\mathrm{adj}}_W(M)$ be a vertex. Observe that N is supported on either $\mathcal{T}_M$ or Σ_M. If N is supported on $\mathcal{T}_M$, then N lies in $\mathcal{C}_{\mathrm{dr}}(\mathcal{T}_M)$. Otherwise, N lies in $\mathcal{C}_{\mathrm{dr}}(\Sigma_M)$, which implies that N is connected. Hence there is a canonical isomorphism

$$d_W(M) \cong \mathcal{C}_{\mathrm{split}}(\Sigma_M) * \mathcal{C}_{\mathrm{dr}}(\mathcal{T}_M).$$

By Proposition 3.1.1, we have $\mathfrak{a}(\mathcal{C}_{\mathrm{split}}(\Sigma_M)) = g(\Sigma_M) - 2$. Then by the inductive hypothesis, we have

$$\mathfrak{a}(\mathcal{C}_{\mathrm{dr}}(\mathcal{T}_M)) = g(\mathcal{T}_M) - 3 + |B_+| - 1 = g(\mathcal{T}_M) - 4 + |B_+|.$$

Hence by Lemma 2.1.1, we have

$$\mathfrak{a}(d_W(M)) = g(\Sigma_M) - 2 + g(\mathcal{T}_M) - 4 + |B_+| + 2 = g(\Sigma) - 4 + |B_+|$$

so this step is complete.

Adding other vertices. We will now add in vertices of $\mathcal{C}_{\mathrm{dr}}(\Sigma) \setminus \mathcal{C}^1_{\mathrm{dr}}(\Sigma', K)$. Let M be a vertex of $\mathcal{C}_{\mathrm{dr}}(\Sigma)$ and let Σ_M be the associated partial cobordism. Let

$$W(M) = \begin{cases} 0 & \text{if } M \in \mathcal{C}^1_{\mathrm{dr}}(\Sigma', K) \\ |\chi(\Sigma_M)| & \text{otherwise.} \end{cases}$$

We will show that W is a $(g-4-|B_+|)$–acyclic PL–Morse function. Since W is sharp, an application of Lemma 3.2.6 will complete the proof. Let M be a vertex in $\mathcal{C}_{\mathrm{dr}}(\Sigma)$ of positive weight. Observe that the cobordism $\Sigma_M = (S_M, B_+^M, B_-^M)$ satisfies the following two properties:

- $|B_+^M| = |B_-^M| + 1$ and
- $p \in B_+^M$.

Indeed, otherwise M would be a vertex of $\mathcal{C}_{\mathrm{dr}}(\Sigma', K)$. Let $\mathcal{T}_M$ be the connected component of $\Sigma \setminus M$ that is not Σ_M. An example of such a vertex M can be found in Figure 3.8.

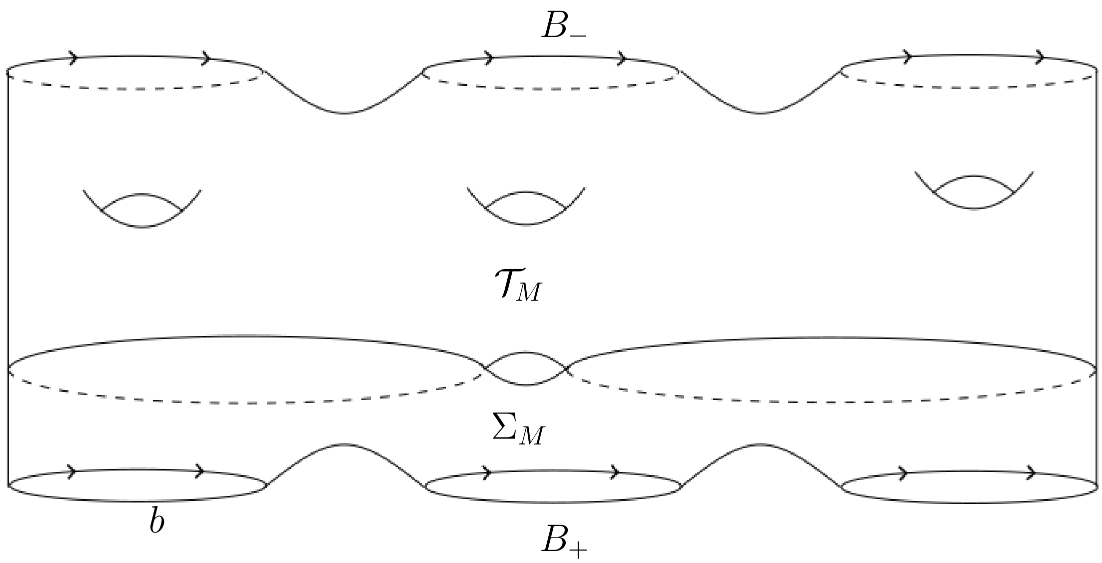

Figure 3.8: A vertex $M \in C^1_{\mathrm{dr}}(\Sigma', K)$. The top boundary components are in B_-, the bottom components are in B_+.

The surface $\mathcal{T}_M$ is naturally a cobordism from a union of M and a subset of B_+ to a subset of B_-. We will show that $d_W^{\mathrm{adj}}(M) \cong \mathcal{C}_{\mathrm{dr}}(\Sigma_M) * \mathcal{C}_{\mathrm{dr}}(\mathcal{T}_M)$. Let $N \in d_W^{\mathrm{adj}}(M)$ be a vertex. Let $P = M \cup N$. By definition N and M are adjacent, so N is either supported on Σ_M or $\mathcal{T}_M$. If $N \subseteq \Sigma_M$, then $N \in \mathcal{C}_{\mathrm{dr}}(\Sigma_M)$. Any vertex $P \in \mathcal{C}_{\mathrm{dr}}(\Sigma_M)$ must have $W(P) < W(M)$, so $d_W^{\mathrm{adj}}(M) \cap \mathcal{C}_{\mathrm{dr}}(\Sigma_M) = \mathcal{C}_{\mathrm{dr}}(\Sigma_M)$. Otherwise, if $N \subseteq \mathcal{T}_M$, then $W(N) = 0$. Therefore the partial cobordism Σ_N realizing N as draining is still draining even if p is filled in with a disk. Then since $\Sigma_N \supseteq \Sigma_M$, we have $N \in \mathcal{C}_{\mathrm{dr}}(\mathcal{T}_M)$.

Hence $d_W(M)$ is at least $(g(\Sigma_M) - 3 + |B_+(\Sigma_M)| + g(\mathcal{T}_M) - 3 + |M| + |\mathcal{T}_M \cap B_+| + 2)$–acyclic by the inductive hypothesis and Lemma 2.1.1. But then $g(\mathcal{T}_M) + g(\Sigma_M) + |M| = g$

and $|B_+^M|+|\mathcal{T}_M \cap B_+| = |B_+|$, so $d_W(M)$ is at least $(g-4+|B_+|)$–acyclic. Therefore since $\mathcal{C}^1_{\text{dr}}(\Sigma', K)$ is $(g-3+|B_+|)$–acyclic, Lemma 3.2.6 implies that $\mathcal{C}_{\text{dr}}(\Sigma)$ is $(g-3+|B_+|)$–acyclic, which completes the proof. □

3.3.3 Completing the proof

We now show that $\mathcal{C}_{\vec{x}}(S_g)$ is $(g-3)$–acyclic. We will prove in Lemma 3.3.3 that the PL–Morse function W defined in Section 3.3.1 is a linear PL–Morse function. We then prove in Lemma 3.3.5 that the descending links of W–constant k–cells are given by joins of the complex of draining cycles. We conclude by proving Theorem B.

Lemma 3.3.3. *The PL–Morse function W is a linear PL–Morse function.*

Proof. Let $\sigma \subseteq \mathcal{B}_{\vec{x}}(S_g)$ be a cell corresponding to a multicurve M. By definition, σ is the convex hull of its vertices $v_0, \ldots, v_m$. Hence every point $v \in \sigma$ is a linear combination

$$v = \sum_{i=0}^{m} t_i v_i$$

with $\sum_{i=0}^{m} t_i = 1$. Set

$$W^\sigma(v) = \sum_{i=0}^{m} t_i W(v_i).$$

It suffices to show that W is well-defined on points in $\mathbb{R}^{\mathfrak{S}}$, i.e., independent of the choice of cell σ. To see this, suppose that

$$v = \sum_{i=0}^{m} t'_i v_i$$

is another linear combination with $\sum_{i=0}^{m} t'_i = 1$. If $a_0, \ldots, a_n$ are the underlying simple closed curves in the multicurve M corresponding to σ, then each v_i is by definition a formal sum

$$\sum_{j=0}^{n} \lambda_{i,j} a_j$$

such that

$$\vec{x} = \sum_{j=0}^{n} \lambda_{i,j} [a_j]$$

in $H_1(S_g; \mathbb{Z})$. Since

$$\sum_{i=0}^{m} t_i v_i = \sum_{i=0}^{m} t'_i v_i$$

there is a relation for each a_j given by

$$\sum_{i=0}^{m} t_i \lambda_{i,j} = \sum_{i=0}^{m} t'_i \lambda_{i,j}.$$

Hence we have

$$\sum_{j=0}^{n} \sum_{i=0}^{m} t_i \lambda_{i,j} = \sum_{j=0}^{n} \sum_{i=0}^{m} t'_i \lambda_{i,j}.$$

But by the definition of W, this is precisely

$$\sum_{i=0}^{m} t_i W(v_i) = \sum_{i=0}^{m} t'_i W(v_i)$$

so the claim holds. □

We now describe the W–constant cells of $\mathcal{B}_{\vec{x}}(S_g)$. If $\sigma \subseteq \mathcal{B}_{\vec{x}}(S_g)$ is a cell, we denote by $\sigma_{\max}$ the convex hull of the vertices in σ of maximal weight.

Lemma 3.3.4. *Let σ be a W–constant cell of $\mathcal{B}_{\vec{x}}(S_g)$. Let M be the oriented multicurve corresponding to σ and let $\Sigma_0 \sqcup \ldots \sqcup \Sigma_k$ be the connected components of $S_g \searrow M$. Each cobordism Σ_i is balanced.*

Proof. Suppose otherwise. Then some cobordism Σ_i is not balanced. Σ_i is a cobordism between vertices v and w of σ. If Σ_i is unbalanced, then $W(v) \neq W(w)$ which contradicts the assumption that σ is W–constant. □

We now explicitly compute $d_W(M)$ in the case that M is a W–constant k–cell.

Lemma 3.3.5. *Let $M \subseteq \mathcal{B}_{\vec{x}}(S_g)$ be a W–constant k–cell. Let $\Sigma_0, \ldots, \Sigma_k$ be the connected components of $S_g \searrow M$. Then the natural inclusion*

$$C_{\mathrm{dr}}(\Sigma_0) * \ldots * C_{\mathrm{dr}}(\Sigma_k) \to d_W^{\mathrm{adj}}(M)$$

is an isomorphism.

Proof. Let N be a vertex in $d_W^{\mathrm{adj}}(M)$. Then $N \searrow M$ is contained in some connected component Σ of $S_g \searrow M$, which is naturally a balanced cobordism. Let $\mathcal{T} \subseteq \Sigma$ be the cobordism between a subset of $B_+(\Sigma)$ and a union of $N \setminus M$ and a subset of $B_-(\Sigma)$. Let $P \subseteq M$ be a vertex containing $B_+(\Sigma)$, so N is adjacent to P. Then the cobordism $\mathcal{T}$ must be draining. Indeed, suppose that

$$\sum_{p \in P} \lambda_p [p] = \vec{x}$$

is the linear relation corresponding to P. Suppose that $\mathcal{T}$ realizes a relation in $H_1(S_g; \mathbb{Z})$ of the form

$$\sum_{p \in P'} [p] = \sum_{n \in N'} [n]$$

for subsets $P' \subseteq P$, $N' \subseteq N$. Then there is a linear relation in $H_1(S_g; \mathbb{Z})$ supported on N given by

$$\sum_{p \in P} \lambda_p [P] - \min_{p \in P'} \lambda_p \left(\sum_{p \in P'} [p] \right) + \min_{p \in P'} \lambda_p \left(\sum_{n \in N'} [n] \right) = \vec{x}$$

Note that this relation is a nonnegative linear combination of curves in N. Indeed, this relation is another nonnegative integral relation on curves contained in $N \cup P$ and N and P are the endpoints of an edge. Then since $W(N) < W(P)$, we have

$$W(N) = W(P) - \min_{p \in P'} \lambda_p |P'| + \min_{p \in P'} \lambda_p |N'| = W(P) - \min_{p \in P'} (|P| - |N'|).$$

Hence $|N'| < |P'|$, so $\mathcal{T}$ is draining as desired. $\square$

We now conclude Section 3.3 by proving that $\mathcal{C}_{\vec{x}}(S_g)$ is $(g-3)$–acyclic.

Proof of Theorem B. Let M be a multicurve that represents a W–constant k-cell σ. Label the connected components of $S \searrow M$ by $\Sigma_0 \sqcup \ldots \sqcup \Sigma_k$. By Lemma 3.3.5, we have

$$d_W^{\mathrm{adj}}(M) = \mathcal{C}_{\mathrm{dr}}(\Sigma_0) * \ldots * \mathcal{C}_{\mathrm{dr}}(\Sigma_k),$$

where $\Sigma_0, \ldots, \Sigma_k$ are the connected components of $S_g \searrow M$. Hence $d_W(\sigma)$ is at least $(g - 3 - k)$–acyclic by Lemma 3.3.1 and Lemma 2.1.1. Now, W is a linear PL–Morse function by Lemma 3.3.3. Hence W is a $(g - 3)$–acyclic linear PL-Morse function, so Theorem B follows by Lemma 3.2.6 and Bestvina, Bux and Margalit's theorem that $\mathcal{B}_{\vec{x}}(S_g)$ is contractible [4, Theorem E]. $\square$

CHAPTER 4

PRELIMINARY REPRESENTATION THEORY RESULTS

4.1 A finiteness result about groups acting on vector spaces

The goal in this section is to prove Proposition 4.1.1, which is a result that we will apply repeatedly to determine that certain representations of subgroups of the symplectic group are finite dimensional. Recall that we have assumed that curves and multicurves are oriented. The statement is as follows.

Proposition 4.1.1. *Let $g \geq 1$, and let $N \subseteq S_g$ be a nonseparating multicurve such that $|N| < g$. Let $G \subseteq \mathrm{Sp}(2g, \mathbb{Z})$ be the image of the map $\mathrm{Stab}_{\mathrm{Mod}(S_g)}(N) \to \mathrm{Sp}(2g, \mathbb{Z})$. Let V be a G–representation over $\mathbb{Q}$. Suppose that there is a constant $0 \leq d \leq g - |N|$ such that the following hold.*

1. *For any $M \subseteq S_g$ such that:*
    - *$|M| \geq d$,*
    - *M is disjoint from N, and*
    - *$M \sqcup N$ is nonseparating,*

    the cokernel of the map $\bigoplus_{c \in M} V^{T_c} \to V$ is finite dimensional.
2. *For any $M \subseteq S_g$ such that:*
    - *$|M| < d$,*
    - *M is disjoint from N, and*
    - *$M \sqcup N$ is nonseparating,*

    the coinvariants module $V_{\mathrm{Stab}_G(M)}$ is finite dimensional.

Then V is finite dimensional.

Before proving the proposition, we will need a result about generating sets of subgroups of the symplectic group, which is Lemma 4.1.2. We will also standardize some terminology that we will use throughout this thesis.

Terminology on symplectic lattices. A *quasi-unimodular lattice* is a finitely generated free abelian group L equipped with an alternating form $\langle \cdot, \cdot \rangle : L \times L \to \mathbb{Z}$. The lattice L is *unimodular* if the form $\langle \cdot, \cdot \rangle$ is nondegenerate. The *genus of* L, denoted $g(L)$, is $\frac{1}{2}\operatorname{rk}(W)$, where $W \subseteq L$ is a maximal free abelian subgroup of L such that the restriction of $\langle \cdot, \cdot \rangle$ to W is nondegenerate. If $v \in L$ is some element, then v is *primitive* if there is no integer $m \geq 2$ and nonzero $w \in L$ with $mw = v$. For any $v \in L$, the *transvection along* v is the homomorphism $T_v : L \to L$ given by

$$T_v(w) = w + \langle v, w \rangle v.$$

We will say that the transvection T_v is *primitive* if v is primitive. We will say that a subgroup $L' \subseteq L$ is *primitive* if L is generated by primitive elements. The set of degenerate elements in L, i.e., elements $v \in L$ such that $\langle v, \cdot \rangle : L \to \mathbb{Z}$ is the zero map, will be denoted L^{degen}. If $\mathcal{V} \subseteq L$ is a set of elements, we will denote by $\mathcal{V}^{\perp}$ the subgroup $\{w \in L : \langle w, v \rangle = 0 \text{ for all } v \in \mathcal{V}\}$. We let

$$\operatorname{Sp}(L, \mathbb{Z}) = \{g \in \operatorname{GL}(L, \mathbb{Z}) : \langle gv, gw \rangle = \langle v, w \rangle \text{ for all } v, w \in L\}.$$

If $L' \subseteq L$ is a primitive, unimodular subgroup, let $\operatorname{proj}_{L'} : L \to L'$ denote the projection map induced by the form $\langle \cdot, \cdot \rangle$.

Mapping class groups of cut open surfaces. Let $M \subseteq S_g$ be a nonseparating multicurve. There is a natural inclusion $S_g \setminus M \hookrightarrow M$ that induces a map $\operatorname{Mod}(S_g \setminus M) \to \operatorname{Mod}(S_g)$ given by extending $\varphi \in \operatorname{Mod}(S_g \setminus M)$ by the identity along $S_g \setminus \operatorname{im}(S_g \setminus M \to S_g)$.

We are now ready to state and prove the following lemma, after which we will conclude Section 4.1 by proving Proposition 4.1.1.

Lemma 4.1.2. *Let $g \geq 2$ and $b \geq 0$, and let $S = S_g^b$. Let $M \subseteq S$ be a nonseparating multicurve such that $g(S \searrow M) \geq 1$. Let G be the image of the composition*

$$\mathrm{Mod}(S \searrow M) \to \mathrm{Mod}(S) \to \mathrm{Aut}(H_1(S_g^b; \mathbb{Z})).$$

Then G has a finite generating set consisting of transvections along classes v represented by nonseparating curves $c \subseteq S_g \searrow M$.

Proof. Korkmaz [26, Theorem 3.1] has shown that if $S_{g'}^{b'}$ is a compact, oriented surface with genus $g' \geq 1$, then $\mathrm{Mod}(S_{g'}^{b'})$ has a finite generating set $\mathcal{D} \subseteq \mathrm{Mod}(S_{g'}^{b'})$ consisting of Dehn twists along nonseparating curves. Then $S \searrow M$ is a compact, oriented surface with $g(S \searrow M) \geq 1$ by hypothesis. Hence there is a finite set of Dehn twists along nonseparating curves $\mathcal{D} \subseteq \mathrm{Mod}(S \searrow M)$ such that $\mathcal{D}$ generates $\mathrm{Mod}(S \searrow M)$. Then the image of $T_d \in \mathcal{D}$ under the symplectic representation is the transvection $T_{[d]}$. Hence G has a finite generating set consisting of transvections along elements represented by nonseparating curves, as desired. □

We are now ready to conclude Section 4.1.

Proof of Proposition 4.1.1. The proof proceeds by induction on d.

Base case: $d = 0$. In this case, the first hypothesis says that if M is the empty multicurve, then the cokernel $\bigoplus_{c \in \emptyset} V^{T_c} \to V$ is finite dimensional. This is just saying that the cokernel of the zero map is finite dimensional, so in particular V is finite dimensional.

Inductive step: $d \geq 1$. Assume that the proposition holds for all $d' < d$. We will show that it holds for d as well. Our aim is to show that if G and V satisfy the hypotheses of the proposition for d, then they satisfy the hypotheses of the proposition for $d - 1$ as well. The inductive hypothesis would then imply that V is finite dimensional. The second hypothesis, namely that the coinvariants module $V_{\mathrm{Stab}_G(M)}$ is finite dimensional for $|M| < d$ also holds for $|M| < d-1$, so it suffices to prove the first hypothesis for $d-1$. Let M be a multicurve such that:

- $|M| \geq d-1$,
- M is disjoint from N, and
- $M \sqcup N$ is nonseparating.

It suffices to show that the cokernel of the map $\bigoplus_{c \in M} V^{T_c} \to V$ is finite dimensional. This holds for $|M| \geq d$ by hypothesis, so it suffices to show that the result holds for $|M| = d-1$.

Let $M \subseteq S_g \searrow N$ be a nonseparating multicurve with $|M| = d-1$. Let $G' = \mathrm{im}(\mathrm{Mod}(S_g \searrow (M \sqcup N) \to \mathrm{Sp}(2g, \mathbb{Z}))$. By Lemma 4.1.2, G' is generated by a finite set of transvections $\mathcal{F}$ along a set of primitive elements $\mathcal{V}$ such that each $v \in \mathcal{V}$ has a nonseparating representative $c \subseteq S_g \searrow (M \sqcup N)$. Let $\mathcal{R}$ be a set of such representatives, one for each element of $\mathcal{V}$. For $c \in \mathcal{R}$, let $M_c = M \sqcup c$. The multicurve M_c satisfies:

- $|M_c| = 1 + |M| = d$,
- M_c is disjoint from N, and
- $M_c \sqcup N$ is nonseparating.

The first part is trivial, while the second follows since c is disjoint from N, and the third follows because we have chosen c to be nonseparating in $S_g \searrow (M \sqcup N)$. Therefore, the first hypothesis in the proposition applied to V as a G representation tells us that

$$\mathrm{cok}\left(\bigoplus_{d \in M_c} V^{T_{[d]}} \to V\right)$$

is finite dimensional. In particular, this implies that, if we let $V_M = \mathrm{cok}(\bigoplus_{d \in M} V^{T_d} \to V)$, then $V_M^{T_v}$ has finite codimension in V_M for any $v \in \mathcal{V}$. Now, consider the filtration of

V_M given by choosing an ordering $\mathcal{V} = \{v_1, \ldots, v_n\}$ and setting $W_{M,i} = \bigcap_{j=1}^{i} V_M^{T_{v_i}}$. Since $V_M^{T_{v_i}}$ has finite codimension in V_M, each $V_{M,i}$ has finite codimension in $V_{M,i-1}$. In particular, this implies that $W_{M,n}$ has finite codimension in V_M. Thus, it suffices to show that $W_{M,n}$ is finite dimensional.

$W_{M,n}$ *is finite dimensional*. Consider the long exact sequence in G'–homology associated to the short exact sequence

$$0 \to W_{M,n} \to V_M \to V_M/W_{M,n} \to 0.$$

Part of this sequence is given by

$$H_1(G'; V_M/W_{M,n}) \to H_0(G'; W_{M,n}) \to H_0(G'; V_M).$$

The group G' is finitely presented by Lemma 4.1.2 and $V_M/W_{M,n}$ is finite dimensional by hypothesis, so $H_1(G'; V_M/W_{M,n})$ is finite dimensional. Furthermore, $H_0(G'; V_M)$ is a quotient of $H_0(G'; V)$. This last vector space is finite dimensional by applying the second hypothesis of the proposition to the multicurve M, since we know $|M| = d - 1 < d$. Therefore, the vector space $H_0(G'; W_{M,n})$ is finite dimensional. But now, $W_{M,n}$ is the intersection $\bigcap_{v \in \mathcal{V}} V_M^{T_v}$. Since the set $\mathcal{F} = \{T_v : v \in \mathcal{V}\}$ generates G' by assumption, we conclude that $W_{M,n}$ is a trivial G'–representation, so $H_0(G'; W_{M,n}) = W_{M,n}$. Hence $W_{M,n}$ is finite dimensional and has finite codimension in V_M, so V_M is finite dimensional as well. The proof is now complete by the inductive hypothesis. □

4.2 Bestvina–Margalit tori

For the remainder of this section, fix a $g \geq 3$ and an $a \subseteq S_g$ a nonseparating curve. Let $[a] = \vec{x}$. Let $X_g = \mathcal{C}_{\vec{x}}(S_g)/\mathcal{I}_g$. A Bestvina–Margalit torus (defined below), is a certain type of 2–torus embedded in X_g. Let $\mathrm{BM}_2(X_g; \mathbb{Q}) \subseteq H_2(X_g; \mathbb{Q})$ denote the subspace of

$H_2(X_g; \mathbb{Q})$ generated by the fundamental classes of the Bestvina–Margalit tori. The goal in this section is to prove basic structural results about X_g and Bestvina–Margalit tori.

Bestvina–Margalit tori, by example. In unpublished work [5], Bestvina and Margalit associate to each 2-cell σ in X_g a corresponding dual cell. The cells σ and σ' form a torus which we refer to as a *Bestvina–Margalit torus*. The idea is that σ and σ' have the same edges, except the edges are in a different cyclic order. Let $x, y, z, w \in \mathcal{C}_{\vec{x}}(S_g)$ be as in Figure 4.1. Let $\sigma \subseteq X_g$ be the 2-cell such that there is a lift $\widehat{\sigma} \subseteq \mathcal{C}_{\vec{x}}(S_g)$ with vertices x, y, and w, and let $\sigma' \subseteq X_g$ be the 2–cell with a lift $\widehat{\sigma}'$ with vertices x, z, and w. Then the edges $e_{x,y}$ and $e_{y,w}$ are in the same orbit under the action of $\mathcal{I}_g$ to $e_{z,w}$ and $e_{x,z}$, respectively [38, Proposition 3.29]. However, the cells $\widehat{\sigma}$ and $\widehat{\sigma}'$ are not in the same orbit under the action of $\mathcal{I}_g$, since they induce decompositions of $H_1(S_g; \mathbb{Z})$ with different cyclic orders (defined precisely in a moment). Hence the cells σ and σ' form a torus in X_g. We will denote this

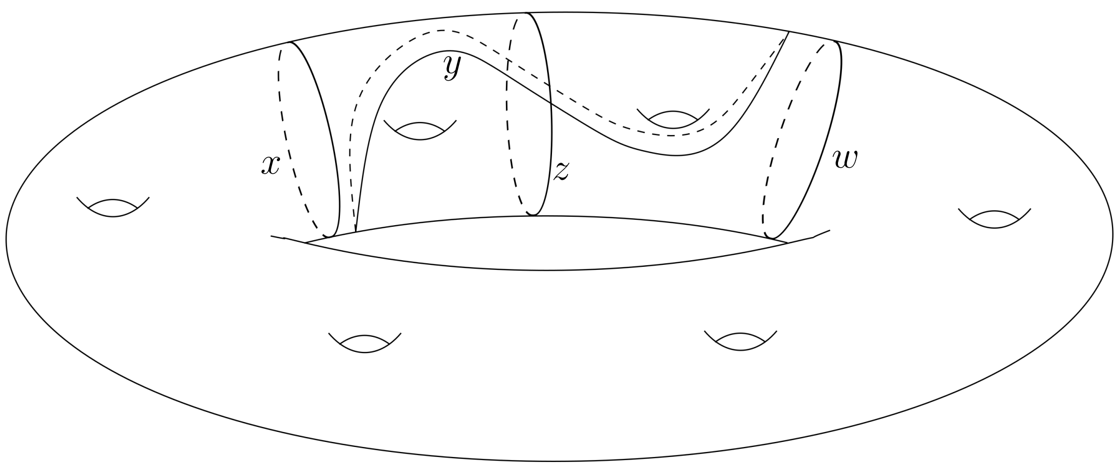

Figure 4.1: The construction of dual cells.

torus BM_σ and its fundamental class by $[\mathrm{BM}_\sigma]$. In general, a *Bestvina–Margalit* torus is any subcomplex of X_g given by the union of a pair of 2–cells σ and σ' such that each edge of σ is in the same $\mathcal{I}_g$–orbit as an edge of σ', but $\sigma \neq \sigma'$. Let $\mathrm{BM}_2(X_g; \mathbb{Q})$ denote the subspace of $H_2(X_g; \mathbb{Q})$ spanned by the set of fundamental classes $\{[\mathrm{BM}_\sigma]\}_{\sigma \in X_g^{(2)}}$. This subspace plays a crucial role in the proof of Proposition 1.3.1.

4.2.1 Structural results about $\mathcal{C}_{\tilde{x}}(S_g)/\mathcal{I}_g$

In this section we prove Lemma 4.2.1, which will tell us when two k–cells $\sigma, \sigma' \subseteq \mathcal{C}_{\vec{x}}(S_g)$ are in the same $\mathcal{I}_g$–orbit for the action of $\mathcal{I}_g$ on $\mathcal{C}_{\vec{x}}(S_g)$. Johnson showed that two nonseparating curves c and d in S_g are in the same orbit under the action of the Torelli group if and only if c and d are homologous (see, e.g., Putman [34, Lemma 6.2]). Hence the complex X_g has one vertex. Therefore each edge $x \in X_g$ is a loop in X_g, and we will let $[x]$ denote the element of $\pi_1(X_g)$ induced by x. If y and z are two edges of a 2–cell $\sigma \subseteq X_g$, we will let yz denote the third edge of σ, with the appropriate orientation so that

$$[yz] = [y][z]$$

in $\pi_1(X_g)$. If $e \subseteq \mathcal{C}_{\vec{x}}(S_g)$ is an edge, there is a bounding pair $T_{c,c'}$ taking one endpoint of e to the other endpoint. See Figure 4.2 for an example. Lemma 4.2.1 describes the cells of

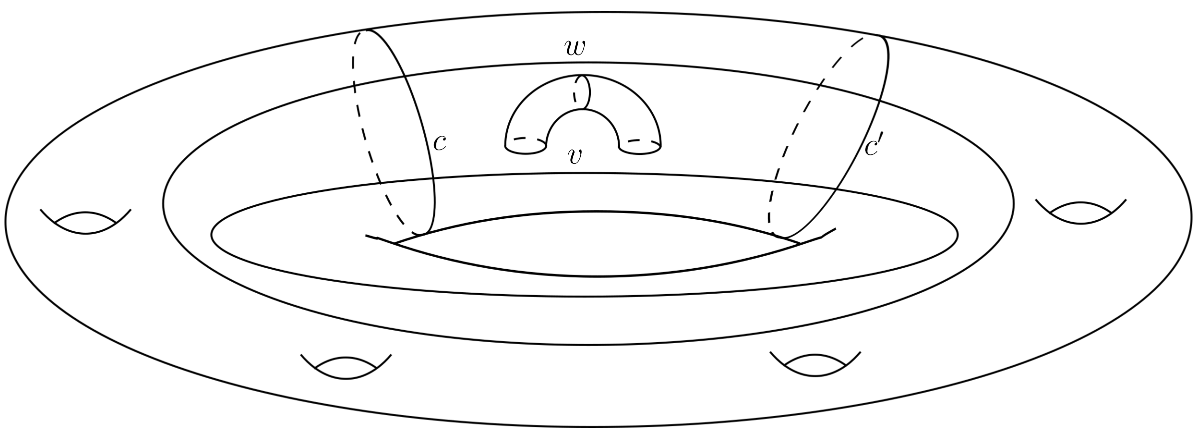

Figure 4.2: The bounding pair map $T_c T_{c'}^{-1}$ and edge $e = e_{v,w}$.

the complex X_g. Before stating and proving the lemma, we introduce some notation.

Cyclic decompositions induced by cells of X_g. Let $\sigma \subseteq X_g$ be a k–cell with a lift $\widehat{\sigma} \subset \mathcal{C}_{\vec{x}}(S_g)$, where $\widehat{\sigma}$ corresponds to a multicurve $a_0 \sqcup \ldots \sqcup a_k$. Let $S_0, \ldots, S_k$ be the connected components of $S_g \setminus \widehat{\sigma}$. Assume that $S_0, \ldots, S_k$ are indexed so that $S_i \cap S_j \neq \emptyset$ if and only if $|i - j| = 1$ or k. For each S_i, let $\mathcal{H}_i^\sigma$ denote the image in $H_1(S_g; \mathbb{Z})$ of the pushforward map $H_1(S_i;\mathbb{Z}) \to H_1(S_g;\mathbb{Z})$. Let $\mathcal{H}(\sigma)$ denote the set of free abelian groups $\mathcal{H}_0^\sigma, \ldots, \mathcal{H}_k^\sigma$. Associated to $\mathcal{H}(\sigma)$ is a directed cycle C. The vertices of C are the connected components S_i and there is a directed edge $S_i \to S_j$ if the following hold:

- $a_\ell \subseteq S_i \cap S_j$ for some $0 \leq \ell \leq k$, and
- a_ℓ is oriented so that S_i is on the left of a_ℓ.

This directed cycle induces a cyclic ordering on the set $\mathcal{H}(\sigma)$. For convenience, we will index the $\mathcal{H}_i$ so that the cyclic order is $\mathcal{H}_0 < \mathcal{H}_1 < \ldots < \mathcal{H}_k < \mathcal{H}_0$. We will refer to this cyclically ordered set $\mathcal{H}(\sigma)$ as the *cyclic decomposition of* $H_1(S_g;\mathbb{Z})$ *induced by* $\widehat{\sigma}$. We will say that the *genus of* σ is the cyclically ordered tuple $(g(\mathcal{H}_0), \ldots, g(\mathcal{H}_k))$, and we will denote this tuple by $g(\widehat{\sigma})$. We will prove the following result.

Lemma 4.2.1. *Let σ and σ' be two k–cells of $\mathcal{C}_{\vec{x}}(S_g)$. Then there is an $f \in \mathcal{I}_g$ such that $f\sigma = \sigma'$ if and only if σ and σ' induce the same decomposition of $H_1(S_g;\mathbb{Z})$.*

Proof. We prove each implication in turn.

The forward implication. Let $\sigma, \sigma' \subseteq \mathcal{C}_{\vec{x}}(S_g)$ be 2–cells in the same $\mathcal{I}_g$ orbit. Let $f \in \mathcal{I}_g$ be an element with $f\sigma = \sigma'$. Since $f \in \mathrm{Mod}(S_g)$, f takes each $\mathcal{H} \in \mathcal{H}(\sigma)$ to some $\mathcal{H}' \in \mathcal{H}(\sigma')$, and since $f \in \mathcal{I}_g$, we must have $\mathcal{H} = \mathcal{H}'$. Therefore $\mathcal{H}(\sigma) = \mathcal{H}(\sigma')$ as unordered sets, so it remains to show that f preserves the cyclic order on $\mathcal{H}(\sigma)$ and $\mathcal{H}(\sigma')$. Let $b \subseteq S_g$ be an oriented curve such that for any vertex $d \in \sigma$, b intersects d once with signed intersection -1. Then since $f(\sigma) = \sigma'$, the image $f(b)$ satisfies the property that for any $d' \in \sigma'$, $f(b)$ intersects d' once with signed intersection number -1. Now, for each $\mathcal{H}_i \in \mathcal{H}(\sigma)$, let c_i be a nonseparating simple closed curve disjoint from σ and not homologous to $[c]$ such that c_i intersects b once with signed intersection -1. An example with σ a 2–cell of the curves σ, b, and c_i can be found in Figure 4.3, if we denote the vertices of σ by d_0, d_1, d_2.

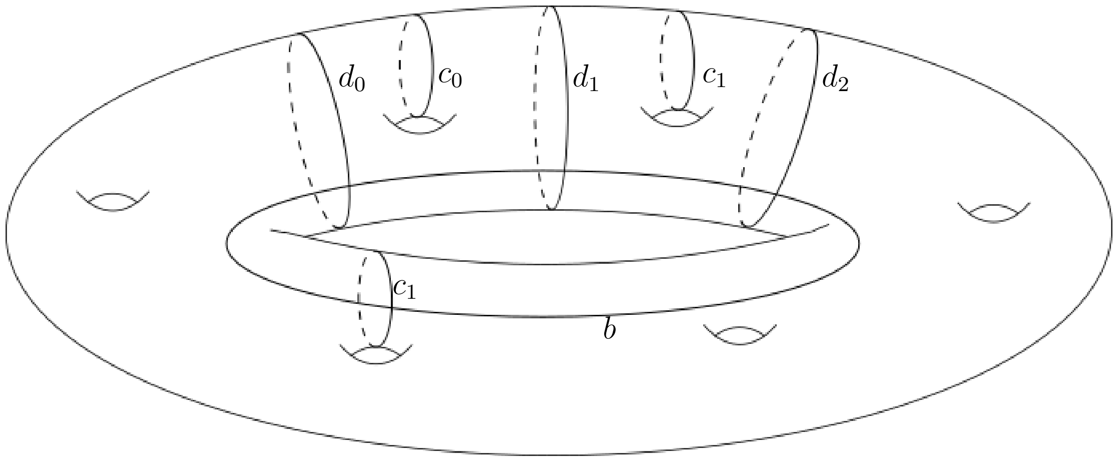

Figure 4.3: The curves a_0, a_1, a_2, b, and c_i

Let $c_i' = f(c_i)$. Then $[c_i'] = [c_i]$ since $f \in \mathcal{I}_g$, and $[c_i'] \in \mathcal{H}_i(\sigma')$. Then the cyclic order of the elements of $\mathcal{H}(\sigma)$ is the same as the cyclic order of the intersections of c_i with b, where the cyclic order on the c_i is induced by the orientation of b, so the lemma follows.

The backwards implication. Suppose now that σ and σ' induce the same decomposition of $H_1(S_g;\mathbb{Z})$. Reindex $\mathcal{H}(\sigma')$ so that $\mathcal{H}_i^\sigma = \mathcal{H}_i^{\sigma'}$ for all $0 \leq i \leq \dim(\sigma)$, and let S_i^σ, $S_i^{\sigma'}$ denote the connected components of $S_g \setminus \sigma$ and $S_g \setminus \sigma'$. Let $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g$ be a symplectic basis for $H_1(S_g;\mathbb{Z})$ such that:

- $\vec{x} = \alpha_1$ and
- there is a partition $\nu \vdash \{2, \ldots, g\}$ where ν has $k+1$ blocks such that, for each $P_i \in \nu$, the set $\{\alpha_j, \beta_j\}_{j \in P_i}$ is a symplectic basis for a maximal unimodular subgroup of $\mathcal{H}_i^\sigma \in \mathcal{H}(\sigma)$.

For each $\mathcal{H}_i^\sigma$ and $\mathcal{H}_i^{\sigma'}$, let $\{a_j, b_j\}_{j \in P_i}$ and $\{a_j', b_j'\}_{P_i}$ be representatives in S_i^σ and $S_i^{\sigma'}$ respectively for $\{\alpha_j, \beta_j\}_{j \in P_i}$ such that $|a_i \cap b_j| = |a_i' \cap b_j'| = |\langle \alpha_i, \beta_j \rangle|$ for any $1 \leq i, j \leq g$. By the change of coordinates principle, there is an $f \in \operatorname{Mod}(S_g)$ taking σ to σ' that also takes $a_j \to a_j', b_j \to b_j'$ for all $2 \leq j \leq g$. Then $f_*(\alpha_i) = \alpha_i$ and $f_*(\beta_i) = \beta_i$ for $2 \leq i \leq g$. Then f takes a vertex of σ to a vertex of σ', so $f_*(\alpha_1) = \alpha_1$. This implies that $\langle f_*(\beta_1), \alpha_i \rangle = \langle f_*(\beta_1), f_*(\beta_i) \rangle = 0$ for any $2 \leq i \leq g$, and $\langle \alpha_1, f_*(\beta_1) \rangle = 1$. Hence

$f(\beta_1) = \beta_1 + n\alpha_1$ for some $n \in \mathbb{Z}$, so f is not *a priori* in the Torelli group, since it may act nontrivially on β_1. We now find another element $h \in \mathrm{Mod}(S_g)$ such that $hf\sigma = \sigma'$ and $hf \in \mathcal{I}_g$.

Let $a' \in \sigma'$ be a curve in the k–cell σ'. Then $T_{a'}^{-n} f\sigma = T_{a'}^{-n}\sigma' = \sigma'$ since $f\sigma = \sigma'$ and $T_{a'}$ acts trivially on σ' by construction. Furthermore, for any $m \in \mathbb{Z}$, we have

$$T_{[a']}(\beta_1 + m\alpha_1) = \beta_1 + m\alpha_1 + \langle [a'], \beta_1 + m\alpha_1 \rangle [a'].$$

Since $[a'] = \alpha_1$, we have $T_{[a']}(\beta_1 + m\alpha_1) = \beta_1 + m\alpha_1 + \alpha_1 = \beta_1 + (m+1)\alpha_1$. Hence $T_{[a']}^{-n}(\beta_1 + m\alpha_1) = \beta_1 + (m-n)\alpha_1$, so we have we have $T_{[a']}^{-n} f_*(\beta_1) = T_{[a']}^{-n}(\beta_1 + n\alpha_1) = \beta_1 + n\alpha_1 - n\alpha_1 = \beta_1$. Then $T_{a'}$ acts trivially on the set $\{\alpha_1, \alpha_2, \beta_2, \ldots, \alpha_g, \beta_g\}$ since $[a'] = \alpha_1$ has trivial algebraic intersection every element in this set. Hence $(T_a^{-n} f)_*$ fixes the set $\{\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g\}$, so $T_a^{-n} f \in \mathcal{I}_g$, and so $T_a^{-n} f$ is an element of the Torelli group taking σ to σ', as desired. □

Given Lemma 4.2.1, if $\sigma \subseteq X_g$ is a cell, the cyclically ordered decomposition $\mathcal{H}(\widehat{\sigma})$ for $\widehat{\sigma}$ a lift of σ to $\mathcal{C}_{\vec{x}}(S_g)$ depends only on σ. We will use the notation $\mathcal{H}(\sigma)$ to refer to the decomposition of $H_1(S_g;\mathbb{Z})$ induced by $\widehat{\sigma}$, and refer to $\mathcal{H}(\sigma)$ as the *decomposition of* $H_1(S_g;\mathbb{Z})$ *induced by* σ.

4.2.2 Bestvina–Margalit tori

We will now give a formal definition of a Bestvina–Margalit torus and prove a collection of results about fundamental classes of Bestvina–Margalit tori.

Bestvina–Margalit tori. Let $\sigma \subseteq X_g$ be a 2–cell. There is a unique cell σ' called the *dual cell to* σ such that $\mathcal{H}(\sigma) = \mathcal{H}(\sigma')$ as unordered sets, but with $\mathcal{H}(\sigma) \neq \mathcal{H}(\sigma')$ as cyclically ordered sets. This σ' is unique by Lemma 4.2.1 since there are exactly two cyclic orders on the set of three elements. The Bestvina–Margalit torus BM_σ is the union $\sigma \cup \sigma'$. Assuming that σ is oriented, the fundamental class of BM_σ is denoted $[\mathrm{BM}_\sigma]$. We have the following lemma that allows us to add fundamental classes of Bestvina–Margalit tori.

Lemma 4.2.2. *Let x, y, z be three edges in X_g such that $\mathrm{BM}_{x,y}$, $\mathrm{BM}_{x,z}$ and $\mathrm{BM}_{y,z}$ are all Bestvina–Margalit tori. Let yz denote the third edge of a 2–cell σ with $y, z \subseteq \sigma$ oriented so that $[y][z] = [yz]$ in $\pi_1(X_g)$. A lift of a 3–cell $\tau \subseteq X_g$ containing these edges can be seen in Figure 4.4. Then in $H_2(X_g;\mathbb{Q})$, the relation $[\mathrm{BM}_{x,y}] + [\mathrm{BM}_{x,z}] = [\mathrm{BM}_{x,yz}]$ is satisfied.*

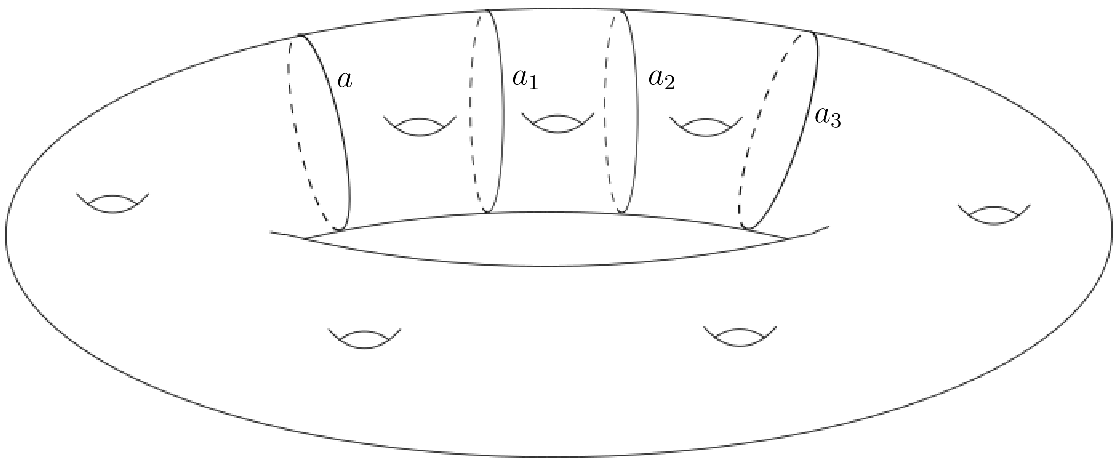

Figure 4.4: A 3–cell containing lifts of the edges x, y, z, yz as in Lemma 4.2.2. x lifts to (a, a_1), y lifts to (a_1, a_2), z lifts to (a_2, a_3) and yz lifts to (a_1, a_3).

Before proving the lemma, we will explain how to compute differentials in X_g. If $\tau \subseteq \mathcal{C}_{\vec{x}}(S_g)$ is a k–cell with vertices $\{c_0, \ldots, c_k\}$, then the differential $\partial\tau$ is given by

$$\partial\tau = \sum_{i=0}^{k} (-1)^i (c_0, \ldots, \widehat{c_i}, \ldots, c_k)$$

where $(c_0, \ldots, \widehat{c_i}, \ldots, c_k)$ denotes the $(k-1)$–cell with vertices $c_0, \ldots, c_{i-1}, c_{i+1}, \ldots, c_k$. This differential descends to X_g by definition. Now, if $\tau \subseteq X_g$ is a k–cell, Lemma 4.2.1 tells us that τ is determined by the decomposition

$$\mathcal{H}(\tau) = \{\mathcal{H}_0, \ldots, \mathcal{H}_k\}.$$

Then, if $\overline{\tau}$ is a lift of τ to $\mathcal{C}_{\vec{x}}(S_g)$, forgetting a vertex $c \subseteq \overline{\tau}$ corresponds to replacing

$\mathcal{H}_i, \mathcal{H}_{i+1}$ in $\mathcal{H}(\tau)$ with $\mathcal{H}_i + \mathcal{H}_{i+1}$. Therefore the differential $\partial\tau$ is given by

$$\partial\tau = \sum_{i=0}^{k}(-1)^i\sigma_i$$

where for $0 \leq i \leq k-1$, each σ_i satisfies

$$\mathcal{H}(\sigma_i) = \{\mathcal{H}_0, \ldots, \mathcal{H}_{i-1}, \mathcal{H}_i + \mathcal{H}_{i+1}, \mathcal{H}_{i+2}, \ldots, \mathcal{H}_k\}.$$

We have that $\mathcal{H}(\sigma_k) = \{\mathcal{H}_1, \ldots, \mathcal{H}_{k-1}, \mathcal{H}_k + \mathcal{H}_0\}$.

Proof of Lemma 4.2.2. We will find 3–dimensional subcomplex $\overline{L} \subseteq X_g$ such that the boundary realizes the relation in the lemma. Let τ be a 3–cell with $\mathcal{H}(\tau) = \{\mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3\}$ such that $\mathcal{H}_0 \in \mathcal{H}(x)$, $\mathcal{H}_1 \in \mathcal{H}(x)$, and $\mathcal{H}_2 \in \mathcal{H}(x)$. The space $\overline{L}$ is given as the union of the following 3–cells:

- $\tau_0 = \tau$,
- τ_1 that satisfies $\mathcal{H}(\tau_1) = \{\mathcal{H}_1, \mathcal{H}_0, \mathcal{H}_2, \mathcal{H}_3\}$, and
- τ_2 that satisfies $\mathcal{H}(\tau_2) = \{\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_0, \mathcal{H}_3\}$.

We can compute the differentials of each 3–cell above as follows, signed appropriately to yield the desired relation in the lemma. For notational convenience, we will conflate σ and $\mathcal{H}(\sigma)$ for $\sigma \subseteq X_g$.

- $\partial\tau_0 = \{\mathcal{H}_0 + \mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3\} - \{\mathcal{H}_0, \mathcal{H}_1 + \mathcal{H}_2, \mathcal{H}_3\} + \{\mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2 + \mathcal{H}_3\} - \{\mathcal{H}_0 + \mathcal{H}_3, \mathcal{H}_1, \mathcal{H}_2\}$,
- $-\partial\tau_1 = -\{\mathcal{H}_0 + \mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3\} + \{\mathcal{H}_1, \mathcal{H}_0 + \mathcal{H}_2, \mathcal{H}_3\} - \{\mathcal{H}_1, \mathcal{H}_0, \mathcal{H}_2 + \mathcal{H}_3\} + \{\mathcal{H}_1 + \mathcal{H}_3, \mathcal{H}_0, \mathcal{H}_2\}$, and
- $\partial\tau_2 = \{\mathcal{H}_1 + \mathcal{H}_2, \mathcal{H}_0, \mathcal{H}_3\} - \{\mathcal{H}_1, \mathcal{H}_2 + \mathcal{H}_0, \mathcal{H}_3\} + \{\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_0 + \mathcal{H}_3\} - \{\mathcal{H}_1 + \mathcal{H}_3, \mathcal{H}_2, \mathcal{H}_0\}$.

Now, we see that in the sum $\partial\tau_0 - \partial\tau_1 + \partial\tau_2$, the term $\{\mathcal{H}_0 + \mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3\}$ in $\partial\tau_0$ cancels with $-\{\mathcal{H}_0 + \mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3\}$ in $-\partial\tau_1$, the term $-\{\mathcal{H}_0 + \mathcal{H}_3, \mathcal{H}_1, \mathcal{H}_2\}$ in $\partial\tau_0$ cancels with $\{\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_0 + \mathcal{H}_3\}$ in $\partial\tau_2$, and $\{\mathcal{H}_1, \mathcal{H}_0 + \mathcal{H}_2, \mathcal{H}_3\}$ in $-\partial\tau_1$ cancels with $-\{\mathcal{H}_1, \mathcal{H}_2 + \mathcal{H}_0, \mathcal{H}_3\}$ in $\partial\tau_2$. This means that

$$\begin{aligned}\partial\tau_0 - \partial\tau_1 + \partial\tau_2 = &- \{\mathcal{H}_0, \mathcal{H}_1 + \mathcal{H}_2, \mathcal{H}_3\} + \{H_0, \mathcal{H}_1, \mathcal{H}_2 + \mathcal{H}_3\} \\ &- \{\mathcal{H}_1, \mathcal{H}_0, \mathcal{H}_2 + \mathcal{H}_3\} + \{\mathcal{H}_1 + \mathcal{H}_3, \mathcal{H}_0, \mathcal{H}_2\} \\ &+ \{\mathcal{H}_1 + \mathcal{H}_2, \mathcal{H}_0, \mathcal{H}_3\} - \{\mathcal{H}_1 + \mathcal{H}_3, \mathcal{H}_2, \mathcal{H}_0\}.\end{aligned}$$

Now, by rearranging terms, we have

$$\begin{aligned}\partial\tau_0 - \partial\tau_1 + \partial\tau_2 = &- \{\mathcal{H}_0, \mathcal{H}_1 + \mathcal{H}_2, \mathcal{H}_3\} + \{\mathcal{H}_1 + \mathcal{H}_2, \mathcal{H}_0, \mathcal{H}_3\} \\ &+ \{H_0, \mathcal{H}_1, \mathcal{H}_2 + \mathcal{H}_3\} - \{\mathcal{H}_1, \mathcal{H}_0, \mathcal{H}_2 + \mathcal{H}_3\} \\ &+ \{\mathcal{H}_1 + \mathcal{H}_3, \mathcal{H}_0, \mathcal{H}_2\} - \{\mathcal{H}_1 + \mathcal{H}_3, \mathcal{H}_2, \mathcal{H}_0\}.\end{aligned}$$

Then each pair of two cells term on the right side of the above equation is a Bestvina–Margalit torus, so we have

$$\partial\tau_0 - \partial\tau_1 + \partial\tau_2 = -[\mathrm{BM}_{x,yz}] + [\mathrm{BM}_{x,y}] + [\mathrm{BM}_{x,z}]$$

and therefore the relationship given in the lemma holds. □

We now prove Lemma 4.2.3, which says when two Bestvina–Margalit tori are equal in X_g.

Lemma 4.2.3. *Let* BM_σ *and* BM_τ *be two Bestvina–Margalit tori. Then* $\mathrm{BM}_\sigma = \mathrm{BM}_\tau$ *if and only if* $\mathcal{H}(\sigma) = \mathcal{H}(\tau)$ *as unordered sets.*

Proof. We prove the two directions in turn.

Backwards direction. If $\mathrm{BM}_\sigma \neq \mathrm{BM}_\tau$, then $\sigma \neq \tau$ since every cell σ has a unique dual 2–cell σ' such that $\sigma \cup \sigma'$ is a Bestvina–Margalit torus. This implies that $\mathcal{H}(\sigma)$ is not equal to $\mathcal{H}(\tau)$ as decompositions of $\vec{x}^\perp$, which implies in particular that $\mathcal{H}(\sigma) \neq \mathcal{H}(\tau)$ as unordered sets.

Forwards direction. Suppose that $\mathrm{BM}_\sigma = \mathrm{BM}_\tau$. By Lemma 4.2.1, σ and τ are identified under the action of $\mathcal{I}_g$ if and only if $\mathcal{H}(\sigma) = \mathcal{H}(\tau)$ as decompositions of $\vec{x}^\perp$. But then if $\mathcal{H}(\sigma) \neq \mathcal{H}(\tau)$, we must have $\mathcal{H}(\sigma) = \mathcal{H}(\tau')$, where τ' is the 2–cell dual to τ, since there are only two cyclic orders on a set of three elements. □

If BM_σ is a Bestvina–Margalit torus, we will use the notation $\mathcal{H}(\mathrm{BM}_\sigma)$ to denote the decomposition $\mathcal{H}(\sigma)$ without the cyclic order, and we will refer to this decomposition as the *decomposition of* $H_1(S_g;\mathbb{Z})$ *induced by* BM_σ. If σ is a 2–cell with edges x and y, we will denote the splitting $\mathcal{H}(\mathrm{BM}_\sigma)$ by $\mathcal{H}(\mathrm{BM}_{x,y})$. We describe how $\mathcal{H}$ interacts with the addition of tori in the following lemma. The idea is that if we have a 3–cell τ as in Figure 4.4, then we can describe the splitting induced by one torus corresponding to a 2–cell $\sigma \subseteq \tau$ in terms of the splitting induced by tori corresponding to other 2–cells contained in τ.

Lemma 4.2.4. *Let* $g \geq 5$ *and let* $a \subseteq S_g$ *be a nonseparating simple closed curve. Let* $\vec{x} = [a]$ *and* $X_g = \mathcal{C}_{\vec{x}}(S_g)/\mathcal{I}_g$. *Let* $x, y, z \subseteq X_g$ *be three edges such that* $\mathrm{BM}_{x,y}$, $\mathrm{BM}_{x,z}$ *and* $\mathrm{BM}_{y,z}$ *are all Bestvina–Margalit tori. Suppose that we have*

$$\mathcal{H}(\mathrm{BM}_{x,y}) = \{\mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2\} \text{ and } \mathcal{H}(\mathrm{BM}_{x,z}) = \{\mathcal{H}_0, \mathcal{H}_1', \mathcal{H}_2'\}$$

with $\mathcal{H}_1 \cap \mathcal{H}_1' = \mathbb{Z}\vec{x}$. *Suppose additionally that we have* $\mathcal{H}_0 \in \mathcal{H}(x)$, $\mathcal{H}_1 \in \mathcal{H}(y)$ *and* $\mathcal{H}_1' \in \mathcal{H}(z)$. *Let* yz *denote the third edge of a 2–cell* σ *with* $y, z \subseteq \sigma$ *with* yz *oriented so that* $y * z = yz$ *in the group generated by edges of* X_g. *Then we have*

$$\mathcal{H}(\mathrm{BM}_{x,yz}) = \{\mathcal{H}_0, \mathcal{H}_1 + \mathcal{H}_1', \mathcal{H}_2 \cap \mathcal{H}_2'\}.$$

Proof. We first want to compute the decomposition $\mathcal{H}(yz)$. The 2–cell σ containing y, z and yz has $\mathcal{H}_1, \mathcal{H}_1' \in \mathcal{H}(\sigma)$. This implies that the third element of $\mathcal{H}(\sigma)$ is $\mathcal{H}_1^\perp \cap (\mathcal{H}_1')^\perp$. Hence

$$\mathcal{H}(yz) = \{\mathcal{H}_1^\perp \cap (\mathcal{H}_1')^\perp, (\mathcal{H}_1^\perp \cap (\mathcal{H}_1')^\perp)^\perp\} = \{\mathcal{H}_1^\perp \cap (\mathcal{H}_1')^\perp, \mathcal{H}_1 + \mathcal{H}_1'\}$$

and in particular, we have $\mathcal{H}_1 + \mathcal{H}_1' \in \mathcal{H}(yz)$. Now, the splitting $\mathcal{H}(\mathrm{BM}_{x,yz})$ contains $\mathcal{H}_0$ and $\mathcal{H}_1 + \mathcal{H}_1'$, so the third element of $\mathcal{H}(\mathrm{BM}_{x,yz})$ is given by

$$\mathcal{H}_0^\perp \cap (\mathcal{H}_1 + \mathcal{H}_1')^\perp = (\mathcal{H}_0 + \mathcal{H}_1)^\perp \cap (\mathcal{H}_0 + \mathcal{H}_1')^\perp = \mathcal{H}_2 \cap \mathcal{H}_2',$$

so the lemma holds. ☐

Before moving forward, we will briefly discuss a notation for cells in X_g, as well as the action of subgroups of $\mathrm{Sp}(2g, \mathbb{Z})$ on X_g.

Graphical representations for Bestvina–Margalit tori. Let $x, y, z, yz \subseteq X_g$ be a set of edges as in Lemma 4.2.4. Let $\widehat{\mathcal{H}} = \{\mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_1', \mathcal{H}_2 \cap \mathcal{H}_2'\}$ be an unordered set. Then $\widehat{\mathcal{H}}$ corresponds to the labeled graph as in Figure 4.5.

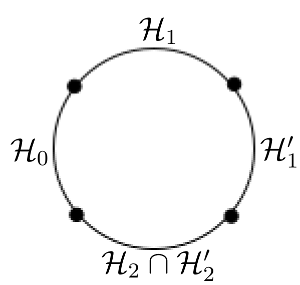

Figure 4.5: The graphical representation of $\widehat{\mathcal{H}}$

The advantage of this notation is that the decompositions corresponding to $\mathcal{H}(x, y)$, $\mathcal{H}(x, z)$ and $\mathcal{H}(x, yz)$ can be read off from the graph. Each of these Bestvina–Margalit tori corresponds to the decomposition given by taking the span of the union of two subgroups not equal to $\mathcal{H}_0$. We will use this notation in Section 5.2 to keep track of certain decompositions of $\vec{x}^\perp \subseteq H_1(S_g;\mathbb{Z})$.

The action of $\mathrm{Sp}(\vec{x}^\perp, \mathbb{Z})$ *on* X_g. Throughout the rest of this thesis, we will consider the action of $\mathrm{Sp}(\vec{x}^\perp, \mathbb{Z})$ on X_g. The subgroup $\Gamma_{\vec{x}} \subseteq \mathrm{Mod}(S_g)$ consisting of all elements that act trivially on the class $\vec{x} \in H_1(S_g;\mathbb{Z})$ acts on $\mathcal{C}_{\vec{x}}(S_g)$. We have $\mathcal{I}_g \subseteq \Gamma_a$ by definition, and so the quotient $\Gamma_a/\mathcal{I}_g = \mathrm{Sp}(\vec{x}^\perp, \mathbb{Z})$ acts on X_g.

CHAPTER 5

THE PROOF THAT $\mathbf{H_2(X_g;\mathbb{Q})}$ IS FINITE DIMENSIONAL

5.1 Abelian cycles in $\mathbf{H_2(\mathcal{I}_g;\mathbb{Z})}$

The goal of this section is to prove Proposition 5.1.1, which says that certain subspaces of $H_2(\mathcal{I}_g;\mathbb{Q})$ generated by abelian cycles of bounding pair maps can be generated by abelian cycles supported on finitely many subsurfaces.

Abelian cycles. Let G be a group and $c_1,\ldots,c_k \subseteq G$ be a collection of pairwise commuting elements of infinite order. Let $A = \langle c_1,\ldots,c_k\rangle \subseteq G$. We have $H^*(A;\mathbb{Z}) \cong \bigwedge^* A$ as graded rings. The *abelian cycle* $\mathrm{ab}(c_1,\ldots,c_k) \in H_k(G;\mathbb{Q})$ is the image of the class $[c_1]\wedge\ldots\wedge[c_k] \in H_k(A;\mathbb{Q})$ under the pushforward map $H_k(A;\mathbb{Q}) \to H_k(G;\mathbb{Q})$. If $k=2$, we will denote the abelian cycle $\mathrm{ab}(c_1,c_2)$ by $[c_1,c_2]$. Let $H_k^{\mathrm{ab}}(G;\mathbb{Q})$ denote the subspace of $H_k(G;\mathbb{Q})$ generated by abelian cycles.

Torelli group of a subsurface. Let S be a compact surface and $\iota : S \hookrightarrow S_g$ an embedding. We will denote the genus of S by $g(S)$. We say that ι is *clean* if no connected component of $S_g \setminus \iota(S)$ is a disk. Let $\mathcal{I}(S,S_g)$ denote the intersection $\iota_*(\mathrm{Mod}(S))\cap\mathcal{I}_g$, where the pushforward map ι_* is given by extending $\varphi\in\mathrm{Mod}(S)$ by the identity over $S_g\setminus\iota(S)$. Let $\mathcal{I}(\iota)$ denote the inclusion map $\mathcal{I}(S,S_g)\to\mathcal{I}_g$.

Bounding pair maps. We will say that two curves $c,c'\subseteq S_g$ are a *bounding pair* if c and c' are disjoinat and homologous. We say that $f\in\mathcal{I}_g$ is a *bounding pair map* if $f = T_cT_{c'}^{-1}$ for c and c' a bounding pair. We will denote the bounding pair map $T_cT_{c'}^{-1}$ by $T_{c,c'}$. Let $H_2^{\mathrm{ab,bp}}(\mathcal{I}(S,S_g);\mathbb{Q})$ denote the subspace of $H_2(\mathcal{I}(S,S_g);\mathbb{Q})$ generated by abelian cycles $[T_{c,c'},T_{d,d'}]$ such that the bounding pairs $c\cup c'$ and $d\cup d'$ are disjoint.

The main goal of the section is to prove the following result.

Proposition 5.1.1. *Let $g \geq 33$ and let $M \subseteq S_g$ be a nonseparating multicurve with $|M| = 9$. Let $\mathcal{V} = \{[c] : c \in M\}$. The natural map*

$$\bigoplus_{v \in \mathcal{V}} H_2^{\mathrm{ab,bp}}(\mathcal{I}_g; \mathbb{Q})^{T_v} \to H_2^{\mathrm{ab,bp}}(\mathcal{I}_g; \mathbb{Q})$$

is surjective.

The idea of Proposition 5.1.1 is that if there is a large enough nonseparating multicurve $M \subseteq S_g$, then any abelian cycle of bounding pairs is a linear combination of abelian cycles that are each algebraically disjoint from at least one of the curves.

Outline of the proof of Proposition 5.1.1. If G is a group acting on a vector space V and $\mathcal{F} \subseteq G$ is a subset, we let $V^{\mathcal{F}}$ denote the subspace of V fixed by all elements of $\mathcal{F}$. If $\mathcal{W} \subseteq H_1(S_g; \mathbb{Z})$, let $T_{\mathcal{W}}$ denote the set of transvections along elements of $\mathcal{W}$. If $F \in \mathcal{I}_g$, let $[F]$ denote the corresponding class in $H_1(\mathcal{I}_g; \mathbb{Q})$. Let $[T_{d,d'}, T_{e,e'}] \in H_2^{\mathrm{ab,bp}}(\mathcal{I}_g; \mathbb{Q})$ be an abelian cycle. The proof of the proposition proceeds in two steps.

1. There is a relation
$$[T_{d,d'}, T_{e,e'}] = \sum_{i=1}^{k} \lambda_i [T_{d,d'}, T_{f_i,f_i'}]$$
for $\lambda_i \in \mathbb{Q}$, with $[T_{f_i,f_i'}] \in H_1(\mathcal{I}_g; \mathbb{Q})^{T_{\mathcal{V}'}}$ for some $\mathcal{V}' \subseteq \mathcal{V}$ and $|\mathcal{V}'| = 4$ and for all $1 \leq i \leq k$.

2. If $[T_{d,d'}, T_{f_i,f_i'}]$ is an abelian cycle with $T_{f_i,f_i'} \in H_1(\mathcal{I}_g; \mathbb{Q})^{T_{\mathcal{V}'}}$ for some $\mathcal{V}' \subseteq V$ and $|\mathcal{V}'| = 4$, then there is a relation
$$[T_{d,d'}, T_{f_i,f_i'}] = \sum_{j=1}^{m} \lambda_j [T_{h_j,h_j'}, T_{f_i,f_i'}]$$
for $\lambda_j \in \mathbb{Q}$, with each $[T_{h_j,h_j'}] \in H_1(\mathcal{I}_g; \mathbb{Q})^{T_v}$ for some $v \in \mathcal{V}'$.

Step (1) is the content of Lemma 5.1.11 and Step (2) is the content of Lemma 5.1.9.

Outline of the section. We will begin by describing the Johnson homomorphism. We will then prove Lemma 5.1.5, which describes the vector space $H_1(\mathcal{I}(S, S_g); \mathbb{Q})$ for $S \hookrightarrow S_g$ a clean embedding with $g(S) \geq 3$. We then prove Lemma 5.1.8 which is a result about generating sets of exterior powers of quasi–unimodular lattices, and then use Lemma 5.1.8 to prove Lemma 5.1.9. We also use Lemma 5.1.8 to prove Lemma 5.1.10, which is another result about generating sets of exterior products of vector spaces. We use Lemma 5.1.10 to prove Lemma 5.1.11. We then combine these lemmas to prove Proposition 5.1.1.

The Johnson homomorphism. Let $a_1, b_1, \ldots, a_g, b_g$ be a symplectic basis for $H_1(S_g; \mathbb{Z})$ and let $\omega = a_1 \wedge b_1 + \ldots + a_g \wedge b_g \in \wedge^2 H_1(S_g; \mathbb{Z})$. There is an inclusion $H_1(S_g; \mathbb{Z}) \hookrightarrow \wedge^3 H_1(S_g; \mathbb{Z})$ given by $[c] \to [c] \wedge \omega$. Johnson [19] constructed a map

$$\tau_g : \mathcal{I}_g \to \wedge^3 H_1(S_g; \mathbb{Z})/H_1(S_g; \mathbb{Z})$$

and showed that the pushforward map

$$(\tau_g)_* : H_1(\mathcal{I}_g; \mathbb{Q}) \to \wedge^3 H_1(S_g; \mathbb{Q})/H_1(S_g; \mathbb{Q})$$

is an isomorphism [22]. We will make use of Lemma 5.1.2, which describes $\tau_g(T_{c,c'})$ for $T_{c,c'}$ a bounding pair map. The lemma is due to Johnson [19, Lemma 4B].

Lemma 5.1.2. *Let $g \geq 3$. Let $T_{c,c'}$ be a bounding pair map and let S be the connected component of $S_g \setminus (c \cup c')$ such that S is on the left side of c. Let $a_1, b_1, \ldots, a_h, b_h$ be a symplectic basis for a maximal nondegenerate subspace of $H_1(S; \mathbb{Z})$. We have*

$$\tau_g(T_{c,c'}) = [c] \wedge \left(\sum_{i=1}^{h} a_i \wedge b_i \right).$$

We use Lemma 5.1.2 to prove Lemma 5.1.5, which describes $H_1(\mathcal{I}(S; S_g); \mathbb{Z})$ for clean embeddings $S \hookrightarrow S_g$. We first prove an auxiliary lemma, which is essentially a corollary of a result of Putman [34, Theorem 1.3].

Lemma 5.1.3. *Let $g \geq 4$ and let $\iota : S \hookrightarrow S_g$ be a clean embedding such that $g(S) \geq 3$. Then $\mathcal{I}(S, S_g)$ is generated by bounding pair maps.*

Proof. A result of Putman [34, Theorem 1.3] tells us that $\mathcal{I}(S, S_g)$ is generated by bounding pair maps and Dehn twists along separating curves. Hence it suffices to show that if $\delta \subseteq \iota(S)$ is a separating curve in S_g, then T_δ is a product of bounding pair maps. Since $g(S) \geq 3$, there is an embedded copy of $S_1^2 \subseteq S$ such that the image of one boundary component of S_1^2 is the curve δ. Then by embedding a copy of $S_0^4 \hookrightarrow S_1^2$, the lantern relation says that T_δ is a product of three bounding pair maps [12, Section 5.1], so the lemma follows. □

We also need one more fact about quasi–unimodular lattices.

Lemma 5.1.4. *Let L be a quasi–unimodular lattice such that $g(L) \geq 1$. The free abelian group $\wedge^2 L$ is spanned by elements of the form $\gamma \wedge \delta$ with γ, δ primitive and $\langle \gamma, \delta \rangle = 1$.*

Proof. Let $\mathcal{S} \subseteq \wedge^2 L$ be the set of all elements $\gamma \wedge \delta$ as in the statement of the lemma. Let

$$\mathcal{B} = \{\gamma_1, \delta_1, \gamma_2, \delta_2, \ldots, \gamma_{g(L)}, \delta_{g(L)}, \gamma_{g(L)+1}, \gamma_{g(L)+2}, \ldots, \gamma_\ell\}$$

be a partial symplectic basis for L. The group $\wedge^2 L$ is generated by all elements of the form $r \wedge s$ for $r, s \in \mathcal{B}$, so it suffices to show that any element $r \wedge s$ with $r, s \in \mathcal{B}$ is a $\mathbb{Z}$–linear combination of elements as in the statement of the lemma.

We begin by proving this in the case $r = \gamma_i, s = \delta_j$. We have $\gamma_i \wedge \delta_j = (\gamma_i + \gamma_j) \wedge \delta_j - (\gamma_j \wedge \delta_j)$. Each of these latter elements is a wedge of primitive elements with algebraic intersection number one, so $r \wedge s$ is in the subgroup of $\wedge^2 L$ generated by $\mathcal{S}$.

Now, suppose that $r \in \mathcal{B}, s = \delta_i$ with $r \neq \gamma_i$. We have $r \wedge s = (r + \gamma_i) \wedge \delta_i - \gamma_i \wedge \delta_i$, and each of these latter elements are in $\mathcal{S}$, since the only element of $\mathcal{B}$ that δ_i intersects nontrivially is γ_i.

Finally, suppose that $r = \gamma_i, s = \gamma_j$. If i or j is less than $g(L)$, then without loss of generality we have $j \leq g(L)$. Then we have $r \wedge s = \gamma_i + \delta_j \wedge \gamma_j - \delta_j \wedge \gamma_j$, and these latter elements are all in $\mathcal{S}$. Otherwise, $i, j > g(L)$. In this case, we have

$$r \wedge s = (\gamma_i + \delta_{g(L)}) \wedge (\gamma_j + \gamma_{g(L)}) - \delta_{g(L)} \wedge \gamma_j - \delta_{g(L)} \wedge \gamma_j - \delta_{g(L)} - \gamma_{g(L)}.$$

The first and last elements are in $\mathcal{S}$, and the middle two elements are $\mathbb{Z}$–linear combinations of elements in $\mathcal{S}$ by the previous cases, so $r \wedge s \in \mathrm{Span}_{\mathbb{Z}}(\mathcal{S})$ and thus the lemma holds. □

We now describe $H(\mathcal{I}(S, S_g); \mathbb{Q})$ for certain clean embeddings $\iota : S \hookrightarrow S_g$.

Lemma 5.1.5. *Let $S = S_g^b$ be a compact surface with $b \in \{1, 2\}$. Let $\iota : S \hookrightarrow S_g$ be a clean embedding such that $S_g \setminus S$ is connected. Suppose additionally that $g(S_g \setminus S) \geq 1$. Then there is a surjection*

$$f : H_1(\mathcal{I}_g(S, S_g); \mathbb{Q}) \twoheadrightarrow \wedge^3 H_1(S; \mathbb{Q})$$

such that the following diagram commutes:

$$\begin{array}{ccc} H_1(\mathcal{I}(S, S_g); \mathbb{Q}) & \xrightarrow{f} & \wedge^3 H_1(S; \mathbb{Q}) \\ \downarrow{\scriptstyle \mathcal{I}(\iota)_*} & & \downarrow{\scriptstyle \wedge^3 \iota_*} \\ H_1(\mathcal{I}_g; \mathbb{Q}) & \xrightarrow{\tau_g \otimes \mathbb{Q}} & \wedge^3 H_1(S_g; \mathbb{Q}) / H_1(S_g; \mathbb{Q}). \end{array}$$

Furthermore, suppose that $g(S) \geq 3$. Then f is an isomorphism, and the map $\wedge^3 \iota_* \circ f$ is injective.

Proof. We begin with the first part of the lemma. We show that the image of the composition the composition

$$H_1(\mathcal{I}_g(S, S_g); \mathbb{Q}) \xrightarrow{\mathcal{I}(\iota)_*} H_1(\mathcal{I}_g; \mathbb{Q}) \xrightarrow{\tau_g \otimes \mathbb{Q}} \wedge^3 H_1(S_g; \mathbb{Q}) / H_1(S_g; \mathbb{Q})$$

is $\mathrm{im}(\wedge^3 \iota_*)$, and that $\wedge^3 \iota_*$ is an injection. If these two statements hold, then f is given by

$$f = (\wedge^3 \iota_*)^{-1}|_{\mathrm{im}(\tau_g \otimes \mathbb{Q} \circ \mathcal{I}(\iota)_*)} \circ \tau_g \otimes \mathbb{Q} \circ \mathcal{I}(\iota)_*.$$

We prove each containment in turn, and then prove injectivity.

Showing that $\mathrm{im}(\tau_g \otimes \mathbb{Q} \circ \mathcal{I}(\iota)_*) \subseteq \mathrm{im}(\wedge^3 H_1(S;\mathbb{Q}) \to \wedge^3 H_1(S_g;\mathbb{Q})/H_1(S_g;\mathbb{Q}))$. The group $\mathcal{I}_g(S, S_g)$ is generated by bounding pair maps by Lemma 5.1.3, so it suffices to show that if $T_{c,c'} \in \mathcal{I}_g(S, S_g)$ is a bounding pair map, then $\tau_g \otimes \mathbb{Q} \circ \mathcal{I}(\iota)_*(T_{c,c'}) \in \mathrm{im}(\wedge^3 H_1(S;\mathbb{Q}) \to \wedge^3 H_1(S_g;\mathbb{Q})/H_1(S_g;\mathbb{Q}))$. Since the bounding pair $c \cup c'$ is contained in S, we have $[c] \in \wedge^3 H_1(S;\mathbb{Q})$. Then there must be a connected component S' of $S_g \searrow (c \cup c')$ such that $S' \subseteq S$. Let $\alpha_1, \beta_1, \ldots, \alpha_h, \beta_h$ be a symplectic basis for a maximal unimodular subgroup of $H_1(S';\mathbb{Z})$. Lemma 5.1.2 says that $\tau_g \circ \mathcal{I}(\iota)_*([T_{c,c'}]) = [c] \wedge (\alpha_1 \wedge \beta_1 + \ldots + \alpha_h \wedge \beta_h) \in \mathrm{im}(\wedge^3 H_1(S;\mathbb{Q}) \to \wedge^3 H_1(S_g;\mathbb{Q})/H_1(S_g;\mathbb{Q}))$, so the $\subseteq$ containment holds.

Showing that $\mathrm{im}(\tau_g \circ \mathcal{I}(\iota)_*) \supseteq \mathrm{im}(\wedge^3 H_1(S;\mathbb{Q}) \to \wedge^3 H_1(S_g;\mathbb{Q})/H_1(S_g;\mathbb{Q}))$. Observe that if $v_1, v_2, v_3 \in H_1(S;\mathbb{Z})$ are primitive classes with $\langle v_1, v_2 \rangle = 1$ and $\langle v_1, v_3 \rangle = \langle v_2, v_3 \rangle = 0$, then Lemma 5.1.2 says that if c, c' is a bounding pair with $[c] = v_3$ and v_1, v_2 a symplectic basis for a maximal nondegenerate subgroup of the first homology of a connected component of $S \searrow (c \cup c')$, then $\tau_g(T_{c,c'}) = v_1 \wedge v_2 \wedge v_3 \in \mathrm{im}(\tau_g \circ \mathcal{I}(\iota)_*)$. Since the set of such triples of elements spans $\wedge^3 H_1(S;\mathbb{Z})$ by Lemma 5.1.4, we have $\mathrm{im}(\tau_g \circ \mathcal{I}(\iota)_*) \supseteq \mathrm{im}(\wedge^3 H_1(S;\mathbb{Q}) \to \wedge^3 H_1(S_g;\mathbb{Q})/H_1(S_g;\mathbb{Q}))$.

Showing that $\wedge^3 H_1(S;\mathbb{Q})$ *injects into* $\wedge^3 H_1(S_g;\mathbb{Q})/H_1(S_g;\mathbb{Q})$. We first show that the exterior power of pushforwards $\wedge^3 H_1(S;\mathbb{Q}) \to \wedge^3 H_1(S_g;\mathbb{Q})$ is injective. Since $S_g \searrow S$ is connected, the pushforward $H_1(S;\mathbb{Q}) \to H_1(S_g;\mathbb{Q})$ is injective, so $\wedge^3 H_1(S;\mathbb{Q}) \to \wedge^3 H_1(S_g;\mathbb{Q})$ is injective as well. Hence it suffices to show that

$$\mathrm{im}(\wedge^3 H_1(S;\mathbb{Q}) \to \wedge^3 H_1(S_g;\mathbb{Q})) \cap H_1(S_g;\mathbb{Q}) \wedge \omega = 0,$$

where $\omega \in \wedge^2 H_1(S_g;\mathbb{Q})$ is the characteristic element. Suppose otherwise, so there is some nonzero $\alpha \wedge \omega \in \mathrm{im}(\wedge^3 H_1(S;\mathbb{Q}) \to \wedge^3 H_1(S_g;\mathbb{Q}))$. Since ω does not depend on the choice of symplectic basis, we can choose a symplectic basis $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g$ with

$\alpha \wedge \omega = m \cdot \alpha_1 \wedge (\alpha_1 \wedge \beta_1 + \alpha_2 \wedge \beta_2 + \ldots + \alpha_g \wedge \beta_g) = m \cdot \alpha_1 \wedge (\alpha_2 \wedge \beta_2 + \ldots + \alpha_g \wedge \beta_g)$ for m some positive integer. But this implies that there are $(2g-1)$ linearly independent elements $\alpha_1, \alpha_2, \beta_2, \ldots, \alpha_g, \beta_g \in \mathrm{im}(H_1(S;\mathbb{Q}) \to H_1(S_g;\mathbb{Q}))$. We have assumed that $g(S_g \setminus S) \geq 1$, so $\dim(H_1(S;\mathbb{Q})) \leq 2g-2$, which is a contradiction.

We now prove the second part of the lemma, where we assume that $g(S) \geq 3$. In this case, since $g(S) \geq 3$ and $\iota : S \hookrightarrow S_g$ is clean, a theorem of Putman [37, Theorem B] states that the pushforward map $\mathcal{I}(\iota)_*$ is an injection. Therefore the composition $(\wedge^3 \iota_*)^{-1}|_{\mathrm{im}(\tau_g \otimes \mathbb{Q} \circ \mathcal{I}(\iota)_*} \circ \tau_g \otimes \mathbb{Q} \circ \mathcal{I}(\iota)_*$ is an injection. By definition this is f, so f is injective as well, and is therefore an isomorphism. $\square$

We will prove Lemma 5.1.7, which is a result that describes how stabilizers of transvections interact with the abelianization of certain $\mathcal{I}(S, S_g)$. We first prove the following result.

Lemma 5.1.6. *Let $g \geq 3$ and let $w \in H_1(S_g;\mathbb{Z})$ be a nonzero primitive element. The fixed space $H_1(\mathcal{I}_g;\mathbb{Q})^{T_w}$ is sent to $\mathrm{im}(\wedge^3[w]^\perp \to \wedge^3 H_1(S_g;\mathbb{Q})/H_1(S_g;\mathbb{Q}))$ under the Johnson homomorphism.*

Proof. Since the Johnson homomorphism is $\mathrm{Sp}(2g,\mathbb{Z})$–equivariant, it suffices to show that $(\wedge^3 H_1(S_g;\mathbb{Q})/H_1(S_g;\mathbb{Q}))^{T_w} = \mathrm{im}(\wedge^3[w]^\perp \to \wedge^3 H_1(S_g;\mathbb{Q})/H_1(S_g;\mathbb{Q}))$. Since T_w acts trivially on $[w]^\perp$ by the definition of transvection, the transvection T_w acts trivially on $\mathrm{im}(\wedge^3[w]^\perp \to \wedge^3 H_1(S_g;\mathbb{Q})/H_1(S_g;\mathbb{Q}))$, and thus

$$\tau_g(H_1(\mathcal{I}_g;\mathbb{Q})^{T_w}) \supseteq \mathrm{im}(\wedge^3[w]^\perp \to \wedge^3 H_1(S_g;\mathbb{Q})/H_1(S_g;\mathbb{Q})).$$

It therefore remains to prove the other containment.

Let $\mathcal{B} = \{\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g\} \in H_1(S_g;\mathbb{Z})$ be a symplectic basis so that $\alpha_1 = w$. The vector space $\wedge^3 H_1(S_g;\mathbb{Q})$ has a basis consisting of all pure wedge products of elements in $\mathcal{B}$. Then after modding out by $H_1(S_g;\mathbb{Q}) \to \wedge^3 H_1(S_g;\mathbb{Q})$, we see that any pure tensor of the form $\alpha_1 \wedge \beta_1 \wedge \gamma$ is a linear combination

$$\alpha_2 \wedge \beta_2 \wedge \gamma + \ldots + \alpha_g \wedge \beta_g \wedge \gamma.$$

Hence $\wedge^3 H_1(S_g;\mathbb{Q})/H_1(S_g;\mathbb{Q})$ has a basis consisting of all pure tensors of elements in $\mathcal{B}$ except those containing $\alpha_1 \wedge \beta_1$. This set spans $\wedge^3 H_1(S_g;\mathbb{Q})/H_1(S_g;\mathbb{Q})$, and is therefore a basis by counting dimensions. Now, T_w acts trivially on any such wedge products except for those containing β_1. For such a basis element $\beta_1 \wedge \gamma \wedge \delta$ for $\gamma, \delta \in \mathcal{B} \setminus \{\alpha_1\}$, we have $T_w(\beta_1 \wedge \gamma \wedge \delta) = \beta_1 \wedge \gamma \wedge \delta + \alpha_1 \wedge \gamma \wedge \delta$. Now, if we have a $\mathbb{Q}$–linear combination of such elements

$$\sum_{i=1}^{n} \lambda_i \beta_1 \wedge \gamma_i \wedge \delta_i,$$

then we have

$$T_w\left(\sum_{i=1}^{n} \lambda_i \beta_1 \wedge \gamma_i \wedge \delta_i\right) = \sum_{i=1}^{n} \lambda_i(\beta_1 + \alpha_1) \wedge \gamma_i \wedge \delta_i = (\beta_1 + \alpha_1) \wedge \sum_{i=1}^{n} \lambda_i \gamma_i \wedge \delta_i.$$

Then to have $\sum_{i=1}^{n} \lambda_i \beta_1 \wedge \gamma_i \wedge \delta_i$ fixed by T_w, we must have

$$\sum_{i=1}^{n} \lambda_i \beta_1 \wedge \gamma_i \wedge \delta_i = (\beta_1 + \alpha_1) \wedge \sum_{i=1}^{n} \lambda_i \gamma_i \wedge \delta_i,$$

or equivalently we must have

$$\alpha_1 \wedge \sum_{i=1}^{n} \lambda_i \gamma_i \wedge \delta_i = 0$$

Since the span of the set $\{\gamma_i, \delta_i : 1 \leq i \leq n\}$ does not contain α_1, we conclude that $\sum_{i=1}^{n} \lambda_i \gamma_i \wedge \delta_i = 0$. In particular, this implies that only elements of the form $\wedge^3 [w]^\perp \to \wedge^3 H_1(S_g;\mathbb{Q})/H_1(S_g;\mathbb{Q})$ are fixed by T_w, as desired. $\square$

We apply Lemma 5.1.6 in the following context.

Lemma 5.1.7. *Let $g \geq 4$, and let $\iota : S \hookrightarrow S_g$ be a clean embedding satisfying the hypotheses of Lemma 5.1.5. Let $M \subseteq S_g$ be a nonseparating multicurve, and let $\mathcal{W} \subseteq H_1(S_g; \mathbb{Z})$ be given by $\mathcal{W} = \{[c] : c \in M\}$. Then, after passing to $\mathbb{Q}$–coefficients, the Johnson homomorphism τ_g restricts to an isomorphism*

$$\mathrm{im}\,(H_1(\mathcal{I}(S, S_g); \mathbb{Q}) \to H_1(\mathcal{I}_g; \mathbb{Q})) \cap H_1(\mathcal{I}_g; \mathbb{Q})^{T_{\mathcal{W}}} \to \wedge^3 \mathrm{im}(H_1(S; \mathbb{Q}) \to H_1(S_g; \mathbb{Q})) \bigcap \wedge^3(\mathcal{W}^\perp).$$

Proof. By Lemma 5.1.5, we have

$$\mathrm{im}\,(H_1(\mathcal{I}(S, S_g); \mathbb{Q}) \to H_1(\mathcal{I}_g; \mathbb{Q})) \cong \mathrm{im}(\wedge^3 \iota_*) \cong \wedge^3 H_1(S; \mathbb{Q}).$$

Now, the Johnson homomorphism is $\mathrm{Sp}(2g, \mathbb{Z})$–equivariant, so if $w \in \mathcal{W}$, the fixed set $H_1(\mathcal{I}_g; \mathbb{Q})^{T_w}$ is isomorphic to $\mathrm{im}(\tau_g \otimes \mathbb{Q})^{T_w}$. Then by Lemma 5.1.6, we have

$$\tau_g(H_1(\mathcal{I}_g; \mathbb{Q})^{T_w}) = \mathrm{im}(\wedge^3 [w]^\perp \to \wedge^3 H_1(S_g; \mathbb{Q})/H_1(S_g; \mathbb{Q})).$$

Then we have $H_1(\mathcal{I}_g; \mathbb{Q})^{T_{\mathcal{W}}} = \bigcap_{w \in \mathcal{W}} H_1(\mathcal{I}_g; \mathbb{Q})^{T_w}$ and $\wedge^3 \mathcal{W}^\perp = \bigcap_{w \in \mathcal{W}} \wedge^3 [w]^\perp$. Therefore we have $\tau_g(H_1(\mathcal{I}_g; \mathbb{Q})^{T_{\mathcal{W}}}) = \mathrm{im}(\wedge^3 \mathcal{W}^\perp \to \wedge^3 H_1(S_g; \mathbb{Q})/H_1(S_g; \mathbb{Q}))$. Hence the intersection

$$\mathrm{im}\,(H_1(\mathcal{I}(S, S_g); \mathbb{Q}) \to H_1(\mathcal{I}_g; \mathbb{Q})) \cap H_1(\mathcal{I}_g; \mathbb{Q})^{T_{\mathcal{W}}}$$

is sent to $\mathrm{im}(\wedge^3 H_1(S; \mathbb{Q}) \to \mathrm{im}(\tau_g)) \cap \mathrm{im}(\wedge^3 \mathcal{W}^\perp \to \mathrm{im}(\tau_g))$ by the Johnson homomorphism, and this last intersection is equal to

$$\wedge^3(\mathrm{im}(H_1(S; \mathbb{Q}) \to H_1(S_g; \mathbb{Q}))) \cap \wedge^3 \mathcal{W}^\perp$$

as desired. □

The remaining steps to do before proving Proposition 5.1.1 are to prove Lemma 5.1.9 and

Lemma 5.1.11, which will allow us to rewrite classes $[T_{d,d'}, T_{e,e'}]$ as linear combinations of abelian cycles of bounding pairs maps contained in $H_2(\mathcal{I}_g;\mathbb{Q})^{T_v}$ for various primitive $v \in H_1(S_g;\mathbb{Z})$.

5.1.1 The proof of Lemma 5.1.9

We now recall a basic fact from linear algebra.

Lemma 5.1.8. *Let $(L, \langle \cdot, \cdot \rangle)$ be quasi–unimodular lattice. Let $\mathcal{W} = \{w_1, \ldots, w_4\} \subseteq L$ be a set of primitive elements such that the image of $\mathcal{W}$ under the adjoint map $L \to \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$ is a linearly independent set. Then the natural map*

$$\varphi : \bigoplus_{w \in \mathcal{W}} \wedge^3 \left(w^\perp \otimes \mathbb{Q}\right) \to \wedge^3 \left(L \otimes \mathbb{Q}\right)$$

is surjective.

Proof. Choose a basis $\mathcal{B} = \{a_1, \ldots, a_n\}$ for $L \otimes \mathbb{Q}$ such that $\langle w_i, a_j \rangle = \delta_{ij}$. Then the set $\mathcal{B}_i = \{a_1, \ldots, \widehat{a_i}, \ldots, a_n\}$ is a basis for $w^\perp \otimes \mathbb{Q}$ for each $w \in \mathcal{W}$. Now, the vector space $\wedge^3(L \otimes \mathbb{Q})$ has a basis consisting of triples of elements in $\mathcal{B}$. Since $|\mathcal{W}| \geq 4$, each of these triples lies in some $\mathcal{B}_i$. Therefore $\mathrm{im}(\varphi)$ contains a basis for $\wedge^3(L \otimes \mathbb{Q})$ and hence is surjective. □

For our purpose, the important takeaway from Lemma 5.1.8 is the following result.

Lemma 5.1.9. *Let $M' \subseteq S_g$ be a nonseparating multicurve with $|M'| \geq 4$, and let $\mathcal{V}' = \{[c] : c \in M'\}$. Let $d, d' \subseteq S_g$ be a bounding pair such that $[T_{d,d'}] \in H_1(\mathcal{I}_g;\mathbb{Q})^{T_{\mathcal{V}'}}$. Let $[T_{d,d'}, T_{f,f'}] \in H_2^{\mathrm{ab,bp}}(\mathcal{I}_g;\mathbb{Q})$ be an abelian cycle. There is a $\mathbb{Q}$–linear relation*

$$[T_{d,d'}, T_{f,f'}] = \sum_{\ell=1}^{m} \lambda_\ell [T_{d,d'}, T_{f_\ell, f'_\ell}]$$

in $H_2^{\mathrm{ab,bp}}(\mathcal{I}_g;\mathbb{Q})$ such that, for each $1 \leq \ell \leq m$, there is a $v \in \mathcal{V}'$ such that $[T_{f_\ell, f'_\ell}] \in H_1(\mathcal{I}_g;\mathbb{Q})^{T_v}$.

Proof. Let S' and S'' be the two connected components of $S_g \setminus (d \cup d')$. Without loss of generality, we may assume that $f, f' \subseteq S'$. Indeed, if f, f' are in distinct connected components of $S_g \setminus (d \cup d')$, then f and f' are homologous to d, so we have a relation

$$[T_{d,d'}, T_{f,f'}] = [T_{d,d'}, T_{f,d}] + [T_{d,d'}, T_{d,f'}]$$

We now have two cases.

Case 1: $\mathcal{V}' \not\subseteq \mathrm{im}(H_1(S'; \mathbb{Z}) \to H_1(S_g; \mathbb{Z}))$. In this case, there is a $v \in \mathcal{V}'$ such that $v \in \mathrm{im}(H_1(S''; \mathbb{Z}) \to H_1(S_g; \mathbb{Z}))$. Then $T_{f,f'} \in H_1(\mathcal{I}_g; \mathbb{Z})^{T_v}$, so we are done.

Case 2: $\mathcal{V}' \subseteq \mathrm{im}(H_1(S'; \mathbb{Q}))$. We see that since $\mathcal{V}'$ is linearly independent and the elements of $\mathcal{V}'$ have pairwise trivial algebraic intersection, we have $g(S') \geq 4$. Lemma 5.1.5 says that $H_1(\mathcal{I}(S', S_g); \mathbb{Q}) \cong \wedge^3 H_1(S'; \mathbb{Q})$. Lemma 5.1.8 implies that there is a surjection

$$\bigoplus_{v \in \mathcal{V}'} \wedge^3 v^{\perp} \cap \wedge^3 H_1(S'; \mathbb{Q}) \to \wedge^3 H_1(S'; \mathbb{Q}).$$

Then Lemma 5.1.7 applied to each term on the summand on the left implies there is a surjection

$$\bigoplus_{v \in \mathcal{V}'} H_1(\mathcal{I}_g; \mathbb{Q})^{T_v} \cap H_1(\mathcal{I}(S', S_g); \mathbb{Q}) \to H_1(\mathcal{I}(S', S_g); \mathbb{Q}).$$

Hence there is a relation in $H_1(\mathcal{I}(S', S_g); \mathbb{Q})$ given by

$$[T_{f,f'}] = \sum_{r=1}^{n} \lambda_r [F_r]$$

where each $F_r \in H_1(\mathcal{I}(S', S_g); \mathbb{Q})^{T_v}$ for some $v \in \mathcal{V}'$. Then each F_r is supported on S', so each F_r commutes with $T_{d,d'}$. Therefore, there is a relation in $H_2(\mathcal{I}_g; \mathbb{Q})$ given by

$$[T_{d,d'}, T_{f,f'}] = \sum_{r=1}^{n} \lambda_r [T_{d,d'}, F_r]$$

where each $[F_r] \in H_1(\mathcal{I}(S', S_g); \mathbb{Q})^{T_v}$ for some $v \in \mathcal{V}'$.

We now show that each $[F_r]$ can be written as a sum of classes represented by bounding pair maps. Choose an r with $1 \leq r \leq n$ and let $v \in \mathcal{V}'$ such that $[F_r] \in H_1(S'; \mathbb{Q})^{T_v}$. Lemma 5.1.7 says that $\wedge^3 H_1(\mathcal{I}(S', S_g); \mathbb{Q})^{T_v} \cong \wedge^3 v^{\perp} \cap \wedge^3 H_1(S'; \mathbb{Q})$, and this isomorphism is induced by restricting the Johnson homomorphism. Then Lemma 5.1.2 and Lemma 5.1.4 combine to tell us that $H_1(\mathcal{I}(S' \searrow c, S_g); \mathbb{Q})^{T_v}$ is generated by classes represented by bounding pair maps, since each element in the basis for $\wedge^3 v^{\perp} \cap H_1(S'; \mathbb{Q})$ in Lemma 5.1.4 is the image under τ_g of a bounding pair map. Therefore we may rewrite $[F_r]$ as a linear combination of classes $[T_{f_{1,r}, f'_{1,r}}], \ldots, [T_{f_{\ell_r, r}, f'_{m_r, r}}] \in H_1(\mathcal{I}(S', S_g); \mathbb{Q})^{T_v}$. Since each $T_{f_{\ell,r}, f'_{\ell,r}}$ is supported on S', each $T_{f_{\ell,r}, f'_{\ell,r}}$ commutes with $T_{d,d'}$. Therefore we have a relation

$$[T_{d,d'}, T_{f,f'}] = \sum_{r=1}^{n} \lambda_r \sum_{\ell=1}^{m_r} [T_{d,d'}, T_{f_{\ell,r}, f'_{\ell,r}}]$$

which by forgetting the r index yields a relation

$$[T_{d,d'}, T_{f,f'}] = \sum_{\ell=1}^{m} \lambda_\ell [T_{d,d'}, T_{f_\ell, f'_\ell}].$$

Now, since each T_{f_ℓ, f'_ℓ} is stabilized by some $T_v \in T_{\mathcal{V}'}$, we have each $[T_{f_\ell, f'_\ell}] \in H_1(\mathcal{I}_g; \mathbb{Q})^{T_v}$ for some $v \in \mathcal{V}'$, as desired. $\square$

5.1.2 The proof of Lemma 5.1.11

We begin by extending Lemma 5.1.8 to the following.

Lemma 5.1.10. *Let L be a quasi–unimodular lattice. Let $\mathcal{W} = \{w_1, \ldots, w_7\}$ be a set of primitive elements in L such that the image of $\mathcal{W}$ under the adjoint map $L \to \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$ is $\mathbb{Q}$–linearly independent. Then the natural map*

$$\varphi : \bigoplus_{\mathcal{W}' \subseteq \mathcal{W} : |\mathcal{W}'| = 4} \wedge^3 \left((\mathcal{W}')^{\perp} \otimes \mathbb{Q} \right) \to \wedge^3 (L \otimes \mathbb{Q})$$

is surjective.

Proof. We will prove inductively that the natural maps

$$\varphi_k : \bigoplus_{\mathcal{W}' \subseteq \mathcal{W} : |\mathcal{W}'| = k} \wedge^3 \left((\mathcal{W}')^\perp \otimes \mathbb{Q} \right) \to \wedge^3 (L \otimes \mathbb{Q})$$

are surjective for $k \leq 4$.

Base case: $k = 1$. This follows by applying Lemma 5.1.8 with any subset $\mathcal{W}' \subseteq \mathcal{W}$ with $|\mathcal{W}'| = 4$.

Inductive step: $2 \leq k \leq 4$. Assume that the map $\varphi_{k'}$ is surjective for all $1 \leq k' < k$. We will show that φ_k is surjective as well. Since $\varphi_{k'}$ is surjective, it suffices to show that, for any $\mathcal{W}'' \subseteq \mathcal{W}$ with $|\mathcal{W}''| = k - 1$, we have

$$\operatorname{im}(\varphi_k) \supseteq \operatorname{im} \left(\wedge^3 (\mathcal{W}'')^\perp \otimes \mathbb{Q} \to \wedge^3 L \otimes \mathbb{Q} \right).$$

Since $k \leq 4$ and $|\mathcal{W}''| < k$, we have $|\mathcal{W} \setminus \mathcal{W}''| \geq 7 - k > 7 - 4 = 3$, so $|\mathcal{W} \setminus \mathcal{W}''| \geq 4$. Hence the natural map

$$\varphi_{\mathcal{W}''} : \bigoplus_{w \in \mathcal{W} \setminus \mathcal{W}''} \wedge^3 (\mathcal{W}'' \cup \{w\})^\perp \otimes \mathbb{Q} \to \wedge^3 (\mathcal{W}'')^\perp \otimes \mathbb{Q}$$

is surjective by Lemma 5.1.8. But then $\operatorname{im}(\varphi_{k'} \circ \varphi_{\mathcal{W}''}) \subseteq \operatorname{im}(\varphi_k)$ for any $\mathcal{W}'' \subseteq \mathcal{W}$, so $\operatorname{im}(\varphi_{k'}) \subseteq \operatorname{im}(\varphi_k)$. Since $\operatorname{im}(\varphi_{k'}) = \wedge^3 L \otimes \mathbb{Q}$ by the inductive hypothesis, the proof is complete. □

We use Lemma 5.1.10 to prove the following result about $H_2^{\mathrm{ab,bp}}(\mathcal{I}_g; \mathbb{Q})$.

Lemma 5.1.11. *Let $g \geq 33$ and let $[T_{d,d'}, T_{e,e'}] \in H_2^{\mathrm{ab,bp}}(\mathcal{I}_g; \mathbb{Q})$ be an abelian cycle. Assume that one connected component of $S_g \setminus (d \cup d')$ has genus one. Let $M \subseteq S_g$ be a nonseparating multicurve with $|M| = 9$ and let $\mathcal{V} = \{[c] : c \in M\}$. Then there is a linear* *relation*

$$[T_{d,d'}, T_{e,e'}] = \sum_{i=1}^{k} \lambda_i [T_{d,d'}, T_{f_i,f_i'}]$$

with each $T_{f_i,f_i'} \in H_1(\mathcal{I}_g;\mathbb{Q})^{T_{\mathcal{V}'}}$ *for some* $\mathcal{V}' \subseteq \mathcal{V}$ *and* $|\mathcal{V}'| = 4$.

Proof. The proof follows the same approach as that of Lemma 5.1.9. Let $S \sqcup S' = S_g \setminus (d \cup d')$ with $g(S) = 1$. Since $e \cup e'$ is disjoint from $d \cup d'$, we have $e, e' \subseteq S'$. Since $g(S) = 1$, we must have $e, e' \subseteq S'$. Now, choose a primitive element $[b] \in H_1(S_g;\mathbb{Z})$ such that $[b] \in [e]^{\perp}$ and $\langle [b], [d] \rangle = 1$. Since $g(S) = 1$ and $\langle [b], [d] \rangle = 1$, we have $g(H_1(S';\mathbb{Z}) \cap [b]^{\perp}) = g - 2$. Then since the elements of $\mathcal{V}$ have pairwise trivial algebraic intersection, the projection of $\mathcal{V}$ to $H_1(S';\mathbb{Z}) \cap [b]^{\perp}$ must contain a set of at least 7 elements whose image under the adjoint map $H_1(S';\mathbb{Z}) \cap [b]^{\perp} \to \mathrm{Hom}_{\mathbb{Z}}(H_1(S';\mathbb{Z}), \mathbb{Z})$ is $\mathbb{Q}$–linearly independent. Let $\overline{\mathcal{V}}$ be the image of the projection of $\mathcal{V}$ to $H_1(S';\mathbb{Z}) \cap [b]^{\perp}$. Let $\mathcal{B} \subseteq \overline{\mathcal{V}}$ be a set of at least 7 linearly independent elements. By applying Lemma 5.1.10 to $H_1(S';\mathbb{Q})$ and $\mathcal{B}$, there is a surjection

$$\bigoplus_{\mathcal{B}' \subseteq \mathcal{B} : |\mathcal{B}'| = 4} \wedge^3 (\mathcal{B}')^{\perp} \to \wedge^3 H_1(S';\mathbb{Q}).$$

Let $\mathcal{V}_{\mathcal{B}}$ be the preimage of $\mathcal{B}$ under the projection map $\mathcal{V} \to \overline{\mathcal{V}}$. Since for any $v \in \mathcal{V}$, we have $v^{\perp} \cap H_1(S';\mathbb{Q}) = \left(\mathrm{proj}_{H_1(S';\mathbb{Q}) \cap [b]^{\perp}} v\right)^{\perp} \cap H_1(S';\mathbb{Q})$, we therefore have a surjection

$$\bigoplus_{\mathcal{V}' \subseteq \mathcal{V}_{\mathcal{B}} : |\mathcal{V}'| = 4} \wedge^3 (\mathcal{V}')^{\perp} \cap H_1(S';\mathbb{Q}) \to \wedge^3 H_1(S';\mathbb{Q}).$$

By construction, the complement $S_g \setminus S'$ is connected and has $g(S_g \setminus S') \geq 1$. Hence by Lemma 5.1.5, there is an isomorphism $H_1(\mathcal{I}(S', S_g);\mathbb{Q}) \cong \wedge^3 H_1(S';\mathbb{Q})$. Furthermore, by applying Lemma 5.1.7, we have $\wedge^3 (\mathcal{V}')^{\perp} \cap H_1(S';\mathbb{Q}) \cong \wedge^3 H_1(S';\mathbb{Q})^{T_{\mathcal{V}'}}$. Then, since $T_{e,e'} \in \mathcal{I}(S', S_g)$, we have a relation in $H_1(\mathcal{I}(S', S_g);\mathbb{Q})$ given by

$$[T_{e,e'}] = \sum_{i=1}^{k} \lambda_i [F_i]$$

where each $[F_i] \in H_1(\mathcal{I}_g;\mathbb{Q})^{T_{\mathcal{V}'}}$ for some $\mathcal{V}' \subseteq \mathcal{V}_{\mathcal{B}}$ with $|\mathcal{V}'| = 4$. Since any $f \in \mathcal{I}(S', S_g)$ commutes with $T_{d,d'}$, we have a relation in $H_2(\mathcal{I}_g;\mathbb{Q})$ given by

$$[T_{d,d'}, T_{e,e'}] = \sum_{i=1}^{k} \lambda_i [T_{d,d'}, F_i]$$

where each $[F_i] \in H_1(\mathcal{I}(S', S_g);\mathbb{Q})^{T_{\mathcal{V}'}}$ for some $\mathcal{V}' \subseteq \mathcal{B}$ with $|\mathcal{V}'| = 4$. By Lemma 5.1.7, we have an isomorphism $H_1(\mathcal{I}(S', S_g);\mathbb{Q})^{T_{\mathcal{V}'}} \cong \wedge^3 \operatorname{im}(H_1(S';\mathbb{Q}) \to H_1(S_g;\mathbb{Q})) \cap \wedge^3(\mathcal{V}')^{\perp}$. Then Lemma 5.1.4 gives us a spanning set for the vector space

$$\wedge^3 \operatorname{im}(H_1(S';\mathbb{Q}) \to H_1(S_g;\mathbb{Q})) \cap \wedge^3(\mathcal{V}')^{\perp},$$

and by Lemma 5.1.2 each element of this spanning set is given by applying τ_g to a bounding pair map supported on a bounding pair $f \cup f' \subseteq S'$. Hence there is a relation in $H_2(\mathcal{I}_g;\mathbb{Q})$ given by

$$[T_{d,d'}, T_{e,e'}] = \sum_{i=1}^{k} \lambda_i [T_{d,d'}, T_{f_i,f_i'}]$$

such that each $T_{f_i,f_i'} \in H_1(\mathcal{I}_g;\mathbb{Q})^{T_{\mathcal{V}'}}$ for some $\mathcal{V}' \subseteq \mathcal{V}$ with $|\mathcal{V}'| = 4$, so the lemma holds. □

We will need one more auxiliary result, which well help us apply Lemma 5.1.11.

Lemma 5.1.12. *Let $g \geq 33$ and let $[T_{d,d'}, T_{e,e'}] \in H_2^{\mathrm{ab,bp}}(\mathcal{I}_g;\mathbb{Q})$. Then there is a linear relation*

$$[T_{d,d'}, T_{e,e'}] = \sum_{j=1}^{r} [T_{d_j,d_j'}, T_{e,e'}]$$

in $H_2^{\mathrm{ab,bp}}(\mathcal{I}_g;\mathbb{Q})$ such that, for each $1 \leq j \leq r$, at least one connected component of $S_g \searrow (d_j \cup d_j')$ has genus one.

Proof. Let $S \cup S'$ be the connected components of $S_g \searrow (d \cup d')$. If e, e' are contained in different connected components of $S_g \searrow (d \cup d')$, we have a relation $T_{e,e'} = T_{e,d} T_{d,e'}$.

Hence there is a relation

$$[T_{d,d'}, T_{e,e'}] = [T_{d,d'}, T_{e,d}] + [T_{d,d'}, T_{d,e'}]$$

in $H_2^{\mathrm{ab,bp}}(\mathcal{I}_g; \mathbb{Q})$. Therefore without loss of generality, we may assume that $e, e' \subseteq S$. Now, let $d_0, \ldots, d_k \subseteq S'$ be curves such that the following hold:

- $d_0 = d$,
- $d_k = d'$,
- d_i and d_j are disjoint for all $0 \leq i < j \leq k$,
- d_i is homologous to d for all $0 \leq i \leq k$, and
- the connected components of $S' \smallsetminus \left(\bigcup_{0 \leq i \leq k} d_i\right)$ all have genus one.

By construction, we have $T_{d,d'} = \prod_{0 \leq i \leq k-1} T_{d_i,d_{i+1}}$. Furthermore, each bounding pair $d_i \cup d_{i+1}$ is contained in S', and hence each $T_{d_i,d_{i+1}}$ commutes with $T_{e,e'}$. Therefore there is a relation in $H_2^{\mathrm{ab,bp}}(\mathcal{I}_g; \mathbb{Q})$ given by

$$[T_{d,d'}, T_{e,e'}] = \sum_{i=0}^{k-1} [T_{d_i,d_{i+1}}, T_{e,e'}],$$

so the lemma holds. □

We now conclude Section 5.1.

Proof of Proposition 5.1.1. Let $[T_{d,d'}, T_{e,e'}] \in H_2^{\mathrm{ab,bp}}(\mathcal{I}_g; \mathbb{Q}))$ be an abelian cycle. The proof of the proposition proceeds in two steps.

1. There is a relation
$$[T_{d,d'}, T_{e,e'}] = \sum_{i=1}^{k} \lambda_i [T_{d,d'}, T_{f_i,f_i'}]$$
with each $[T_{f_i,f_i'}] \in H_1(\mathcal{I}_g; \mathbb{Q})^{T_{\mathcal{V}'}}$ with $\mathcal{V}' \subseteq \mathcal{V}$ and $|\mathcal{V}'| = 4$.

2. If $[T_{d,d'}, T_{f_i,f_i'}]$ is an abelian cycle with $T_{f_i,f_i'} \in H_1(\mathcal{I}_g;\mathbb{Q})^{T_{\mathcal{V}'}}$ with $\mathcal{V}' \subseteq V$ and $|\mathcal{V}'| = 4$, then there is a relation

$$[T_{d,d'}, T_{f_i,f_i'}] = \sum_{j=1}^{m} \lambda_j [T_{h_j,h_j'}, T_{f_i,f_i'}]$$

with each $[T_{h_j,h_j'}] \in H_1(\mathcal{I}_g;\mathbb{Q})^{T_v}$ for some $v \in \mathcal{V}'$.

Lemma 5.1.12 says that we may assume without loss of generality that at least one connected component of $S_g \searrow (d \cup d')$ has genus one. Then Step (1) is the content of Lemma 5.1.11 and Step (2) is the content of Lemma 5.1.9. □

5.2 Finiteness of coinvariants in $\mathrm{BM}_2(\mathbf{X_g};\mathbb{Q})$

The main goal of this section is to prove the following result.

Lemma 5.2.1. *Let $g \geq 33$ and $a, b \subseteq S_g$ nonseparating curves with $|a \cap b| = 1$. Let $M \subseteq S_g \searrow (a \cup b)$ be a nonseparating multicurve with $|M| = 9$. Let $\vec{x}[a]$ and let $X_g = \mathcal{C}_{\vec{x}}(S_g)/\mathcal{I}_g$. Let G be the image of the map $\mathrm{Mod}(S_g \searrow (a \cup b \cup M)) \to \mathrm{Sp}(2g,\mathbb{Z})$. The vector space*

$$H_0(G;\mathrm{BM}_2(X_g;\mathbb{Q}))$$

is finite dimensional.

Outline of Section 5.2. Let $g \geq 33$ and a, b, M be as in the statement of Lemma 5.2.1 and let $\mathcal{V} = \{[c] : c \in M\}$. We will begin by describing some algebraic invariants of Bestvina–Margalit tori, which record how the elements in $\mathcal{V}$ project onto the different subgroups in $\mathcal{H}(\mathrm{BM}_\sigma)$. We will then prove Lemma 5.2.2, which allows us to use these algebraic invariants to determine when two Bestvina–Margalit tori BM_σ and BM_τ are in the same G–orbit for certain subgroups $G \subseteq \mathrm{Sp}(2g,\mathbb{Z})$. We will then prove Lemma 5.2.3, which describes how these algebraic invariants interact with addition of fundamental classes. We will then prove Lemma 5.2.5 and Lemma 5.2.7. These two results will allow decompose a class $[\mathrm{BM}_\sigma]$ into "simpler" classes, where "simpler" means roughly that the algebraic invariants of the torus BM_σ are bounded. These latter two lemmas make up the main work of the section.

Algebraic invariants of Bestvina–Margalit tori. We begin by describing the invariants of Bestvina–Margalit tori that we will use to prove Lemma 5.2.1. Let $\mathcal{H}(\mathrm{BM}_\sigma) = \{\mathcal{H}_0^\sigma, \mathcal{H}_1^\sigma, \mathcal{H}_2^\sigma\}$ as in Section 4.2. For each $v_i \in \mathcal{V}$, let $\mathrm{rk}^{\mathcal{V}}(v_{i,k}^\sigma)$ denote the maximal $m \in \mathbb{Z}$ such that that $\mathrm{proj}_{\mathcal{H}_k^\sigma \cap [b]^\perp}(v_i) = mw$ for some nonzero $w \in \mathcal{H}_k^\sigma$. For each $v_i, v_j \in \mathcal{V}$, let

$$\theta(\mathcal{V})_{i,j,k}(\sigma) = \langle \mathrm{proj}_{\mathcal{H}_k^\sigma \cap [b]^\perp}(v_i), \mathrm{proj}_{\mathcal{H}_k^\sigma \cap [b]^\perp}(v_j)\rangle.$$

We now prove a result that describes the orbits of Bestvina–Margalit tori under the action of stabilizer subgroups of $\mathrm{Sp}(2g, \mathbb{Z})$.

Lemma 5.2.2. *Let $g \geq 33$, and let $a, b \subseteq S_g$ with $|a \cap b| = 1$. Let $\vec{x} = [a]$. Let $M \subseteq S_g$ be a nonseparating multicurve disjoint from a and b. Let $\mathcal{V} = \{[c] : c \in M\}$, and assume that the elements of $\mathcal{V}$ are indexed as $\mathcal{V} = \{v_1, \dots, v_n\}$. Let G be the group $\mathrm{im}(\mathrm{Mod}(S_g \setminus (a \cup b \cup M)) \to \mathrm{Sp}(2g, \mathbb{Z}))$. Let $\sigma, \tau \subseteq X_g$ be 2–cells and let $\mathrm{BM}_\sigma, \mathrm{BM}_\tau$ be the corresponding Bestvina–Margalit tori. The tori BM_σ and BM_τ are in the same G–orbit if, after possibly reindexing $\mathcal{H}(\mathrm{BM}_\tau)$, we have:*

- $g(\sigma) = g(\tau)$,
- $\mathrm{rk}^{\mathcal{V}}(v_{i,k}^\sigma) = \mathrm{rk}^{\mathcal{V}}(v_{i,k}^\tau)$ *for* $1 \leq i \leq n, 0 \leq k \leq 2$, *and*
- $\theta(\mathcal{V})_{i,j,k}(\sigma) = \theta(\mathcal{V})_{i,j,k}(\tau)$ *for* $1 \leq i, j \leq n, 0 \leq k \leq 2$.

Proof. Let $\mathrm{BM}_\sigma, \mathrm{BM}_\tau$ be two Bestvina–Margalit tori that satisfy the hypotheses of the lemma. Because $g(\sigma) = g(\tau)$, there is an $f \in \mathrm{Stab}_{\mathrm{Sp}(2g,\mathbb{Z})}(\vec{x}, [b])$ satisfying $f\,\mathrm{BM}_\sigma = \mathrm{BM}_\tau$. Therefore, after possibly reindexing, we have $f(\mathcal{H}_k^\sigma) = \mathcal{H}_k^\tau$ for every $0 \leq k \leq 2$.

We claim that this f sends $\mathrm{proj}_{\mathcal{H}_k^\sigma \cap [b]^\perp}(v_i)$ to $\mathrm{proj}_{\mathcal{H}_k^\tau \cap [b]^\perp}(v_i)$ for every i, k. Indeed, such an element exists since $g(\mathcal{H}(\sigma)) = g(\mathcal{H}(\tau))$ and BM_σ and BM_τ have $\mathrm{rk}^{\mathcal{V}}(v_{i,k}^\sigma) = \mathrm{rk}^{\mathcal{V}}(v_{i,k}^\tau)$

and $\theta(\mathcal{V})_{i,j,k}(\sigma) = \theta(\mathcal{V})_{i,j,k}(\tau)$. Hence for each $\mathcal{H}_k(\tau)$ there is an $f_k \in \mathrm{Stab}_{\mathrm{Sp}(2g,\mathbb{Z})}(\vec{x}, [b])$ which is the identity on $\mathcal{H}_{k'}(\tau)$ for $k' \neq k$ and which takes $f(v^\sigma_{i,k})$ to $v^\tau_{i,k}$. Then we can replace f with $f_0 f_1 f_2 f$, which satisfies $f\,\mathrm{proj}_{\mathcal{H}^\sigma_k \cap [b]^\perp}(v_i) = \mathrm{proj}_{\mathcal{H}^\tau_k \cap [b]^\perp}(v_i)$. But then this new f fixes every v_i, since

$$f(v_i) = \sum_{k \in \{1,2,3\}} f(\mathrm{proj}_{\mathcal{H}^\sigma_k \cap [b]^\perp}(v_i)) = \sum_{k \in \{1,2,3\}} \mathrm{proj}_{\mathcal{H}^\tau_k \cap [b]^\perp}(v_i) = v_i.$$

It now remains to show that $f \in G$, i.e., that there is some $F \in \mathrm{Mod}(S_g \smallsetminus (a \cup b \cup M))$ such that the image of F under the symplectic representation is f. Since the symplectic representation $\mathrm{Mod}(S_g) \to \mathrm{Sp}(2g, \mathbb{Z})$ is surjective, there is some $F' \in \mathrm{Mod}(S_g)$ such that F' is sent to f by the symplectic representation. There is a multicurve $M \subseteq S_g \smallsetminus (a \cup b)$ with $[M] = \mathcal{V}$. Thus, we can extend $\mathcal{V} \cup \{\vec{x}, [b]\}$ to a symplectic basis $\mathcal{B} = \{\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g\}$. Choose a set of curves $\widehat{\mathcal{B}}$ such that:

- $M, \{a, b\} \subseteq \widehat{\mathcal{B}}$,
- $\mathcal{B} = \{[c] : c \in \widehat{\mathcal{B}}\}$, and
- $|c \cap c'| = |\langle [c], [c'] \rangle|$ for all $c, c' \in \widehat{\mathcal{B}}$.

Let $\widehat{\mathcal{B}}'$ be another set of representatives for $\mathcal{B}$ satisfying the above conditions, except we have

$$F'(M), \{F'(a), F'(b)\} \subseteq \widehat{\mathcal{B}}'.$$

But now by the change of coordinates principle, there is some $F'' \in \mathrm{Mod}(S_g)$ taking $\widehat{\mathcal{B}}' \to \widehat{\mathcal{B}}$ in such a way that F''_* acts trivially on $\mathcal{B}$. But then $F'' \in \mathcal{I}_g$, so the image of $F'' \cdot F'$ under the symplectic representation is the same as F', which is f. Then we have $F'' \cdot F'(M) = M$, $F'' \cdot F'(a) = a$, and $F'' \cdot F'(b) = b$. Hence $F'' \cdot F' \in \mathrm{Mod}(S_g \smallsetminus (M \cup a \cup b))$ and $F'' \cdot F'$ maps to f under the symplectic representation, so $f \in G$, as desired. □

We now prove a lemma that describes how the $\theta(\mathcal{V})$ and $\mathrm{rk}^{\mathcal{V}}$ interact with the Bestvina–Margalit tori

Lemma 5.2.3. *Let $g \geq 33$, $a, b, M \subseteq S_g$, $\vec{x} \in H_1(S_g;\mathbb{Z})$ and $\mathcal{V} \subseteq H_1(S_g;\mathbb{Z})$ be as in Lemma 5.2.2. Let $x, y, z \subseteq X_g$ and $\mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2, \mathcal{H}'_1, \mathcal{H}'_2$ be as in Lemma 4.2.4. For any choice of $* = \{y, z, yz\}$, let σ_* denote the 2–cell containing x and $*$. Let $v^*_{i,k}$ denote $\mathrm{proj}_{\mathcal{H}_i \in \mathcal{H}(\sigma_*) \cap [b]^\perp}(v_i)$, and let $\theta(\mathcal{V})_{i,j,k}(*)$ denote $\theta(\mathcal{V})_{i,j,k}(\sigma_*)$. Then the following equalities hold for all $1 \leq i, j \leq 9$:*

1. $\mathrm{rk}^{\mathcal{V}}(v^y_{i,0}) = \mathrm{rk}^{\mathcal{V}}(v^z_{i,0}) = \mathrm{rk}^{\mathcal{V}}(v^{yz}_{i,0})$,
2. $\mathrm{rk}^{\mathcal{V}}(v^{yz}_{i,1}) = \gcd(\mathrm{rk}^{\mathcal{V}}(v^y_{i,1}), \mathrm{rk}^{\mathcal{V}}(v^z_{i,1}))$,
3. $\mathrm{rk}^{\mathcal{V}}(v^y_{i,2}) = \gcd(\mathrm{rk}^{\mathcal{V}}(v^z_{i,1}), \mathrm{rk}^{\mathcal{V}}(v^{yz}_{i,2}))$,
4. $\mathrm{rk}^{\mathcal{V}}(v^z_{i,2}) = \gcd(\mathrm{rk}^{\mathcal{V}}(v^y_{i,1}), \mathrm{rk}^{\mathcal{V}}(v^{yz}_{i,2}))$,
5. $\theta(\mathcal{V})_{i,j,0}(y) = \theta(\mathcal{V})_{i,j,0}(z) = \theta(\mathcal{V})_{i,j,0}(yz)$,
6. $\theta(\mathcal{V})_{i,j,1}(y) + \theta(\mathcal{V})_{i,j,1}(z) = \theta(\mathcal{V})_{i,j,1}(yz)$,
7. $\theta(\mathcal{V})_{i,j,1}(z) + \theta(\mathcal{V})_{i,j,2}(yz) = \theta(\mathcal{V})_{i,j,2}(y)$, *and*
8. $\theta(\mathcal{V})_{i,j,1}(y) + \theta(\mathcal{V})_{i,j,2}(yz) = \theta(\mathcal{V})_{i,j,2}(z)$.

Proof. Since we have chosen the multicurve M to be disjoint from a and b, we have $\mathcal{V} \subseteq [b]^\perp$. Then, if $X, Y \subseteq \vec{x}^\perp$ are two quasi–unimodular lattices with $X \cap Y = \mathbb{Z}\vec{x}$ and $\mathrm{rk}^{\mathcal{V}}(X) = 2g(X) + 1$, $\mathrm{rk}^{\mathcal{V}}(Y) = 2g(Y) + 1$, we see that for any $v \in \vec{x}^\perp \cap [b]^\perp$, we have

$$\mathrm{proj}_{(X+Y)\cap[b]^\perp}(v) = \mathrm{proj}_{X\cap[b]^\perp}(v) + \mathrm{proj}_{Y\cap[b]^\perp}(v).$$

Now, Lemma 4.2.4 says that the following hold:

1. $\mathcal{H}^y_0 = \mathcal{H}^z_0 = \mathcal{H}^{yz}_0$,
2. $\mathcal{H}^y_1 + \mathcal{H}^z_1 = \mathcal{H}^{yz}_1$,
3. $\mathcal{H}^y_1 + \mathcal{H}^{yz}_2 = \mathcal{H}^z_2$, and

4. $\mathcal{H}_1^z + \mathcal{H}_2^{yz} = \mathcal{H}_2^y$.

Hence the above observation about projections tells us that the following equalities among $v_{i,k}^y$, $v_{i,k}^z$ and $v_{i,k}^{yz}$ hold for all $1 \leq i \leq 9$:

(a) $v_{i,0}^y = v_{i,0}^z = v_{i,0}^{yz}$,

(b) $v_{i,1}^y + v_{i,1}^z = v_{i,1}^{yz}$,

(c) $v_{i,1}^y + v_{i,2}^{yz} = v_{i,2}^z$, and

(d) $v_{i,1}^z + v_{i,2}^{yz} = v_{i,2}^y$.

The relations in the statement of the lemma are derived as follows.

Relations (1) and (5). Since $v_{i,0}^y = v_{i,0}^z = v_{i,0}^{yz}$ by equality (a), we must have $\mathrm{rk}^{\mathcal{V}}(v_{i,0}^y) = \mathrm{rk}^{\mathcal{V}}(v_{i,0}^z) = \mathrm{rk}^{\mathcal{V}}(v_{i,0}^{yz})$, and similarly for $\theta(\mathcal{V})_{i,j,0}(y) = \theta(\mathcal{V})_{i,j,0}(z) = \theta(\mathcal{V})_{i,j,0}(yz)$.

Relation (2) and (6). We have $v_{i,1}^y + v_{i,1}^z = v_{i,1}^{yz}$ by relation (b). If $w_{i,1}^*$ is a primitive class with $\mathrm{rk}^{\mathcal{V}}(v_{i,1}^*)w_{i,1}^* = v_{i,1}^*$ for some choice of $* = y, z, yz$, then by relation (b) we have $\mathrm{rk}^{\mathcal{V}}(v_{i,1}^y)w_{i,1}^y + \mathrm{rk}^{\mathcal{V}}(v_{i,1}^z)w_{i,1}^z = v_{i,1}^{yz}$. By Lemma 4.2.4, the group $\mathcal{H}_1^{x,yz} \in \mathcal{H}(\sigma_{yz})$ is given by $\mathcal{H}_1^{x,y} + \mathcal{H}_1^{x,z}$ with $\mathcal{H}_1^{x,y} \cap \mathcal{H}_1^{x,z} = \mathbb{Z}\vec{x}$, so we may represent $w_{i,1}^y$ and $w_{i,1}^z$ using disjoint curves $c_y, c_z \subseteq S_g$. Then the homology class

$$\mathrm{rk}^{\mathcal{V}}(v_{i,1}^y)/\gcd(\mathrm{rk}^{\mathcal{V}}(v_{i,1}^y), \mathrm{rk}^{\mathcal{V}}(v_{i,1}^z)) \cdot [c_y] + \mathrm{rk}^{\mathcal{V}}(v_{i,1}^z)/\gcd(\mathrm{rk}^{\mathcal{V}}(v_{i,1}^y), \mathrm{rk}^{\mathcal{V}}(v_{i,1}^z)) \cdot [c_z]$$

is primitive, since

$$\mathrm{rk}^{\mathcal{V}}(v_{i,1}^y)/\gcd(\mathrm{rk}^{\mathcal{V}}(v_{i,1}^y), \mathrm{rk}^{\mathcal{V}}(v_{i,1}^z)) \text{ and } \mathrm{rk}^{\mathcal{V}}(v_{i,1}^z)/\gcd(\mathrm{rk}^{\mathcal{V}}(v_{i,1}^y), \mathrm{rk}^{\mathcal{V}}(v_{i,1}^z))$$

are relatively prime. Therefore $\mathrm{rk}^{\mathcal{V}}(v_{i,1}^{yz}) = \gcd(\mathrm{rk}^{\mathcal{V}}(v_{i,1}^y), \mathrm{rk}^{\mathcal{V}}(v_{i,1}^z))$. For relation (6), relation (b) implies that

$$\theta(\mathcal{V})_{i,j,1}^{yz} = \langle v_{i,1}^y + v_{i,1}^z, v_{j,1}^y + v_{j,1}^z \rangle.$$

As above, $v_{i,1}^y$ and $v_{j,1}^z$, can be represented by multiples of disjoint curves, and similarly for

$v^z_{j,1}$ and $v^y_{i,1}$. Therefore we have

$$\begin{aligned}\theta(\mathcal{V})^{yz}_{i,j,1} &= \langle v^y_{i,1}, v^y_{j,1}\rangle + \langle v^y_{i,1}, v^z_{j,1}\rangle + \langle v^z_{i,1}, v^y_{j,1}\rangle + \langle v^z_{i,1}, v^z_{j,1}\rangle \\ &= \langle v^y_{i,1}, v^y_{j,1}\rangle + \langle v^z_{i,1}, v^z_{j,1}\rangle \\ &= \theta(\mathcal{V})_{i,j,1}(y) + \theta(\mathcal{V})_{i,j,1}(z).\end{aligned}$$

Relations (3) and (7). These two relations follow from relation (c) using an argument similar to that for relations (2) and (6).

Relations (4) and (8). These two relations follow from relation (d) using an argument to that for relations (2) and (6). □

5.2.1 The proof of Lemma 5.2.5

We begin by proving an auxiliary result, which we will need to prove Lemma 5.2.5 and Lemma 5.2.7.

Lemma 5.2.4. *Let $g \geq 33$, $a, b, M \subseteq S_g$, $\vec{x} \in H_1(S_g;\mathbb{Z})$ and $\mathcal{V} \subseteq H_1(S_g;\mathbb{Z})$ be as in Lemma 5.2.2. Let $\sigma \subseteq X_g$ be a 2–cell. Suppose that there is a $1 \leq m \leq 9$ such that the following hold:*

- $\mathrm{rk}^{\mathcal{V}}(v^\sigma_{i,k}) \leq 1$ *for all* $1 \leq i < m$ *and* $0 \leq k \leq 2$*,*
- $\mathrm{rk}^{\mathcal{V}}(v^\sigma_{m,2}) \geq 2$*,*
- $\mathrm{rk}^{\mathcal{V}}(v^\sigma_{m,2}) = \max\{\mathrm{rk}^{\mathcal{V}}(v^\sigma_{m,k}) : 0 \leq k \leq 2\}$*, and*
- $\mathrm{rk}^{\mathcal{V}}(v^\sigma_{m,1}) \neq 0, \mathrm{rk}^{\mathcal{V}}(v^\sigma_{m,2})$.

Then there is a relation in $\mathrm{BM}_2(X_g;\mathbb{Q})$ *given by*

$$[\mathrm{BM}_\sigma] = \sum_{\iota=1}^{q} \lambda_\iota [\mathrm{BM}_{\sigma_\iota}]$$

such that, for all $1 \leq \iota \leq q$*:*

1. *we have* $\mathrm{rk}^{\mathcal{V}}(v_{i,k}^{\sigma_\iota}) \leq 1$ *for all* $1 \leq i < m$ *and* $0 \leq k \leq 2$*,*

2. *the set* $\{\mathrm{rk}^{\mathcal{V}}(v_{m,k}^{\sigma_\iota}) : 0 \leq k \leq 2\}$ *is bounded above* $\mathrm{rk}^{\mathcal{V}}(v_{m,2}^{\sigma})$*,*

3. *the set* $\{\mathrm{rk}^{\mathcal{V}}(v_{m,k}^{\sigma_\iota}) : 0 \leq k \leq 2\}$ *has no more elements equal to* $\mathrm{rk}^{\mathcal{V}}(v_{m,2}^{\sigma_\iota})$ *than* $\{\mathrm{rk}^{\mathcal{V}}(v_{m,k}^{\sigma}) : 0 \leq k \leq 2\}$*,*

4. *if* $\max_{0\leq k\leq 2}\{\mathrm{rk}^{\mathcal{V}}(v_{m,k}^{\sigma_\iota})\} = \mathrm{rk}^{\mathcal{V}}(v_{m,2}^{\sigma})$*, we have* $0 < \mathrm{rk}^{\mathcal{V}}(v_{m,1}^{\sigma_\iota}) < \max\{\mathrm{rk}^{\mathcal{V}}(v_{m,k}^{\sigma_\iota}) : 0 \leq k \leq 2\}$*, and*

5. *we have* $g(\mathcal{H}_1^{\sigma_\iota}) \geq 10$.

Proof. If $g(\mathcal{H}_1^{\sigma}) \geq 10$, then we are done by taking $q = 1$, $\lambda_1 = 1$ and $[\mathrm{BM}_{\sigma_1}] = [\mathrm{BM}_{\sigma}]$. Otherwise, since $g \geq 33$, there is a $\kappa = 0, 2$ such that $g(\mathcal{H}_\kappa^{\sigma}) \geq 19$. Then since $|\mathcal{V}| = 9$, the fact that $g(\mathcal{H}_\kappa^{\sigma}) \geq 19$ implies that there is a primitive $\mathcal{H} \subseteq \mathcal{H}_\kappa^{\sigma}$ such that the following hold:

- $a \in \mathcal{H}$,
- $v_{i,\kappa}^{\sigma} \in \mathcal{H}^{\perp}$ for all $1 \leq i \leq 9$,
- $g(\mathcal{H}) = 9$,
- $g(\mathcal{H}) + g(\mathcal{H}_1^{\sigma}) \geq 10$, and
- $2 * g(\mathcal{H}) + 1 = \mathrm{rk}^{\mathcal{V}}(\mathcal{H})$.

Let $\widehat{\mathcal{H}} = \{\mathcal{H}_0^{\sigma}, \mathcal{H}, \mathcal{H}^{\perp} \cap \mathcal{H}_1^{\sigma}, \mathcal{H}_2^{\sigma}\}$. We can describe $\widehat{\mathcal{H}}$ using the graphical notation of Section 4.2 as in Figure 5.1.

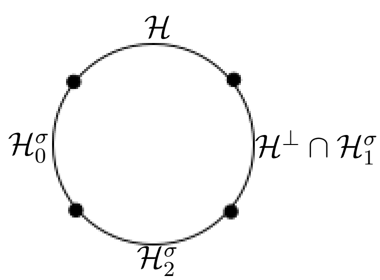

Figure 5.1: The graphical representation of $\widehat{\mathcal{H}}$

Let $z \subseteq X_g$ be the edge with $\mathcal{H} \in \mathcal{H}(z)$. Let $y \subseteq X_g$ the edge with $\mathcal{H}^{\perp} \cap \mathcal{H}_1^{\sigma} \in \mathcal{H}(y)$. Let $x \subseteq X_g$ be the edge with $\mathcal{H}_{\kappa'}^{\sigma} \in \mathcal{H}(y)$, where $0 \leq \kappa' \leq 2$ and $\kappa' \neq 1, \kappa$. After orienting

y and z correctly, we let yz denote the third edge of a 2–cell in X_g containing y and z. We have $\mathcal{H} + \mathcal{H}^\perp \cap \mathcal{H}_1^\sigma = \mathcal{H}_1^\sigma \in \mathcal{H}(yz)$, so x and yz are two edges of the 2–cell σ. Now, Lemma 4.2.2 says that there is a relation

$$[\mathrm{BM}_{x,yz}] = [\mathrm{BM}_{x,y}] + [\mathrm{BM}_{x,z}].$$

Now, let $\sigma_y, \sigma_z \subseteq X_g$ be 2–cells containing x and then y and z respectively. We will now show that σ_z and σ_y desired properties for σ_ι.

The 2–cell σ_z satisfies the desired properties of σ_ι. Assume that $\mathcal{H}(\sigma_z)$ is indexed so that $\mathcal{H} = \mathcal{H}_\kappa^{\sigma_z}$, $\mathcal{H}_1^{\sigma_z} = \mathcal{H}^\perp \cap \mathcal{H}_\kappa^\sigma + \mathcal{H}_1^\sigma$, and $\mathcal{H}_{\kappa'}^{\sigma_z} = \mathcal{H}_{\kappa'}^\sigma$. By our assumption that $v_{i,\kappa}^\sigma \in \mathcal{H}^\perp$ for all $1 \leq i \leq 9$, the following hold:

1. $v_{i,\kappa}^{\sigma_z} = 0$ for all $1 \leq i \leq 9$,
2. $v_{i,1}^{\sigma_z} = v_{i,\kappa}^\sigma + v_{i,1}^\sigma$ for all $1 \leq i \leq 9$, and
3. $v_{i,\kappa'}^{\sigma_z} = v_{i,\kappa'}^\sigma$ for all $1 \leq i \leq 9$.

Then we can compute $\mathrm{rk}^{\mathcal{V}}(v_{i,\kappa}^{\sigma_z})$ as follows.

1. For $1 \leq i < m$ and $0 \leq k \leq 2$, we have $v_{i,k}^{\sigma_z}$ primitive. The class $v_{i,k}^\sigma$ is primitive for all $1 \leq i < m$ and $0 \leq k \leq 2$ by hypothesis, so $v_{i,\kappa'}^{\sigma_z}$ is primitive. Additionally, for each $1 \leq i < m$, the class $v_{i,\kappa}^\sigma$ can be represented by a simple closed curve disjoint from a representative for the class $v_{i,1}^\sigma$, so $v_{i,1}^{\sigma_z}$ is primitive. Hence $\mathrm{rk}^{\mathcal{V}}(v_{i,k}^{\sigma_z}) \leq 1$ for $1 \leq i < m$ and $0 \leq k \leq 2$.

2. For $i = m$, we have the following:

   - $\mathrm{rk}^{\mathcal{V}}(v_{i,\kappa}^{\sigma_z}) = 0$,
   - $\mathrm{rk}^{\mathcal{V}}(v_{i,\kappa'}^{\sigma_z}) = \mathrm{rk}^{\mathcal{V}}(v_{i,\kappa'}^{\sigma_z})$, and
   - $\mathrm{rk}^{\mathcal{V}}(v_{i,1}^{\sigma_z}) = \gcd(\mathrm{rk}^{\mathcal{V}}(v_{i,\kappa}^\sigma), \mathrm{rk}^{\mathcal{V}}(v_{i,1}^\sigma))$.

The computation of $\mathrm{rk}^{\mathcal{V}}(v_{i,k}^{\sigma_z})$ above implies that σ_z satisfies property (1) of σ_ι. Then the computation of $\mathrm{rk}^{\mathcal{V}}(v_{m,k}^{\sigma_z})$ implies that σ_z satisfies properties (2)–(4) of σ_ι. Indeed, properties (2) and (3) follow from the fact that $\gcd(\mathrm{rk}^{\mathcal{V}}(v^{\sigma}_{i,\kappa}), \mathrm{rk}^{\mathcal{V}}(v^{\sigma}_{i,1})) \leq \mathrm{rk}^{\mathcal{V}}(v^{\sigma}_{i,\kappa}), \mathrm{rk}^{\mathcal{V}}(v^{\sigma}_{i,1})$. Property (4) follows from the fact that $\mathrm{rk}^{\mathcal{V}}(v^{\sigma_z}_{i,1}) = \gcd(\mathrm{rk}^{\mathcal{V}}(v^{\sigma}_{i,\kappa}), \mathrm{rk}^{\mathcal{V}}(v^{\sigma}_{i,1})) \leq \mathrm{rk}^{\mathcal{V}}(v^{\sigma}_{i,1}) < \mathrm{rk}^{\mathcal{V}}(v^{\sigma}_{i,2})$. Finally, property (5) follows from the fact that $g(\mathcal{H}^{\sigma_z}_1) = g(\mathcal{H}^{\sigma}_1) + g(\mathcal{H}^{\sigma}_{\kappa}) - g(\mathcal{H})$. Indeed, we have assumed that $g(\mathcal{H}^{\sigma}_{\kappa}) \geq 19$, so $g(\mathcal{H}^{\sigma_z}_1) \geq 19 - g(\mathcal{H})$. Then since $g(\mathcal{H}) = 9$ by hypothesis, we have $g(\mathcal{H}^{\sigma_z}_1) \geq 19 - 9 \geq 10$, so σ_z satisfies property (5) of σ_ι.

The 2–cell σ_y satisfies the desired properties of σ_ι. Assume that $\mathcal{H}(\sigma_y)$ is indexed so that $\mathcal{H}^{\sigma_y}_1 = \mathcal{H}^{\sigma}_1 + \mathcal{H}$, $\mathcal{H}^{\sigma_y}_{\kappa'} = \mathcal{H}^{\sigma}_{\kappa}$, and $\mathcal{H}^{\sigma_y}_{\kappa} = \mathcal{H}^{\perp} \cap \mathcal{H}^{\sigma_y}_{\kappa}$. The assumption that $v^{\sigma}_{i,1} \in \mathcal{H}^{\perp}$ for all $1 \leq i \leq 9$ implies that $v^{\sigma}_{i,k} = v^{\sigma_y}_{i,k}$ for all $1 \leq i \leq 9$, $0 \leq k \leq 2$. Therefore since σ satisfies properties (1)–(4) of σ_ι by hypothesis, then σ_ι does as well. Then $g(\mathcal{H}^{\sigma_y}_1) = g(\mathcal{H}^{\sigma}_1) + g(\mathcal{H}) \geq 1 + 9 = 10$ by assumption, so σ_y satisfies property (5) of σ_ι.

Now, we have shown that $[\mathrm{BM}_{x,yz}] = [\mathrm{BM}_{x,y}] + [\mathrm{BM}_{x,z}]$. By our choice of x and y, we have $\mathrm{BM}_{x,yz} = \mathrm{BM}_{\sigma}$. Therefore we have a relation

$$[\mathrm{BM}_{x,y}] = [\mathrm{BM}_{x,z}] - [\mathrm{BM}_{x,yz}].$$

By taking $s = 2$, $\tau_1 = \sigma_z$, $\tau_2 = \sigma_{yz}$, $\lambda_1 = 1$, and $\lambda_2 = -1$, the proof is complete. □

We now prove Lemma 5.2.5, which says that any fundamental class $[\mathrm{BM}_{\sigma}]$ is a linear combination of classes $[\mathrm{BM}_{\tau_\ell}]$ with $\mathrm{rk}^{\mathcal{V}}(v^{\tau_\ell}_{i,k}) \leq 1$.

Lemma 5.2.5. *Let $g \geq 33$, $a, b, M \subseteq S_g$, $\vec{x} \in H_1(S_g;\mathbb{Z})$, and $\mathcal{V} \subseteq H_1(S_g;\mathbb{Z})$ be as in Lemma 5.2.2. Let BM_{σ} be a Bestvina–Margalit torus. Then there is a collection of 2–cells $\tau_1, \ldots, \tau_p \subseteq X_g$ that satisfy*

$$[\mathrm{BM}_{\sigma}] = \sum_{\ell=1}^{p} \lambda_\ell [\mathrm{BM}_{\tau_\ell}]$$

and such that $\mathrm{rk}^{\mathcal{V}}(v^{\tau_\ell}_{i,k}) \leq 1$ for every $1 \leq \ell \leq p$, $1 \leq i \leq 9$ and $k = 0, 1, 2$.

Proof. We induct on the number m with $0 \leq m \leq 9$ such that we can write $[\mathrm{BM}_{\sigma}]$ as a linear combination of classes $[\mathrm{BM}_{\tau_\ell}]$ that satisfy, for $1 \leq \ell \leq p$ and $1 \leq i \leq m$, the inequality $\mathrm{rk}^{\mathcal{V}}(v^{\tau_s}_{i,k}) \leq 1$.

Base case: $m = 0$. In this case, the result holds trivially, since there are no i with $1 \leq i$ and $i \leq m$.

Inductive step: $m \geq 1$. Inductively, assume that $[\mathrm{BM}_\sigma]$ can be written as a linear combination of classes $[\mathrm{BM}_{\tau_\ell}]$ that satisfy, for $1 \leq \ell \leq p$ and $1 \leq i \leq m$, the inequality $\mathrm{rk}^{\mathcal{V}}(v_{i,k}^{\tau_s}) \leq 1$. Hence without loss of generality, we may assume that BM_σ satisfies $\mathrm{rk}^{\mathcal{V}}(v_{i,k}^{\sigma}) \leq 1$ for $1 \leq i < m$. We will show that $[\mathrm{BM}_\sigma]$ is a linear combination of fundamental classes $[\mathrm{BM}_\tau]$, each of which has $\mathrm{rk}^{\mathcal{V}}(v_{i,k}^{\tau}) \leq 1$ for all $1 \leq i \leq m$ and $0 \leq k \leq 2$. We will perform a double induction on two quantities associated to σ:

- $\mathrm{maxrk}_m(\sigma) = \max_{0 \leq k \leq 2}\{\mathrm{rk}^{\mathcal{V}}(v_{m,k}^{\sigma})\}$, and
- $\mathrm{nummaxrk}_m(\sigma) = \left|\{k : 0 \leq k \leq 2, \mathrm{rk}^{\mathcal{V}}(v_{m,k}^{\sigma}) = \mathrm{maxrk}_m(\sigma)\}\right|$.

In particular, we suppose that σ is a 2–cell with $\mathrm{rk}^{\mathcal{V}}(v_{i,k}^{\sigma}) \leq 1$ for all $1 \leq i < m$ and $0 \leq k \leq 2$. We suppose that for each 2–cell τ with:

- $\mathrm{rk}^{\mathcal{V}}(v_{i,k}^{\tau}) \leq 1$ for $1 \leq i < m$ and $0 \leq k \leq 2$, and either:
  - $\mathrm{maxrk}_m(\tau) < \mathrm{maxrk}_m(\sigma)$, or
  - $\mathrm{maxrk}_m(\tau) \leq \mathrm{maxrk}_m(\sigma)$ and $\mathrm{nummaxrk}_m(\tau) < \mathrm{nummaxrk}_m(\sigma)$,

we know that $[\mathrm{BM}_\tau]$ is a $\mathbb{Q}$–linear combination of classes $[\mathrm{BM}_{\tau_\ell}]$ such that $\mathrm{rk}^{\mathcal{V}}(v_{i,k}^{\tau_\ell}) \leq 1$ for all $1 \leq i \leq m$ and $0 \leq k \leq 2$. We will show that this implies that the same holds for $[\mathrm{BM}_\sigma]$.

Base case: $\mathrm{maxrk}_m(\sigma) = 1$. In this case, BM_σ satisfies $\mathrm{rk}^{\mathcal{V}}(v_{i,k}^{\sigma}) \leq 1$ for all $1 \leq i \leq m$ and $0 \leq k \leq 2$, so the inductive step for the induction on m is complete.

Inductive step for $\mathrm{maxrk}_m(\sigma)$ *and* $\mathrm{nummaxrk}_m(\sigma)$*: The inductive hypothesis for the induction on* m *holds for all* $\tau \subseteq X_g$ *with:*

- *either* $\mathrm{maxrk}_m(\tau) < \mathrm{maxrk}_m(\sigma)$ *or*

- *both* $\mathrm{maxrk}_m(\tau) \leq \mathrm{maxrk}_m(\sigma)$ *and* $\mathrm{nummaxrk}_m(\tau) < \mathrm{nummaxrk}_m(\sigma)$.

We will show that

$$[\mathrm{BM}_\sigma] = \sum_{\ell=1}^{p} \lambda_\ell [\mathrm{BM}_{\tau_\ell}]$$

such that the following hold:

- at least one of the following holds:
  - $\mathrm{maxrk}_m(\sigma_\iota) < \mathrm{maxrk}_m(\sigma)$ or
  - $\mathrm{maxrk}_m(\sigma_\iota) \leq \mathrm{maxrk}_m(\sigma)$ and $\mathrm{nummaxrk}_m(\sigma_\iota) < \mathrm{nummaxrk}_m(\sigma)$,
- $\lambda_\ell \in \mathbb{Q}$, and
- $\mathrm{rk}^{\mathcal{V}}(v_{i,k}^{\tau_\ell}) \leq 1$ for all$1 \leq i < m$ and $k = 0, 1, 2$.

This completes the proof, since every τ_ℓ as above satisfies the inductive hypothesis for the induction on m by the inductive hypothesis for the double induction on $\mathrm{maxrk}_m(\sigma)$ and $\mathrm{nummaxrk}_m(\sigma)$.

If $\mathrm{rk}^{\mathcal{V}}(v_{m,k}^{\sigma}) = 0$ for two distinct choices of k, then $\mathrm{rk}^{\mathcal{V}}(v_{m,k}^{\sigma}) = 1$ for the third choice of k, since $v_m = v_{m,0}^{\sigma} + v_{m,1}^{\sigma} + v_{m,2}^{\sigma}$ and v_m is primitive by assumption. Otherwise, reindex $\mathcal{H}(\mathrm{BM}_\sigma)$ such that $\mathrm{rk}^{\mathcal{V}}(v_{m,2}^{\sigma})$ is maximal among all $\mathrm{rk}^{\mathcal{V}}(v_{m,k}^{\sigma})$. Note that at least one remaining $\mathrm{rk}^{\mathcal{V}}(v_{m,i}^{\sigma})$ must have $\mathrm{rk}^{\mathcal{V}}(v_{m,i}^{\sigma}) \neq 0, \neq \mathrm{rk}^{\mathcal{V}}(v_{m,2}^{\sigma})$ since v_m is primitive by assumption. Hence, we may further reindex so that $\mathrm{rk}^{\mathcal{V}}(v_{m,1}^{\sigma}) \neq 0, \mathrm{rk}^{\mathcal{V}}(v_{m,2}^{\sigma})$. Then σ satisfies the hypothesis of Lemma 5.2.4, so the conclusion of Lemma 5.2.4 implies that we may rewrite $[\mathrm{BM}_\sigma]$ as a linear combination of fundamental classes of Bestvina–Margalit tori such that each torus $\mathrm{BM}_{\sigma_\iota}$ satisfies the following:

1. $\mathrm{rk}^{\mathcal{V}}(v_{i,k}^{\sigma_\iota}) \leq 1$ for all $1 \leq i < m$ and $0 \leq k \leq 2$,
2. $\mathrm{maxrk}_m(\sigma_\iota) \leq \mathrm{maxrk}_m(\sigma)$,
3. if $\mathrm{maxrk}_m(\sigma_\iota) = \mathrm{maxrk}_m(\sigma)$, then $\mathrm{nummaxrk}_m(\sigma_\iota) \leq \mathrm{nummaxrk}_m(\sigma)$,

4. if $\mathrm{maxrk}_m(\sigma_\iota) = \mathrm{rk}^{\mathcal{V}}(v^{\sigma}_{m,2})$, then $0 < \mathrm{rk}^{\mathcal{V}}(v^{\sigma_\iota}_{m,1}) < \mathrm{rk}^{\mathcal{V}}(v^{\sigma_\iota}_{m,2})$, and

5. $g(\mathcal{H}^{\sigma_\iota}_1) \geq 10$.

Therefore we may assume without loss of generality that $g(\mathcal{H}^{\sigma}_1) \geq 10$.

Now, since $g(\mathcal{H}^{\sigma}_1) \geq 10$ and $|\mathcal{V}| \leq 9$, there must be some nonzero primitive $h \in \mathcal{H}^{\sigma}_1 \cap [b]^{\perp}$ such that there is a multicurve $M_1 \subseteq S_g$ where $\{[c] : c \in M_1\} = \{h, w^{\sigma}_{1,1}, \ldots, w^{\sigma}_{9,1}\}$, where $w^{\sigma}_{i,1}$ is a primitive element with $v^{\sigma}_{i,1} = \lambda w^{\sigma}_{i,1}$ for some $\lambda \in \mathbb{Z}$. For $1 \leq i \leq 9$, let $h_i = v^{\sigma}_{i,1} - h$. Since $g(\mathcal{H}^{\sigma}_1) \geq 10$, there is a primitive subgroup $\mathcal{H} \subseteq \mathcal{H}^{\sigma}_1$ such that the following hold:

- $a \in \mathcal{H}$,
- $2g(\mathcal{H}) + 1 = \mathrm{rk}(\mathcal{H})$,
- $h_i \in \mathcal{H}$ for all $1 \leq i \leq 9$, and
- $h \in \mathcal{H}^{\perp}$.

Let $\widehat{\mathcal{H}} = \{\mathcal{H}^{\sigma}_0, \mathcal{H}, \mathcal{H}^{\sigma}_1 \cap \mathcal{H}^{\perp}, \mathcal{H}^{\sigma}_2\}$, which can be graphically represented as in Figure 5.2.

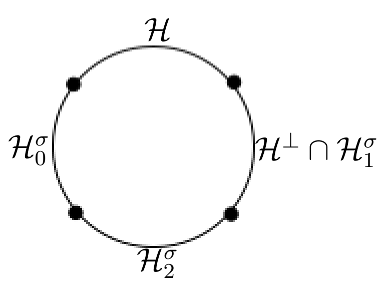

Figure 5.2: The graphical representation of $\widehat{\mathcal{H}}$

Let $y \subseteq X_g$ be the unique edge such that $\mathcal{H} \in \mathcal{H}(y)$. Let x be the edge of σ with $\mathcal{H}^{\sigma}_0 \in \mathcal{H}(x)$. There is a unique edge $z \subseteq X_g$ such that:

- y and z are two edges in a 2–cell τ and
- the third edge of τ, denoted yz, is the unique edge with $\mathcal{H}^{\sigma}_1 \in \mathcal{H}(yz)$.

Let σ_y and σ_z denote 2–cells with $x, y \subseteq \sigma_y$, $x, z \subseteq \sigma_z$. Assume that $\mathcal{H}(\sigma_y)$ is indexed so that $\mathcal{H}_0^{\sigma_y} = \mathcal{H}_0^{\sigma}$, $\mathcal{H}_1^{\sigma_y} = \mathcal{H}$, and $\mathcal{H}_2^{\sigma_y} = \mathcal{H}^{\perp} \cap \mathcal{H}_1^{\sigma} + \mathcal{H}_2^{\sigma}$. Our choice of $\mathcal{H}$ implies that $\mathrm{proj}_{\mathcal{H}_1^{\sigma_y} \cap [b]^{\perp}}(v_i) = h_i$ for all $1 \leq i \leq 9$. Since h is nonzero and primitive by assumption, the element h_i is nonzero and primitive as well for an $1 \leq i \leq 9$, so $\mathrm{rk}^{\mathcal{V}}(v_{i,1}^{\sigma_y}) = 1$ for all $1 \leq i \leq 9$. Assume now that $\mathcal{H}(\sigma_z)$ is indexed such that $\mathcal{H}_0^{\sigma_z} = \mathcal{H}_0^{\sigma}$, $\mathcal{H}_1^{\sigma_z} = \mathcal{H}^{\perp} \cap \mathcal{H}_1^{\sigma}$, and $\mathcal{H}_2^{\sigma_z} = \mathcal{H} + \mathcal{H}_2^{\sigma}$. This means that $\mathrm{proj}_{\mathcal{H}_1^{z} \cap [b]^{\perp}}(v_i) = h$ for all $1 \leq i \leq 9$, so $\mathrm{rk}^{\mathcal{V}}(v_{i,1}^{\sigma_z}) = 1$ for all $1 \leq i \leq 9$, since h is primitive by assumption. Furthermore, for such y and z, after possibly reorienting σ_y and σ_z, we have

$$[\mathrm{BM}_{\sigma}] = [\mathrm{BM}_{\sigma_y}] + [\mathrm{BM}_{\sigma_z}].$$

We have assumed that $\mathcal{H}(\sigma_y)$ and $\mathcal{H}(\sigma_z)$ are indexed so that $\mathcal{H}_0^{\sigma_y} = \mathcal{H}_0^{\sigma_z} = \mathcal{H}_0^{\sigma}$. Relations (1)–(4) of Lemma 5.2.3 and the above computations of $\mathrm{rk}^{\mathcal{V}}(v_{i,1}^{\sigma_y})$ and $\mathrm{rk}^{\mathcal{V}}(v_{i,1}^{\sigma_z})$ imply that the following hold for all $1 \leq i \leq 9$:

1. $\mathrm{rk}^{\mathcal{V}}(v_{i,0}^{\sigma_y}) = \mathrm{rk}^{\mathcal{V}}(v_{i,0}^{\sigma_z}) = \mathrm{rk}^{\mathcal{V}}(v_{i,0}^{\sigma})$,
2. $\mathrm{rk}^{\mathcal{V}}(v_{i,1}^{\sigma}) = \gcd(\mathrm{rk}^{\mathcal{V}}(v_{i,1}^{\sigma_y}), \mathrm{rk}^{\mathcal{V}}(v_{i,1}^{\sigma_z})) = 1$,
3. $\mathrm{rk}^{\mathcal{V}}(v_{i,2}^{\sigma_y}) = \gcd(\mathrm{rk}^{\mathcal{V}}(v_{i,1}^{\sigma_z}), \mathrm{rk}^{\mathcal{V}}(v_{i,2}^{\sigma})) = \gcd(1, \mathrm{rk}^{\mathcal{V}}(v_{i,2}^{\sigma})) = 1$, and
4. $\mathrm{rk}^{\mathcal{V}}(v_{i,2}^{\sigma_z}) = \gcd(\mathrm{rk}^{\mathcal{V}}(v_{i,1}^{\sigma_y}), \mathrm{rk}^{\mathcal{V}}(v_{i,2}^{\sigma})) = \gcd(1, \mathrm{rk}^{\mathcal{V}}(v_{i,2}^{\sigma})) = 1$.

We have assumed that $1 \leq \mathrm{rk}^{\mathcal{V}}(v_{m,1}^{\sigma}) < \mathrm{rk}^{\mathcal{V}}(v_{m,2}^{\sigma})$, so we must have:

- $1 = \mathrm{rk}^{\mathcal{V}}(v_{m,2}^{\sigma_y}) < \mathrm{rk}^{\mathcal{V}}(v_{m,2}^{\sigma})$ and
- $1 = \mathrm{rk}^{\mathcal{V}}(v_{m,2}^{\sigma_z}) < \mathrm{rk}^{\mathcal{V}}(v_{m,2}^{\sigma})$.

Then relations (1)–(4) together imply the following:

- For any pair i, k with $1 \leq i \leq 9$ and $0 \leq k \leq 2$ with $\mathrm{rk}^{\mathcal{V}}(v_{i,k}^{\sigma}) \leq 1$, we have $\mathrm{rk}^{\mathcal{V}}(v_{i,k}^{\sigma_y}), \mathrm{rk}^{\mathcal{V}}(v_{i,k}^{\sigma_z}) \leq 1$, and

- for any pair i, k with $1 \leq i \leq 9$ and $0 \leq k \leq 2$ with $\mathrm{rk}^{\mathcal{V}}(v^{\sigma}_{i,k} \geq 1$, we have $\mathrm{rk}^{\mathcal{V}}(v^{\sigma_y}_{i,k}), \mathrm{rk}^{\mathcal{V}}(v^{\sigma_z}_{i,k}) \leq \mathrm{rk}^{\mathcal{V}}(v^{\sigma}_{i,k})$.

Therefore, since we have assumed that $\mathrm{rk}^{\mathcal{V}}(v^{\sigma}_{m,2})$ was maximal over all $\mathrm{rk}^{\mathcal{V}}(v^{\sigma}_{m,k})$, we see that the two sets $\{\mathrm{rk}^{\mathcal{V}}(v^{\sigma_y}_{i,k})\}_{1 \leq i \leq m, k=0,1,2}$ and $\{\mathrm{rk}^{\mathcal{V}}(v^{\sigma_z}_{i,k})\}_{1 \leq i \leq m, k=0,1,2}$ each have strictly fewer elements equal to or exceeding $\mathrm{rk}^{\mathcal{V}}(v^{\sigma}_{m,2})$ than does the set $\{\mathrm{rk}^{\mathcal{V}}(v^{\sigma}_{i,k})\}_{1 \leq i \leq m, k=0,1,2}$. Hence for y, we have:

- $\mathrm{rk}^{\mathcal{V}}(v^{\sigma_y}_{i,k}) \leq 1$ for all $1 \leq i < m$ and $0 \leq k \leq 2$, and
- either:
  - $\mathrm{maxrk}_m(\sigma_y) < \mathrm{maxrk}_m(\sigma)$, or
  - $\mathrm{maxrk}_m(\sigma_y) \leq \mathrm{maxrk}_m(\sigma)$ and $\mathrm{nummaxrk}_m(\sigma_y) < \mathrm{nummaxrk}_m(\sigma)$

and similarly for z. Therefore we have

$$[\mathrm{BM}_{\sigma}] = [\mathrm{BM}_{\sigma_y}] + [\mathrm{BM}_{\sigma_z}]$$

such that σ_y and σ_z satisfy the inductive hypothesis for the induction on maxrk_m and $\mathrm{nummaxrk}_m$. The inductive hypothesis for maxrk_m and $\mathrm{nummaxrk}_m$ says that there are relations

$$[\mathrm{BM}_{\sigma_y}] = \sum_{\ell_y=1}^{p_y} \lambda_{\ell_y} [\mathrm{BM}_{\tau_{\ell_y}}] \text{ and } [\mathrm{BM}_{\sigma_z}] = \sum_{\ell_z=1}^{p_z} \lambda_{\ell_z} [\mathrm{BM}_{\tau_{\ell_z}}]$$

such that for each $1 \leq \ell_y \leq p_y$ and for each $1 \leq i \leq m$, $0 \leq k \leq 2$, we have $\mathrm{rk}^{\mathcal{V}}(v^{\tau_{\ell_y}}_{i,k}) \leq 1$, and similarly for z. By combining these two relations, we have

$$[\mathrm{BM}_{\sigma}] = \sum_{\ell_y=1}^{p_y} \lambda_{\ell_y} [\mathrm{BM}_{\tau_{\ell_y}}] + \sum_{\ell_z=1}^{p_z} \lambda_{\ell_z} [\mathrm{BM}_{\tau_{\ell_z}}] = \sum_{\ell=1}^{p} \lambda_{\ell} [\mathrm{BM}_{\tau_{\ell}}]$$

where for each $1 \leq \ell \leq p$ and $1 \leq i \leq m$ and $0 \leq k \leq 2$, we have $\mathrm{rk}^{\mathcal{V}}(v^{\tau_\ell}_{i,k}) \leq 1$, so the inductive step is complete. □

5.2.2 The proof of Lemma 5.2.7

We are now almost ready to show that fundamental classes $[\mathrm{BM}_\sigma]$ with $\left|\mathrm{rk}^{\mathcal{V}}(v^\sigma_{i,k})\right| \leq 1$ can be written as linear combinations of classes $[\mathrm{BM}_{\tau_s}]$ with $|\theta(\mathcal{V})_{i,j,k}(\tau_s)| \leq 1$. We first prove the following lemma.

Lemma 5.2.6. *Let $g \geq 33$, $a, b, M \subseteq S_g$, $\vec{x} \in H_1(S_g;\mathbb{Z})$ and $\mathcal{V} \subseteq H_1(S_g;\mathbb{Z})$ be as in Lemma 5.2.2. Let $\mathrm{BM}_\sigma \subseteq X_g$ be a Bestvina–Margalit torus such that $\mathrm{rk}^{\mathcal{V}}(v^\sigma_{i,k}) \leq 1$ for every $1 \leq i \leq 9$ and $0 \leq k \leq 2$. Assume that not all pairs $1 \leq i, j \leq 9$ satisfy $|\theta(\mathcal{V})_{i,j,k}(\sigma)| \leq 1$. Choose a pair (i', j') with $1 \leq i' < j' \leq 9$ such that, after possibly reindexing $\mathcal{H}(\mathrm{BM}_\sigma)$, $|\theta(\mathcal{V})_{i',j',1}(\sigma)| \geq 2$ and $|\theta(\mathcal{V})_{i',j',1}(\sigma)|$ is maximal in the set $\{|\theta(\mathcal{V})_{i',j',k}(\sigma)| : 0 \leq k \leq 2\}$. Then BM_σ is a $\mathbb{Q}$–linear combination of classes $[\mathrm{BM}_{\sigma_\iota}]$ for $1 \leq \iota \leq s$ such that the following hold for all $1 \leq \iota \leq s$:*

1. *$\mathrm{rk}^{\mathcal{V}}(v^{\sigma_\iota}_{i,k}) \leq 1$ for all $1 \leq i \leq 9$ and $0 \leq k \leq 2$,*
2. *the set $\{|\theta(\mathcal{V})_{i',j',k}(\sigma_\iota)| : 0 \leq k \leq 2\}$ is bounded above by $|\theta(\mathcal{V})_{i',j',1}(\sigma)|$ and has no more elements equal to $|\theta(\mathcal{V})_{i',j',1}(\sigma)|$ than does the set $\{|\theta(\mathcal{V})_{i',j',k}(\sigma)| : 0 \leq k \leq 2\}$,*
3. *if the set $\{|\theta(\mathcal{V})_{i',j',k}(\sigma_\iota)| : 0 \leq k \leq 2\}$ has as many elements equal to $|\theta(\mathcal{V})_{i',j',1}(\sigma)|$ as does the set $\{|\theta(\mathcal{V})_{i',j',k}(\sigma)| : 0 \leq k \leq 2\}$, then $|\theta(\mathcal{V})_{i',j',1}(\sigma_\iota)| = |\theta(\mathcal{V})_{i',j',1}(\sigma)|$,*
4. *for every pair $1 \leq i < j \leq 9$ with $|\theta(\mathcal{V})_{i,j,k}(\sigma)| \leq 1$ for every $0 \leq k \leq 2$, we have $|\theta(\mathcal{V})_{i,j,k}(\sigma_\iota)| \leq 1$ for all $0 \leq k \leq 2$, and*
5. *$g(\mathcal{H}^{\sigma_\iota}_1) \geq 11$.*

Proof. If $g(\mathcal{H}^\sigma_1) \geq 11$, then the lemma is trivially true by taking $s = 1$ and $\mathrm{BM}_{\sigma_1} = \mathrm{BM}_\sigma$. Otherwise, since $g \geq 33$ there is a $k = 0, 2$ with $g(\mathcal{H}^\sigma_k) \geq 21$. Without loss of generality, assume that $g(\mathcal{H}^\sigma_0) \geq 21$. Since $g(\mathcal{H}^\sigma_0) \geq 21$ and $|\mathcal{V}| = 9$, there is a primitive subgroup $\mathcal{H} \subseteq \mathcal{H}^\sigma_0$ such that the following hold:

- $a \in \mathcal{H}$,
- $v^{\sigma}_{i,k} \in \mathcal{H}^{\perp}$ for all $1 \leq i \leq 8$,
- $g(\mathcal{H}) - g(\mathcal{H}^{\sigma}_1) \geq 9$,
- $g(\mathcal{H}) \geq 11$, and
- $2 * g(\mathcal{H}) + 1 = \mathrm{rk}^{\mathcal{V}}(\mathcal{H})$.

Let $\widehat{\mathcal{H}} = \{\mathcal{H}^{\sigma}_0, \mathcal{H}, \mathcal{H}^{\perp} \cap \mathcal{H}^{\sigma}_1, \mathcal{H}^{\sigma}_2\}$. We can describe $\widehat{\mathcal{H}}$ using the graphical notation of Section 4.2 as in Figure 5.3.

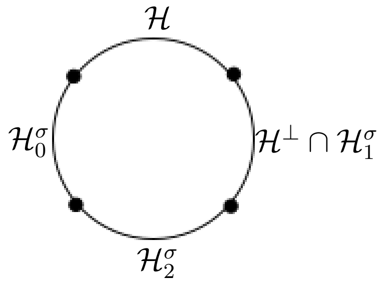

Figure 5.3: The graphical representation of $\widehat{\mathcal{H}}$

Now, let $z \subseteq X_g$ be the unique edge with $\mathcal{H} \in \mathcal{H}(z)$. Let $y \subseteq X_g$ be the unique edge with $\mathcal{H}^{\perp} \cap \mathcal{H}^{\sigma}_0 \in \mathcal{H}(y)$. Reorient y and z so that yz is the unique edge with $\mathcal{H}^{\sigma}_0 \in \mathcal{H}(yz)$. Let $x \subseteq X_g$ be the unique edge with $\mathcal{H}^{\sigma}_2 \in \mathcal{H}(x)$. For $* = y, z$, let $\sigma_* \subseteq X_g$ be a 2–cell containing x and $*$. By Lemma 4.2.2, there is a relation

$$[\mathrm{BM}_{\sigma}] = [\mathrm{BM}_{\sigma_y}] + [\mathrm{BM}_{\sigma_z}].$$

We now show that BM_{σ_y} and BM_{σ_z} have the desired properties of the tori $\mathrm{BM}_{\sigma_\iota}$.

BM_{σ_z} *satisfies properties (1) through (5) of* σ_ι. Assume that $\mathcal{H}(\sigma_z)$ is indexed so that $\mathcal{H} = \mathcal{H}^{\sigma_z}_0$, $\mathcal{H}^{\perp} \cap \mathcal{H}^{\sigma}_0 + \mathcal{H}^{\sigma}_1 = \mathcal{H}^{\sigma_z}_1$, and $\mathcal{H}^{\sigma}_2 = \mathcal{H}^{\sigma_z}_2$. Now, we have chosen $\mathcal{H}$ such that $\mathrm{proj}_{\mathcal{H} \cap [b]^{\perp}}(v_i) = 0$ for all $1 \leq i \leq 9$. Furthermore, for $k = 1$, we have $\mathrm{proj}_{\mathcal{H}^{\sigma_z}_1 \cap [b]^{\perp}}(v_i) = v^{\sigma}_{i,0} + v^{\sigma}_{i,1}$. For $k = 2$, we have $\mathrm{proj}_{\mathcal{H}^{\sigma_z}_2 \cap [b]^{\perp}}(v_i) = v^{\sigma}_{i,2}$. Each $v^{\sigma}_{i,k}$ is primitive by our hypothesis that $\mathrm{rk}^{\mathcal{V}}(v^{\sigma}_{i,k}) \leq 1$, so each $v^{\sigma_z}_{i,k}$ is primitive as well. Therefore BM_{σ_z} satisfies property (1) of σ_ι. Furthermore, the following hold for σ_z:

1. $\theta(\mathcal{V})_{i,j,0}(\sigma_z) = 0$ for all $1 \leq i, j \leq 9$,

2. $|\theta(\mathcal{V})_{i,j,1}(\sigma_z)| = |\theta(\mathcal{V})_{i,j,1}(\sigma) + \theta(\mathcal{V})_{i,j,0}(\sigma)|$ for all $1 \leq i, j \leq 9$,

3. $\theta(\mathcal{V})_{i,j,2}(\sigma_z) = \theta(\mathcal{V})_{i,j,2}(\sigma)$ for all $1 \leq i, j \leq 9$, and

4. $g(\mathcal{H}_1^{\sigma_z}) = g(\mathcal{H}_1^{\sigma}) + g(\mathcal{H}_0^{\sigma}) - g(\mathcal{H})$.

Now, observe that since $\langle v_i, v_j \rangle = 0$, if $1 \leq i < j \leq 9$ is a pair such that $\theta(\mathcal{V})_{i,j,k}(\sigma) \in \{-1, 0, 1\}$ for every $0 \leq k \leq 2$, then either $\theta(\mathcal{V})_{i,j,k}(\sigma) = 0$ for $0 \leq k \leq 2$, or $\{\theta(\mathcal{V})_{i,j,k}(\sigma) : 0 \leq k \leq 2\} = \{-1, 0, 1\}$. Therefore, relations (1), (2) and (3) in the above list for relations among $\theta(\mathcal{V})_{i,j,k}(\sigma_z)$ imply that BM_{σ_z} satisfies property (4) of the lemma. For property (2), observe that the fact that $\langle v_{i'}, v_{j'} \rangle = 0$ implies that

$$\theta(\mathcal{V})_{i',j',0}(\sigma) + \theta(\mathcal{V})_{i',j',1}(\sigma) + \theta(\mathcal{V})_{i',j',2}(\sigma) = 0$$

Since $|\theta(\mathcal{V})_{i',j',1}(\sigma)|$ is maximal among $|\theta(\mathcal{V})_{i',j',k}(\sigma)|$, we see that the integers $\theta(\mathcal{V})_{i',j',1}(\sigma)$ and $\theta(\mathcal{V})_{i',j',0}(\sigma)$ must have opposite signs. Therefore

$$|\theta(\mathcal{V})_{i',j',1}(\sigma_z)| = |\theta(\mathcal{V})_{i',j',1}(\sigma) + \theta(\mathcal{V})_{i',j',1}(\sigma)| \leq |\theta(\mathcal{V})_{i',j',1}(\sigma)|,$$

so relations (1), (2), and (3) imply that BM_{σ_z} satisfies property (2) of σ_ι. For property (3), relations (1)-(3) imply that the only way for the set

$$\{|\theta(\mathcal{V})_{i',j',k}(\sigma_z)| : 0 \leq k \leq 2\}$$

to have as many elements equal to $|\theta(\mathcal{V})_{i',j',1}(\sigma)|$ as does $\{|\theta(\mathcal{V})_{i',j',k}(\sigma)| : 0 \leq k \leq 2\}$ is for $\theta(\mathcal{V})_{i',j',1}(\sigma_z) = \theta(\mathcal{V})_{i',j',1}(\sigma)$, so property (3) must hold. Then since $g(\mathcal{H}_0^{\sigma}) - g(\mathcal{H}) \geq 21 - 11 = 10$ by assumption, relation (4) in the list of relations above implies that $g(\mathcal{H}_1^{\sigma_z}) = g(\mathcal{H}_0^{\sigma}) - g(\mathcal{H}) + g(\mathcal{H}_1^{\sigma}) \geq 10 + 1 \geq 11$, so σ_z satisfies property (5) of σ_ι.

BM_{σ_y} *satisfies hypothesis (1)-(5) of* σ_ι. Assume that $\mathcal{H}(\sigma_y)$ is indexed so that $\mathcal{H}^{\perp} \cap \mathcal{H}_0^{\sigma} = \mathcal{H}_0^{\sigma_y}$, $\mathcal{H} + \mathcal{H}_1^{\sigma} = \mathcal{H}_1^{\sigma_y}$, and $\mathcal{H}_2^{\sigma_y} = \mathcal{H}_2^{\sigma}$. We have $\mathrm{proj}_{\mathcal{H}^{\perp} \cap \mathcal{H}_0(\sigma) \cap [b]^{\perp}}(v_i) = v_{i,0}^{\sigma}$ for all

$1 \le i \le 9$ by our choice of $\mathcal{H}$. Since we have $\mathcal{H}_1^{\sigma_y} = \mathcal{H} + \mathcal{H}_1^{\sigma}$, our choice of $\mathcal{H}$ implies that $\mathrm{proj}_{\mathcal{H}_1^{\sigma_y} \cap [b]^\perp}(v_i) = v_{i,1}^{\sigma}$. Then we also have $\mathcal{H}_2^{\sigma_y} = \mathcal{H}_2^{\sigma}$ by construction, so the following hold:

1. $\theta(\mathcal{V})_{i,j,k}(\sigma) = \theta(\mathcal{V})_{i,j,k}(\sigma_y)$ for all $1 \le i \le 9$ and $0 \le k \le 2$, and

2. $g(\mathcal{H}_1^{\sigma_y}) = g(\mathcal{H}_1^{\sigma}) + g(\mathcal{H})$.

Relation (1) implies that σ_y satisfies hypotheses (1), (2), (3), and (4) of σ_ι. Then since we have assumed that $g(\mathcal{H}) \ge 10$ and we must have $g(\mathcal{H}_1^{\sigma}) \ge 1$, we have $g(\mathcal{H}_1^{\sigma_y}) \ge 1 + 10 = 11$, so BM_{σ_y} satisfies property (4) of $\mathrm{BM}_{\sigma_\iota}$.

Now, we have shown that $[\mathrm{BM}_\sigma] = [\mathrm{BM}_{x,y}] + [\mathrm{BM}_{x,z}]$. The lemma now follows by taking $s = 2$ and $\tau_1 = \sigma_y$, $\tau_2 = \sigma_z$. □

We now show that we can rewrite $[\mathrm{BM}_\sigma]$ as a linear combination of classes $[\mathrm{BM}_\tau]$ with $|\theta(\mathcal{V})_{i,j,k}(\tau)|$ bounded.

Lemma 5.2.7. *Let $g \ge 33$, $a, b, M \subseteq S_g$ and $\mathcal{V} \subseteq H_1(S_g; \mathbb{Z})$ be as in Lemma 5.2.2. Let $\mathrm{BM}_\sigma \subseteq X_g$ be a Bestvina–Margalit torus such that $\mathrm{rk}^{\mathcal{V}}(v_{i,k}^{\sigma}) \le 1$ for every $1 \le i \le 9$ and $0 \le k \le 2$. Then BM_σ is a linear combination of classes $[\mathrm{BM}_{\tau_s}]$ for $1 \le s \le m$ such that*

$$|\theta(\mathcal{V})_{i,j,k}(\tau_s)| \le 1 \text{ and } \mathrm{rk}^{\mathcal{V}}(v_{i,k}^{\tau_s}) \le 1$$

for every $1 \le i \le j \le 9$, $0 \le k \le 2$ and $1 \le s \le m$.

Proof. The proof follows by double induction on i and j. In particular, for some (i, j), we assume that BM_σ has the property that

$$|\theta(\mathcal{V})_{i',j',k}(\tau_s)| \le 1$$

for every $0 \le k \le 2$ and for every $(i', j') < (i, j)$, where $<$ is the dictionary ordering. We will show that $[\mathrm{BM}_\sigma]$ is a linear combination of fundamental classes $[\mathrm{BM}_{\tau_1}], \ldots, [\mathrm{BM}_{\tau_m}]$

such that

$$|\theta(\mathcal{V})_{i',j',k}(\tau_s)| \leq 1$$

for all $0 \leq k \leq 2$, for all $1 \leq s \leq m$ and for all $(i', j') \leq (i, j)$, and $\mathrm{rk}^{\mathcal{V}}(v^{\tau_s}_{i',k}) \leq 1$ for all $1 \leq i' \leq 9$ and $0 \leq k \leq 2$.

Base case: $(i, j) = (1, 1)$. We have $|\theta(\mathcal{V})_{1,1,k}(\sigma)| = 0$.

Inductive step: $(i, j) > (1, 1)$. If $j = i$ then we have $\theta(\mathcal{V})_{i,j,k}(\sigma) = 0$ for $0 \leq k \leq 2$, so assume $i < j$. We will perform a double induction on the following quantities:

1. $\mathrm{maxalg}_{i,j}(\sigma) = \max_{0 \leq k \leq 2} \{|\theta(\mathcal{V})_{i,j,k}(\sigma)|\}$,
2. $\mathrm{nummaxalg}_{i,j}(\sigma) = \big|\{0 \leq k \leq 2 : |\theta(\mathcal{V})_{i,j,k}(\sigma)| = \mathrm{maxalg}_{i,j}(\sigma)\}\big|$.

Reindex $\mathcal{H}(\sigma)$ so that $|\theta(\mathcal{V})_{i,j,1}(\sigma)|$ is maximal in the set $\{|\theta(\mathcal{V})_{i,j,k}(\sigma)| : 0 \leq k \leq 2\}$. If $|\theta(\mathcal{V})_{i,j,1}(\sigma)| \leq 1$ we are done, so assume $|\theta(\mathcal{V})_{i,j,1}(\sigma)| \geq 2$. Lemma 5.2.6 says that we can rewrite $[\mathrm{BM}_\sigma]$ as a linear combination of fundamental classes $[\mathrm{BM}_{\sigma_\iota}]$, where:

- hypothesis (1) implies that $\mathrm{rk}^{\mathcal{V}}_{i',k}(\sigma_\iota) \leq 1$ for $1 \leq i' \leq 9$,
- hypothesis (2) implies that $\theta(\mathcal{V})_{i',j',k}(\sigma_\iota) \leq 1$ for all $(i', j') < (i, j)$ and $0 \leq k \leq 2$,
- hypothesis (3) implies that $\mathrm{maxalg}_{i,j}(\sigma_\iota) \leq \mathrm{maxalg}_{i,j}(\sigma)$,
- hypothesis (3) implies that $\mathrm{nummaxalg}_{i,j}(\sigma_\iota) \leq \mathrm{nummaxalg}_{i,j}(\sigma_\iota)$,
- hypothesis (4) implies that if both:
  - $\mathrm{maxalg}_{i,j}(\sigma) = \mathrm{maxalg}_{i,j}(\sigma_\iota)$ and
  - $\mathrm{nummaxalg}_{i,j}(\sigma) = \mathrm{nummaxalg}_{i,j}(\sigma_\iota)$,

  then $|\theta(\mathcal{V})_{i,j,1}(\sigma_\iota)|$ is maximal among the set $\{|\theta(\mathcal{V})_{i,j,k}(\sigma_\iota)|\, 0 \leq k \leq 2\}$, and
- hypothesis (4) implies $g(\mathcal{H}_1^{\sigma_\iota}) \geq 11$.

Hence we may assume without loss of generality that $g(\mathcal{H}_1^\sigma) \geq 10$. Choose elements $v'_{i,1}, v''_{i,1}, v'_{j,1}, v''_{j,1} \in \mathcal{H}_1$ such that the following hold:

1. $v_{i,1} = v'_{i,1} + v''_{i,1}$,

2. $v_{j,1} = v'_{j,1} + v''_{j,1}$,

3. for all (i, ℓ) and (ℓ, j) with $(i, \ell), (\ell, j) < (i, j)$, we have

$$\theta(\mathcal{V})_{i,\ell,1}(\sigma) = \langle v'_{i,1}, v_{\ell,1}\rangle \text{ and } \theta(\mathcal{V})_{\ell,j,1}(\sigma) = \langle v_{\ell,1}, v'_{j,1}\rangle.$$

4. $\left|\langle v'_{i,1}, v'_{j,1}\rangle\right| = |\theta(\mathcal{V})_{i,j,1}(\sigma)| - 1$,

5. $\langle v''_{i,1}, v'_{j,1}\rangle = \langle v''_{j,1}, v'_{i,1}\rangle = \langle v''_{i,1}, v'_{i,1}\rangle = \langle v''_{j,1}, v'_{j,1}\rangle = 0$,

6. $\langle v''_{i,1}, v_{\ell,1}\rangle = \langle v''_{j,1}, v_{\ell,1}\rangle = 0$ for all $\ell \neq i, j$, and

7. $\left|\langle v''_{i,1}, v''_{j,1}\rangle\right| = 1$

Such elements exist since $g(\mathcal{H}) \geq 11$ and $|\mathcal{V}| = 9$, so we can find $v''_{i,1}, v''_{j,1}$ with their desired algebraic intersections, and then $v'_{i,1} = v_{i,1} - v''_{i,1}$, $v'_{j,1} = v_{j,1} - v''_{j,1}$. Since $g(\mathcal{H}_1) \geq 11$ and $v_{i,1}$, $v_{j,1}$ are primitive by hypothesis, we may choose primitive subgroups $\mathcal{H}'_1, \mathcal{H}''_1 \subseteq \mathcal{H}_1$ such that the following hold:

1. for any $x \in \mathcal{H}'_1$ and $y \in \mathcal{H}''_1$, we have $\langle x, y\rangle = 0$,

2. $\mathcal{H}''_1 = (\mathcal{H}'_1)^\perp \cap \mathcal{H}_1$,

3. $\mathcal{H}'_1 + \mathcal{H}''_1 = \mathcal{H}_1$,

4. $\mathcal{H}'_1 \cap \mathcal{H}''_1 = \mathbb{Z}\vec{x}$,

5. $v'_{i,1}, v'_{j,1}, v_{\ell,1} \in \mathcal{H}'_1$ for all $\ell \neq i, j$ and

6. $v''_{i,1}, v''_{j,1} \in \mathcal{H}''_1$.

Let $\widehat{\mathcal{H}} = \{\mathcal{H}_0^\sigma, \mathcal{H}_1', \mathcal{H}_1'', \mathcal{H}_2\}$. We can describe $\widehat{\mathcal{H}}$ using the graphical notation of Section 4.2 as in Figure 5.4.

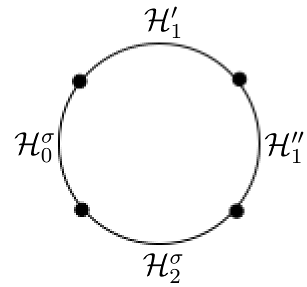

Figure 5.4: The graphical representation of $\widehat{\mathcal{H}}$

Let $\mathrm{BM}_{\sigma'}$ denote the Bestvina–Margalit torus corresponding to $\{\mathcal{H}_0, \mathcal{H}_1', \mathcal{H}_1''+\mathcal{H}_2\}$ and let $\mathrm{BM}_{\sigma''}$ denote the Bestvina–Margalit torus corresponding to $\{\mathcal{H}_0, \mathcal{H}_1'', \mathcal{H}_1' + \mathcal{H}_2\}$. We will compute $\theta(\mathcal{V})_{i',j',k}(\sigma')$ and $\theta(\mathcal{V})_{i',j',k}(\sigma'')$ as follows.

Computing $\theta(\mathcal{V})_{i',j',k}(\sigma')$. By our choice of $\mathcal{H}_1'$, we have

1. $v_{\ell,0}^{\sigma'} = v_{\ell,0}^{\sigma}$ for all $1 \leq \ell \leq 9$,

2. $v_{\ell,1}^{\sigma'} = v_{\ell,1}^{\sigma}$ for all $\ell \neq i, j$,

3. $v_{\ell,1}^{\sigma'} = v_{\ell,1}'$ for $\ell = 1, j$,

4. $v_{\ell,2}^{\sigma'} = v_{\ell,2}^{\sigma}$ for all $\ell \neq i, j$, and

5. $v_{\ell,2}^{\sigma'} = v_{\ell,2} + v_{\ell,1}''$ for $\ell = 1, j$.

Now, computation (1) implies that $\theta(\mathcal{V})_{i,j,0}(\sigma') = \theta(\mathcal{V})_{i,j,0}(\sigma)$ for all $1 \leq i, j \leq 9$. Computations (2) and (3) and the fact that $\langle v_{i,1}', v_{\ell,1}^{\sigma}\rangle = \langle v_{i,1}, v_{\ell,1}^{\sigma}\rangle$ and $\langle v_{\ell,1}^{\sigma}, v_{j,1}'\rangle = \langle v_{\ell,1}^{\sigma}, v_{j,1}\rangle$ for $\ell \neq i, j$ tell us that $\theta(\mathcal{V})_{i',j',1}(\sigma') = \theta(\mathcal{V})_{i',j',1}(\sigma)$ for all pairs $(i', j') \neq (i, j)$. Computation (3), our assumption that $\theta(\mathcal{V})_{i,j,1}(\sigma) \geq 1$, and the fact that $\langle v_{i,1}', v_{j,1}'\rangle = \theta(\mathcal{V})_{i,j,1}(\sigma) - 1$ implies that $|\theta(\mathcal{V})_{i,j,1}(\sigma')| < |\theta(\mathcal{V})_{i,j,1}(\sigma)|$. For $k = 2$, computations (4) and (5) say that $\theta(\mathcal{V})_{i',j',2}(\sigma') = \theta(\mathcal{V})_{i',j',2}(\sigma)$ for all $1 \leq i', j' \leq 9$. Finally, we see that $|\theta(\mathcal{V})_{i,j,2}(\sigma')| <$

$|\theta(\mathcal{V})_{i,j,2}(\sigma)|$ if $|\theta(\mathcal{V})_{i,j,2}(\sigma)| \geq 1$. Indeed, since $\theta(\mathcal{V})_{i,j,1}(\sigma)$ is positive and has

$$|\theta(\mathcal{V})_{i,j,1}(\sigma)| \geq |\theta(\mathcal{V})_{i,j,2}(\sigma)|,$$

the fact that $\langle v_i, v_j \rangle = 0$ allows us to conclude that $\theta(\mathcal{V})_{i,j,2}(\sigma) \leq 0$. Then $\langle v''_{i,1}, v''_{j,1} \rangle = 1$, so $|\theta(\mathcal{V})_{i,j,2}(\sigma')| = |\theta(\mathcal{V})_{i,j,2}(\sigma) + 1| < |\theta(\mathcal{V})_{i,j,2}(\sigma)|$ if $|\theta(\mathcal{V})_{i,j,2}(\sigma)| \leq 1$ by computation (5). Hence σ' has either $\mathrm{maxalg}(\sigma') < \mathrm{maxalg}(\sigma)$ or $\mathrm{maxalg}(\sigma') = \mathrm{maxalg}(\sigma)$ and $\mathrm{nummaxalg}(\sigma') < \mathrm{nummaxalg}(\sigma)$, so σ' satisfies the inductive hypothesis for the induction on maxalg and $\mathrm{nummaxalg}$.

Computing $\theta(\mathcal{V})_{i',j',k}(\sigma'')$. By our choice of $\mathcal{H}''_1$, we have

1. $v^{\sigma''}_{\ell,0} = v^{\sigma}_{\ell,0}$ for all $1 \leq \ell \leq 9$,
2. $v^{\sigma''}_{\ell,1} = 0$ for all $\ell \neq i, j$,
3. $v^{\sigma''}_{\ell,1} = v''_{\ell,1}$ for $\ell = 1, j$,
4. $v^{\sigma''}_{\ell,2} = v^{\sigma}_{\ell,2} + v^{\sigma}_{\ell,1}$ for all $\ell \neq i, j$, and
5. $v^{\sigma''}_{\ell,2} = v_{\ell,2} + v'_{\ell,1}$ for $\ell = 1, j$.

Then we have $\theta(\mathcal{V})_{i',j',0}(\sigma'') = \theta(\mathcal{V})_{i',j',0}(\sigma)$ for all $1 \leq i', j' \leq 9$ by computation (1), $\theta(\mathcal{V})_{i',j',0}(\sigma'') = 0$ for $(i', j') \neq (i, j)$ by computation (2), and $\theta(\mathcal{V})_{i,j,1}(\sigma'') = 1$ by computation (3). Then as in the previous paragraph, we see that $\theta(\mathcal{V})_{i',j',1}(\sigma)$ and $\theta(\mathcal{V})_{i',j',2}(\sigma)$ have opposite signs, so for any $1 \leq i', j' \leq 9$ we have either $|\theta(\mathcal{V})_{i',j',1}(\sigma'')| < |\theta(\mathcal{V})_{i',j',1}(\sigma)|$ or $|\theta(\mathcal{V})_{i',j',1}(\sigma'')| \leq 1$. Hence σ'' satisfies the inductive hypothesis as well.

Since $[\mathrm{BM}_\sigma] = [\mathrm{BM}_{\sigma'}] + [\mathrm{BM}_{\sigma''}]$ by Lemma 4.2.4, the inductive hypothesis applied to $\mathrm{BM}_{\sigma'}$ and $\mathrm{BM}_{\sigma''}$ completes the proof. □

We now complete the section.

Proof of Lemma 5.2.1. By Lemmas 5.2.5 and 5.2.7, any fundamental class $[\mathrm{BM}_\tau]$ is a linear combination of fundamental classes of Bestvina–Margalit tori BM_σ with $\left|\mathrm{rk}^{\mathcal{V}}(v^{\sigma}_{i,k})\right| \leq$

1 and $|\theta(\mathcal{V})_{i,j,k}(\sigma)| \leq 1$ for every $1 \leq i, j \leq 8$ and $0 \leq k \leq 2$. Hence by Lemma 5.2.2, the vector space $H_0(G; \mathrm{BM}_2(X_g; \mathbb{Q}))$ is finite dimensional. $\square$

5.3 Finite dimensionality of $\mathbf{H_2}(\mathcal{C}_{\vec{x}}(\mathbf{S_g})/\mathcal{I}_\mathbf{g}; \mathbb{Q})$

In this section, we will finish the proof of Proposition 1.3.1. If $w \in \vec{x}^\perp$, let X_g^w denote the full subcomplex of X_g generated by edges e such that the decomposition $\mathcal{H}(e)$ is *compatible* with w, i.e., $w \in \mathcal{H}_i^e$ for some $\mathcal{H}_i^e \in \mathcal{H}(e)$. The bulk of Section 5.3 is devoted to proving the following lemma.

Lemma 5.3.1. *Let $g \geq 33$ and $a \subseteq S_g$ a nonseparating simple closed curve. Let $\vec{x} = [a]$ and let $X_g = \mathcal{C}_{\vec{x}}(S_g)/\mathcal{I}_g$. Let $c \subseteq S_g \setminus a$ be a nonseparating simple closed curve, and let $w = [c]$. The cokernel of the pushforward map $H_2(X_g^w; \mathbb{Q}) \to H_2(X_g; \mathbb{Q})$ is a subquotient of* $\mathrm{BM}_2(X_g; \mathbb{Q})$.

For the remainder of this section, fix g, a, and c as in Lemma 5.3.1. Let $X_g^{w,2} \subseteq X_g$ be the subcomplex given by the union of all cells σ such that $\dim(\sigma \cap X_g^w) \geq \dim(\sigma) - 1$. Lemma 5.3.1 will proceed in two steps.

Step (1). We show that $\mathrm{cok}(H_2(X_g^{w,2}; \mathbb{Q}) \to H_2(X_g; \mathbb{Q}))$ is generated by the images of fundamental classes of Bestvina–Margalit tori.

Step (2). We show that $\mathrm{cok}(H_2(X_g^w; \mathbb{Q}) \to H_2(X_g^{w,2}; \mathbb{Q}))$ is generated by the images of fundamental classes of Bestvina–Margalit tori.

Outline of the proof of Lemma 5.3.10. Step (1) is recorded as Lemma 5.3.2, and Step (2) is recorded as Lemma 5.3.7. We will prove Lemma 5.3.2 in Section 5.3.1 and Lemma 5.3.7 on Section 5.3.2. Additionally in Section 5.3.2, we will assemble Lemmas 5.3.2 and 5.3.7 into the proof of Lemma 5.3.1. We will conclude with Section 5.3.3, where we prove Lemma 5.3.10 and Proposition 1.3.1.

5.3.1 Step (1) of the proof of Lemma 5.3.1

This step is recorded as the following lemma.

Lemma 5.3.2. *Let $g \geq 33$ and $a \subseteq S_g$ a nonseparating curve. Let $\vec{x} = [a]$. Let $w \in \vec{x}^{\perp}$ be a nonzero primitive homology class such that the image of w under the adjoint map $\vec{x}^{\perp} \to \mathrm{Hom}_{\mathbb{Z}}(\vec{x}^{\perp}, \mathbb{Z})$ is nontrivial. The cokernel of the pushforward map*

$$H_2(X_g^{w,2}; \mathbb{Q}) \to H_2(X_g; \mathbb{Q})$$

is generated by images of fundamental classes of Bestvina–Margalit tori.

Before proving Lemma 5.3.2, we will prove a collection of auxiliary lemmas. The main goal is to prove Lemma 5.3.3, which describes the cokernel of the H_2–pushforward of certain subcomplexes of X_g. Let $g, a, \vec{x}$, and w be as in the statement of Lemma 5.3.2. If $e \subseteq X_g$ is an edge with w incompatible with $\mathcal{H}(e)$, let $\widehat{U}_e$ denote the union of all 3-cells $\tau \subseteq X_g$ such that:

- $e \subseteq \tau$,
- there is $\mathcal{H}_0 \in \mathcal{H}(e)$ such that $\mathcal{H}_0 \in \mathcal{H}(\tau)$,
- there is a $\mathcal{H}' \in \mathcal{H}(\tau)$ with $\{\mathcal{H}', (\mathcal{H}')^{\perp}\}$ compatible with w, and
- $\dim(\tau \cap X_g^w) = 1$.

Let U_e denote the union of all 2–cells $\sigma \subseteq \widehat{U}_e$ such that the following hold:

- $\dim(\sigma \cap e) = 0$ and
- $\dim(\sigma \cap X_g^w) \geq 1$.

We will prove the following lemma.

Lemma 5.3.3. *Let $g \geq 33$, $a \subseteq S_g$, $\vec{x} = [a]$, and $w \in \vec{x}^{\perp}$ be as in Lemma 5.3.1. Let $e, f \subseteq X_g$ be two edges such that $\mathcal{H}(e)$ and $\mathcal{H}(f)$ are incompatible with w. Furthermore, assume that $g(e) = g(f) = \{1, g-2\}$. Then the following hold:*

1. *the cokernel of the pushforward map*

$$H_2(U_e; \mathbb{Q}) \to H_2(\widehat{U}_e; \mathbb{Q})$$

   is generated by the images of fundamental classes of Bestvina–Margalit tori,

2. *the pushforward*

$$H_1(U_e; \mathbb{Q}) \to H_1(\widehat{U}_e; \mathbb{Q})$$

   is an isomorphism, and

3. *the pushforward $H_1(U_e \cap U_f; \mathbb{Q}) \to H_1(\widehat{U}_e \cap \widehat{U}_f; \mathbb{Q})$ is surjective.*

We begin by recording the following result.

Lemma 5.3.4. *Let $\tau \subseteq X_g$ be a 3–cell. Let T denote the union of all $\tau' \subseteq X_g$ such that $\mathcal{H}(\tau) = \mathcal{H}(\tau')$ as unordered sets. Then T is a 3–torus.*

Proof. This is the 3–dimensional analogue of the standard description of the 2–torus as a union of two 2–cells. □

If T is a torus as in the statement of Lemma 5.3.4, we will use $\mathcal{H}(T)$ to denote the decomposition $\mathcal{H}(\tau)$, except with the order forgotten. In order to prove Lemma 5.3.3, we will prove the following result.

Lemma 5.3.5. *Let $g \geq 33$, $a \subseteq S_g$ be a nonseparating simple closed curve, $\vec{x} = [a]$, and $w \in \vec{x}^{\perp}$ as in Lemma 5.3.1. Let $e \subseteq X_g$ be an edge with $\mathcal{H}(e)$ not compatible with w. Let $\tau, \tau' \subseteq \widehat{U}_e$ be two 3–cells. Let T denote the union of all 3–cells $\tau'' \subseteq \widehat{U}_e$ such that $H(\tau) = H(\tau'')$ as unordered sets, and similarly T' and τ'. Then the following hold:*

1. *the pushforward* $H_1(T \cap U_e; \mathbb{Q}) \to H_1(T; \mathbb{Q})$ *is an isomorphism, and the cokernel of the map* $H_2(T \cap U_e; \mathbb{Q}) \to H_2(T; \mathbb{Q})$ *is generated by fundamental classes of Bestvina–Margalit tori,*

2. *if* $T \cap T'$ *contains an edge* f *such that* $f \neq e$ *and* $f \not\subseteq X_g^w$*, then pushforward* $H_1(T \cap T' \cap U_e; \mathbb{Q}) \to H_1(T \cap T'; \mathbb{Q})$ *is a surjection, and*

3. *if* $T \cap T'$ *does not contain an* f *as above, then the cokernel of the pushforward* $H_1(T \cap T' \cap U_e; \mathbb{Q}) \to H_1(T \cap T'; \mathbb{Q})$ *is generated by the image of the class* $[e]$.

Proof. We prove each statement in turn.

Statement (1) . Let $\mathcal{H}(\tau) = \{\mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3\}$ such that $\mathcal{H}_0 \in \mathcal{H}(e)$. Since T is independent of the cyclic ordering on $\mathcal{H}(\tau)$ by definition, we may assume without loss of generality that the decomposition of $\vec{x}^\perp$ given by $\{\mathcal{H}_2, \mathcal{H}_2^\perp\}$ is compatible with w. Note that if $\dim(\tau \cap X_g^w) = 2$, then this may not be the only edge of τ compatible with w. Given that $\{\mathcal{H}_2, \mathcal{H}_2^\perp\}$ is compatible with w, the intersection $T \cap U_e$ is given by 2–cells in X_g with the following decompositions: $\{\mathcal{H}_0 + \mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3\}$, $\{\mathcal{H}_0 + \mathcal{H}_3, \mathcal{H}_1, \mathcal{H}_2\}$, $\{\mathcal{H}_2, \mathcal{H}_0 + \mathcal{H}_1, \mathcal{H}_3\}$, and $\{\mathcal{H}_0 + \mathcal{H}_3, \mathcal{H}_2, \mathcal{H}_1\}$. This is a pair of Bestvina–Margalit tori corresponding to the unordered decompositions $\{\mathcal{H}_0 + \mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3\}$ and $\{\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_0 + \mathcal{H}_3\}$ that intersect in the edge $\{\mathcal{H}_2, \mathcal{H}_2^\perp\}$. In particular, the Mayer–Vietoris sequence implies that $H_1(T \cap U_e; \mathbb{Q}) \cong \mathbb{Q}^3$ and $H_2(T \cap U_e; \mathbb{Q}) \cong \mathbb{Q}^2$. Hence the map $H_1(T \cap U_e; \mathbb{Q}) \to H_1(T; \mathbb{Q})$ is an isomorphism and the cokernel of $H_2(T \cap U_e; \mathbb{Q}) \to H_2(T; \mathbb{Q})$ is generated by the image of the fundamental class $[\mathrm{BM}_\sigma]$ with $\mathcal{H}(\sigma) = \{\mathcal{H}_0, \mathcal{H}_1 + \mathcal{H}_2, \mathcal{H}_3\}$.

Statement (2). Reuse the indexing of the elements of $\mathcal{H}(\tau)$. Let $\mathcal{H}(\tau') = \{\mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2', \mathcal{H}_3'\}$ where $\mathcal{H}(f) = \{\mathcal{H}_1, \mathcal{H}_1^\perp\}$. We have $\mathcal{H}(T \cap T') = \{\mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_0^\perp \cap \mathcal{H}_1^\perp\}$. In particular, $H_1(T \cap T'; \mathbb{Q})$ is generated by the class $[f]$ and the class $[h]$, with $\mathcal{H}(h) = \{\mathcal{H}_0 + \mathcal{H}_1, \mathcal{H}_0^\perp \cap \mathcal{H}_1^\perp\}$. Then the edge f is contained in 2–cell corresponding to $\{\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_1 \cap \mathcal{H}_2^\perp\}$, which has an edge fixed by T_w and hence lies in U_e. Similarly h is contained in the 2–cell $\{\mathcal{H}_0 +$

$\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3\} \subseteq U_e$. There $[f], [h] \in H_1(T \cap T' \cap U_e; \mathbb{Q})$, and hence $H_1(T \cap T' \cap U_e; \mathbb{Q}) \to H_1(T \cap T'; \mathbb{Q})$ is surjective.

Statement (3). Reuse the indexing of $\mathcal{H}(\tau)$ from the previous case. If $\mathcal{H}_2 \in \mathcal{H}(\tau')$, then $T \cap T \cap U_e$ is the single edge corresponding to $\{\mathcal{H}_2, \mathcal{H}_2^\perp\}$, while the intersection $T \cap T$ is the Bestvina–Margalit torus containing this edge and e. Otherwise, $T \cap T' = e$ and so $T \cap T' \cap U_e$ is the unique vertex of X_g. In either case the lemma holds. □

Proof of Lemma 5.3.3. If $\sigma \subseteq U_e$ is a 2–cell with $e, \sigma \subseteq \tau \subseteq \widehat{U}_e$ for some 3–cell τ, then the union

$$\bigcup_{\tau' \subseteq \widehat{U}_e : \mathcal{H}(\tau') = \tau} \tau'$$

is a 3–torus. Let $\widehat{\mathcal{R}}_e$ denote the set of such 3–tori. This set is a simplicial cover of $\widehat{U}_e$ by construction. Let

$$\mathcal{R}_e = \{T \cap U_e : T \in \widehat{\mathcal{R}}_e\}.$$

By the definition of U_e, $\mathcal{R}_e$ is a cover of U_e. Let $\mathbb{E}^r_{p,q}(\mathcal{R}_e; \mathbb{Q})$ and $\mathbb{E}^r_{p,q}(\widehat{\mathcal{R}}_e; \mathbb{Q})$ denote the Čech–to–singular spectral sequences for $\mathcal{R}_e$ covering U_e and $\widehat{\mathcal{R}}_e$ covering $\widehat{U}_e$ respectively. Then, for $k = 1, 2$ the cokernel of $H_k(U_e; \mathbb{Q}) \to H_k(\widehat{U}_e; \mathbb{Q})$ is noncanonically isomorphic to

$$\bigoplus_{p+q=k} \operatorname{cok}(\mathbb{E}^\infty_{p,q}(\mathcal{R}_e; \mathbb{Q}) \to \mathbb{E}^\infty_{p,q}(\widehat{\mathcal{R}}_e; \mathbb{Q})).$$

Hence to prove the first statements in the lemma, it suffices to prove the following.

1. The vector space $\operatorname{cok}(\mathbb{E}^\infty_{0,2}(\mathcal{R}_e; \mathbb{Q}) \to \mathbb{E}^\infty_{0,2}(\widehat{\mathcal{R}}_e; \mathbb{Q}))$ is generated by fundamental classes of Bestvina–Margalit tori, and the maps $\mathbb{E}^\infty_{p,q}(\mathcal{R}_e; \mathbb{Q}) \to \mathbb{E}_{p,q}(\widehat{\mathcal{R}}_e; \mathbb{Q})$ are surjective for $p > 0$ and $p + q = 2$.

The proof of statement (1). We first show that $\operatorname{cok}(\mathbb{E}^\infty_{0,2}(\mathcal{R}_e; \mathbb{Q}) \to \mathbb{E}^\infty_{0,2}(\widehat{\mathcal{R}}_e; \mathbb{Q})$ is generated by the images of fundamental classes of Bestvina–Margalit tori. Since the differentials out of $\mathbb{E}^r_{0,2}(\mathcal{R}_e; \mathbb{Q})$ and $\mathbb{E}^r_{0,2}(\widehat{\mathcal{R}}_e; \mathbb{Q})$ vanish for all $r \geq 1$, it suffices to show that

$\mathrm{cok}(\mathbb{E}^1_{0,2}(\mathcal{R}_e;\mathbb{Q}) \to \mathbb{E}^1_{0,2}(\widehat{\mathcal{R}}_e;\mathbb{Q}))$ is generated by the images of fundamental classes of Bestvina–Margalit tori. Hence it is enough to show that for any $T \in \widehat{\mathcal{R}}_e$, the cokernel $H_2(T \cap U_e;\mathbb{Q}) \to H_2(T;\mathbb{Q})$ is generated by the images of fundamental classes of Bestvina–Margalit tori. This follows from case (1) of Lemma 5.3.3.

We now show that $\mathbb{E}^\infty_{p,q}(\mathcal{R}_e;\mathbb{Q}) \to \mathbb{E}^\infty_{p,q}(\widehat{\mathcal{R}}_e;\mathbb{Q})$ is surjective for $p > 0$ and $p+q=2$. Consider the map of chain complexes

$$\mathbb{E}^1_{*,1}(\mathcal{R}_e;\mathbb{Q}) \to \mathbb{E}^1_{*,1}(\widehat{\mathcal{R}}_e;\mathbb{Q}).$$

By Lemma 5.3.3, the map $\mathbb{E}^1_{0,1}(\mathcal{R}_e;\mathbb{Q}) \to \mathbb{E}^1_{*,1}(\widehat{\mathcal{R}}_e;\mathbb{Q})$ is injective. Hence it suffices to show that the vector space $H_1(\mathbb{E}^1_{*,1}(\widehat{\mathcal{R}}_e;\mathbb{Q}), \mathbb{E}^1_{*,1}(\widehat{\mathcal{R}}_e;\mathbb{Q}))$ is trivial. Let C_* denote the quotient complex $\mathbb{E}^1_{*,1}(\widehat{\mathcal{R}}_e;\mathbb{Q})/\mathbb{E}^1_{*,1}(\mathcal{R}_e;\mathbb{Q})$. By Lemma 5.3.3, if $T \cap T' \cap U_e$ contains an edge $f \not\subseteq X^w_g$, then the image of a class in $\mathbb{E}^1_{*,1}(\widehat{\mathcal{R}}_e;\mathbb{Q})$ supported on the index $T \cap T'$ is 0 in C_*. Then if no such f exists, Lemma 5.3.3 says that the cokernel $H_1(T \cap T' \cap U_e;\mathbb{Q}) \to H_1(T \cap T';\mathbb{Q})$ is generated by the image of the class $[e]$. Therefore it suffices to show that if $T, T' \in \widehat{\mathcal{R}}_e$, then there is a sequence $T_0, \ldots, T_n \in \widehat{\mathcal{R}}_e$ such that $T = T_0$, $T' = T_n$, and $T_i \cap T_{i+1}$ contains an edge $f_i \neq e$ and $f \not\subseteq X^w_g$. This will imply that the element in C_* given by $[e]$ supported on $T \cap T'$ is homologous to the element $[e]$ supported on $T_0 \cap T_1 + \ldots + T_{n-1} \cap T_n$, since the edge connecting T to T' is homologous to the path $T_0 \to T_1 \to \ldots \to T_n$ in the clique complex on the set $\widehat{\mathcal{R}}_e$. Since the latter classes are all trivial in C_*, this implies that $T \cap T'$ is trivial in $H_1(C_*)$ as well.

Let $\mathcal{H}(e) = \{\mathcal{H}_0, \mathcal{H}_0^\perp\}$ such that $g(\mathcal{H}_0) = g-2$. Let $\mathcal{H}(T) = \{\mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3\}$ such that $\{\mathcal{H}_3, \mathcal{H}_3^\perp\}$ is compatible with w. Choose $\mathcal{H}_1' \subseteq \mathcal{H}_1$ such that $g(\mathcal{H}_1') = 1$, $\{\mathcal{H}_1', (\mathcal{H}_1')^\perp\}$ is incompatible with w, and $w^\perp \supseteq (\mathcal{H}_1')^\perp \cap \mathcal{H}_1$. Let $\mathcal{H}(T_1) = \{\mathcal{H}_0, \mathcal{H}_1', \mathcal{H}_2, \mathcal{H}_3 + (\mathcal{H}_1 \cap (\mathcal{H}_1')^\perp)\}$. Note that $T_1 \cap T_0$ contains the edge $\{\mathcal{H}_2, \mathcal{H}_2^\perp\}$ which is not e and is not in X^w_g. Now, since $g(\mathcal{H}_1') = 1$ and $g \geq 33$, there is some $\mathcal{H}_1'' \in \mathcal{H}(T')$ such that there is an edge $h \subseteq U_e$ with $h \not\subseteq X^w_g$ such that for some $\mathcal{H}_h \in \mathcal{H}$, we have $\mathcal{H}_h \supseteq \mathcal{H}_1', \mathcal{H}_1''$,

$\mathcal{H}_h \subseteq \mathcal{H}_0^\perp$, and $w \in \mathcal{H}_h^\perp \cap \mathcal{H}_0^\perp$. Let $T_2 = \{\mathcal{H}_0, \mathcal{H}_1', \mathcal{H}_h, \mathcal{H}_h^\perp \cap \mathcal{H}_0^\perp \cap (\mathcal{H}_1')^\perp\}$ and $T_3 = \{\mathcal{H}_0, \mathcal{H}_1'', \mathcal{H}_h, \mathcal{H}_h^\perp \cap \mathcal{H}_0^\perp \cap (\mathcal{H}_1'')^\perp\}$. Then $T_1 \cap T_2$ contains the edge $\{\mathcal{H}_1', (\mathcal{H}_1')^\perp\}$, $T_2 \cap T_3$ contains the edge $\{\mathcal{H}_h, \mathcal{H}_h^\perp\}$, and $T_3 \cap T'$ contains the edge $\{\mathcal{H}_1'', (\mathcal{H}_1'')^\perp\}$. Hence the desired path between T and T' exists, so $H_1(C_*) = 0$ and thus statement (1) holds.

For $p = 2$, $q = 0$, note that any intersection $T_0 \cap \ldots \cap T_k$ of $T_0, \ldots, T_k \in \widehat{\mathcal{R}}_e$ contains the unique vertex of X_g, and in particular any such intersection is nonempty and connected. The same holds for $\mathcal{R}_e$, so we have $\mathbb{E}^2_{2,0}(\mathcal{R}_e; \mathbb{Q}) = \mathbb{E}^2_{2,0}(\widehat{\mathcal{R}}_e; \mathbb{Q}) = 0$, so the pushforward on the $r = \infty$ page is surjective.

The proof of statement (2). As a consequence of Lemma 5.3.3, the map $\mathbb{E}^1_{0,1}(\mathcal{R}_e; \mathbb{Q}) \to \mathbb{E}^1_{0,1}(\widehat{\mathcal{R}}_e; \mathbb{Q})$ is an isomorphism. The argument given in the previous case for $p = 1, q = 1$ implies that any equivalence induced by the differential $\widehat{d}^1_{1,1}$ is also induced by $d^1_{1,1}$ by replacing the image of the differential $\widehat{d}^1_{1,1}$ on some $H_1(T_1 \cap T_2; \mathbb{Q})$ with $H_1(T_1 \cap T_2 \cap U_e; \mathbb{Q}) \to H_1(T_1 \cap T_2; \mathbb{Q})$ not surjective with a path between T_1 and T_2 consisting of tori T, T' with $H_1(T \cap T' \cap U_e; \mathbb{Q}) \to H_1(T \cap T'; \mathbb{Q})$ surjective. Therefore the map $\mathbb{E}^2_{0,1}(\mathcal{R}_e; \mathbb{Q}) \to \mathbb{E}^2_{0,1}(\widehat{\mathcal{R}}_e; \mathbb{Q})$ is an isomorphism. Then $\mathbb{E}^2_{2,0}(\mathcal{R}_e; \mathbb{Q}) = \mathbb{E}^2_{2,0}(\widehat{\mathcal{R}}_e; \mathbb{Q})$ by the argument in the previous statement with $p = 2$ and $q = 0$, so $\mathbb{E}^\infty_{0,1}(\mathcal{R}_e; \mathbb{Q}) \to \mathbb{E}^\infty_{0,1}(\widehat{\mathcal{R}}_e; \mathbb{Q})$ is an isomorphism. The case $p = 1$ and $q = 0$ follows by the same argument as $p = 2$ and $q = 0$ from the previous statement, except with 2 replaced by 1.

The proof of statement (3). The vector space $H_1(\widehat{U}_e \cap \widehat{U}_f; \mathbb{Q})$ is generated by classes represented by edges, so it suffices to show that any class in $H_1(\widehat{U}_e \cap \widehat{U}_f; \mathbb{Q})$ represented by an edge is in the image of the pushforward $H_1(U_e \cap U_f; \mathbb{Q}) \to H_1(\widehat{U}_e \cap \widehat{U}_f; \mathbb{Q})$. Now, by construction we have $U_e^{(1)} \cup e = \widehat{U}_e^{(1)}$, and similarly for f. Hence if $h \subseteq \widehat{U}_e \cap \widehat{U}_f$ is an edge with $h \neq e, f$, the class $[h]$ is in the image of $H_1(U_e \cap U_f; \mathbb{Q}) \to H_1(\widehat{U}_e \cap \widehat{U}_f; \mathbb{Q})$. Hence it suffices to show that $[e], [f] \in H_1(U_e \cap U_f; \mathbb{Q}) \to H_1(\widehat{U}_e \cap \widehat{U}_f; \mathbb{Q})$ if $e, f \subseteq \widehat{U}_e \cap \widehat{U}_f$. If $e \subseteq \widehat{U}_e \cap \widehat{U}_f$, this implies that there is a 3–cell τ containing both e and f. But now every edge in τ besides e and f lies in both U_e and U_f by the above argument, so $[e]$ is a linear combination of classes in $U_e \cap U_f$, and the lemma holds. □

We need one more auxiliary result before proving Lemma 5.3.2.

Lemma 5.3.6. *Let $g \geq 33$, $a \subseteq S_g$ be a nonseparating simple closed curve, $\vec{x} = [a]$, and $w \in \vec{x}^\perp$ such that w is not sent to zero under the adjoint map $\vec{x}^\perp \to \mathrm{Hom}_{\mathbb{Z}}(\vec{x}^\perp, \mathbb{Z})$. Let $\sigma \subseteq X_g$ be a 2–cell such that $\sigma \nsubseteq X_g^{w,2}$. Then σ is homologous to a linear combination of 2–cells $\sigma_1, \ldots, \sigma_n$ such that each $\sigma_i \nsubseteq X_g^{w,2}$, and each σ_i contains an edge $e_i \subseteq \sigma_i$ such that $\mathcal{H}(e_i)$ is not compatible with w and $g(e_i) = \{1, g-2\}$.*

Proof. Inductively, it suffices to show that σ is homologous to $\sigma_1 + \sigma_2 + \sigma_3$ such that each σ_i contains some edge $e_i \subseteq \sigma_i$ with $\min g(e_i) < g(\mathcal{H})$, where $\mathcal{H} \in \mathcal{H}(\sigma)$ is minimal among $\mathcal{H}' \in \mathcal{H}(\sigma)$ such that $\{\mathcal{H}', (\mathcal{H}')^\perp\}$ is not compatible with w. Choose such an $\mathcal{H} \subseteq \mathcal{H}(\sigma)$. Let $\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2$ such that there are edges $e_i \nsubseteq X_g^w$ with $\mathcal{H}(e_i) = \{\mathcal{H}_i, \mathcal{H}_i^\perp\}$. Let τ be the 3–cell containing σ, e_1 and e_2. Then every other 2–cell in $\partial\tau$ besides σ contains either e_1 or e_2. We have $\min g(e_i) < g(\mathcal{H})$ and $e_i \nsubseteq X_g^w$, so the lemma holds. □

We are now ready to move forward with Step (1) of the proof of Lemma 5.3.1.

Proof of Lemma 5.3.2. Let

$$\mathcal{E}^w = \{e \subseteq X_g : \mathcal{H}(e) \text{ is incompatible with } w \text{ and } g(e) = \{1, g-2\}\}.$$

For each $e \in \mathcal{E}^w$, let U_e and $\widehat{U}_e$ be as above. Let

$$\mathcal{U} = \{U_e\}_{e \in \mathcal{E}} \text{ and } \widehat{\mathcal{U}} = \{\widehat{U}_e\}_{e \in \mathcal{E}}.$$

Let $\mathcal{U}^+ = \mathcal{U} \cup \{X_g^w\}$ and $\widehat{\mathcal{U}}^+ = \widehat{\mathcal{U}} \cup \{X_g^w\}$. The pushforward $H_2(\bigcup_{\widehat{U} \in \widehat{\mathcal{U}}} \widehat{U} \cup X_g^w) \to H_2(X_g; \mathbb{Q}))$ is a surjection as a consequence of Lemma 5.3.6. For either of the covers $* = \mathcal{U}^+, \widehat{\mathcal{U}}^+$, let $\mathbb{E}^r_{p,q}(*; \mathbb{Q})$ denote the Čech–to–singular spectral sequence corresponding to the cover $*$. By construction, each $U \in \mathcal{U}^+$ has $U \subseteq X_g^{w,2}$. In particular, this implies that the cokernel of the pushforward $H_2(X_g^{w,2}; \mathbb{Q}) \to H_2(X_g; \mathbb{Q})$ is noncanonically a quotient of direct sum

$$\bigoplus_{p+q=2} \operatorname{cok}(\mathbb{E}^{\infty}_{p,q}(\mathcal{U}^+;\mathbb{Q}) \to \mathbb{E}^{\infty}_{p,q}(\widehat{\mathcal{U}}^+;\mathbb{Q})).$$

Therefore, it suffices to prove the following three facts:

1. The cokernel of the map $\mathbb{E}^1_{0,2}(\mathcal{U}^+;\mathbb{Q}) \to \mathbb{E}^1_{0,2}(\widehat{\mathcal{U}}^+,\mathbb{Q})$ is generated by the images of fundamental classes of Bestvina–Margalit tori,

2. the cokernel of the map $\mathbb{E}^2_{1,1}(\mathcal{U}^+;\mathbb{Q}) \to \mathbb{E}^2_{1,1}(\widehat{\mathcal{U}}^+;\mathbb{Q})$ is trivial, and

3. the cokernel of the map $\mathbb{E}^3_{2,0}(\mathcal{U}^+;\mathbb{Q}) \to \mathbb{E}^3_{2,0}(\widehat{\mathcal{U}}^+;\mathbb{Q})$ is trivial.

We prove each of these in turn.

Proof of Fact (1). This follows from (1) of Lemma 5.3.3.

Proof of Fact (2). By statements (2) and (3) respectively of Lemma 5.3.3, the maps $\mathbb{E}^1_{0,1}(\mathcal{U}^+;\mathbb{Q}) \to \mathbb{E}^1_{0,1}(\widehat{\mathcal{U}}^+;\mathbb{Q})$ and $\mathbb{E}^1_{1,1}(\mathcal{U}^+;\mathbb{Q}) \to \mathbb{E}^1_{1,1}(\widehat{\mathcal{U}}^+;\mathbb{Q})$ are an isomorphism and a surjection respectively. Hence the map $\mathbb{E}^2_{1,1}(\mathcal{U}^+;\mathbb{Q}) \to \mathbb{E}^2_{1,1}(\widehat{\mathcal{U}}^+;\mathbb{Q})$ is surjective.

Proof of Fact (3). For any choice of $e_0,\ldots,e_k \in \mathcal{E}^w$, both $U_{e_0}\cap\ldots\cap U_{e_k}$ and $\widehat{U}_{e_0}\cap\ldots\cap\widehat{U}_{e_k}$ contain the unique vertex of X_g, and hence are connected. The same applies if we include X^w_g, so $\mathbb{E}^2_{2,0}(\mathcal{U}^+;\mathbb{Q}) = \mathbb{E}^2_{2,0}(\widehat{\mathcal{U}}^+;\mathbb{Q}) = 0$.

Completing the proof. Given the above three statements, $\operatorname{cok}(H_2(X^{w,2}_g;\mathbb{Q}) \to H_2(X_g;\mathbb{Q}))$ is generated by a quotient of the image $\mathbb{E}^1_{0,2}(\widehat{\mathcal{U}}^+;\mathbb{Q}) \to \mathbb{E}^1_{0,2}(\overline{\mathcal{U}}^+;\mathbb{Q})$, and the image of this map is generated by the images of the fundamental classes of Bestvina–Margalit tori, so the lemma holds. □

5.3.2 Step (2) of the proof of Lemma 5.3.1

We now prove Lemma 5.3.7. This will complete the proof of Lemma 5.3.1. We will also prove Lemma 5.3.8, which is an auxiliary result about the acyclicity of a complex where the vertices are edges $e \subseteq X^{w,2}_g$.

Lemma 5.3.7. *Let $g \geq 33$ and $a \subseteq S_g$ a nonseparating simple closed curve. Let $\vec{x} = [a]$ and let $w \in \vec{x}^\perp$ be a nonzero primitive homology class such that the image of w under the adjoint map $\operatorname{Hom}_{\mathbb{Z}}(\vec{x}^\perp, \mathbb{Z})$ is nontrivial. The vector space*

$$\operatorname{cok}(H_2(X_g^w; \mathbb{Q}) \to H_2(X_g^{w,2}; \mathbb{Q}))$$

is generated by the images of fundamental classes of Bestvina–Margalit tori.

We begin by defining an auxiliary complex. Let $e \subseteq X_g$ be an edge with $g(e) = \{1, g-2\}$. Let $A_e \subseteq H_1(X_g; \mathbb{Q})$ be the affine space given by

$$A_e = \left\{ [f] \in H_1(X_g; \mathbb{Q}) : [f] - [e] \in H_1(X_g^w; \mathbb{Q}) \right\}.$$

Let $Y(e)$ denote the connected component containing the edge e of the complex $C(e)$, where a k–cell of $C(e)$ is a set of *ordered* $(k+1)$ edges $e_0, \ldots, e_k \subseteq X_g$ such that:

- $g(e_i) = \{1, g-2\}$ for every $0 \leq i \leq k$,
- $[e_i] \in A_e$ for every $0 \leq i \leq k$, and
- there is an edge e' such that $[e'] \in A_e$ and such that e' shares a 2–cell $\sigma_i \subseteq X_g^{w,2}$ with each e_i.

Remark. Note that the cells of $Y(e)$ are ordered collections of vertices. This is to avoid certain technical complications later in the section.

We will prove the following auxiliary result.

Lemma 5.3.8. *Let $g \geq 33$ and $a \subseteq S_g$ a nonseparating simple closed curve. Let $w \in \vec{x}^\perp$ be a primitive nonzero class such that w is not in the kernel of the adjoint map $\vec{x}^\perp \to \operatorname{Hom}_{\mathbb{Z}}(\vec{x}^\perp, \mathbb{Z})$. Let $e \subseteq X_g$ be an edge with $e \not\subseteq X_g^w$ and $g(e) = \{1, g-2\}$. The complex $Y(e)$ is 1–acylic.*

Proof. Let $\mathcal{H}(e) = \{V_0, V_1\}$. Let w_0, w_1 be two nonzero elements in $\vec{x}^{\perp}$ such that $w_i \in V_i$ and $w_0 + w_1 = w$. We will prove the following.

Claim. Let $f \subseteq X_g$ be an edge with f a vertex of $Y(e)$ and let $\mathcal{H}(f) = \left\{V_0^f, V_1^f\right\}$. Then after possibly reindexing, we have $w_i \in V_i^f$ for $i = 0, 1$.

Proof of claim. For any 2–cell $\sigma \subseteq X_g$ such that w_0, w_1 compatible with $\mathcal{H}(\sigma)$, there is some $\mathcal{H} \in \mathcal{H}(\sigma)$ with $w_0, w_1 \in \mathcal{H}$ or $\mathcal{H}^{\perp}$, so such a σ is contained in $X_g^{w,2}$. Likewise, if $\sigma \subseteq X_g^{w,2}$ is a 2–cell containing e, then there is $\mathcal{H} \in \mathcal{H}(\sigma)$ compatible with w, so in particular we have w_0, w_1 both compatible with $\mathcal{H}(\sigma)$. Hence if $e_0, \ldots, e_k$ is a cell in $Y(e)$ with e' as in the definition of $Y(e)$, then $e_0, \ldots, e_k$ and e' are all compatible with v_0 and v_1, so the claim holds.

Now, given the claim, we see that since $g \geq 33$, any triple of edges $e_0, e_1, e_2 \in Y(e)$ are the vertices of a 2–cell in $Y(e)$. Indeed, for any three $e_0, e_1, e_2 \subseteq X_g$ with $e_0, e_1, e_2 \in Y(e)$, we have w_0 and w_1 compatible with $\mathcal{H}(e_0)$, $\mathcal{H}(e_1)$ and $\mathcal{H}(e_2)$ by the claim. Since $g \geq 33$ and $g(e_k) = (1, g-2)$ for $k = 0, 1, 2$, we see that there is some primitive $\mathcal{H} \subseteq \vec{x}^{\perp}$ with $\mathcal{H}^{\perp} \cap \mathcal{H} = \mathbb{Z}\vec{x}$, $\mathcal{H}^{\perp} + \mathcal{H} = \vec{x}^{\perp}$, where $w_0 \in \mathcal{H}$, $w_1 \in \mathcal{H}^{\perp}$, and $\mathcal{H}$ compatible with $\mathcal{H}(e_k)$ for $k = 0, 1, 2$. Let $e' \subseteq X_g$ be the unique edge with $\mathcal{H} \in \mathcal{H}(e')$. For each e_k, let σ_k be a 2–cell containing e_k and e'. We see that since w_0, w_1 both compatible with e_k and e', then the third edge z_k of σ_k must have $w_0, w_1 \in \mathcal{H}_0^{z_k}$ or $\mathcal{H}_1^{z_k}$. Hence $\sigma_k \in X_g^{w,2}$, so $[e'] \in A_e$ since $[e'] + [z_k] = [e_k]$ and $[z_k] \in H_1(X_g^w; \mathbb{Q})$. Therefore by the definition of $Y(e)$, we have that e_0, e_1, e_2 is a 2–cell. This implies that $Y(e)$ is the 2–skeleton of a flag complex on the complete graph of the vertices of $Y(e)$, so in particular we have $H_1(Y(e); \mathbb{Q}) = 0$. $\square$

We are now ready to complete Step 2 of the proof of Lemma 5.3.1.

Proof of Lemma 5.3.7. Let $\mathcal{E}_1^w = \{e \subseteq X_g : e \text{ is incompatible with } w, g(e) = \{1, g-2\}\}$. Let $U_e \subseteq X_g^w$ be the subcomplex generated by all 2–cells $\sigma \subseteq X_g^w$ such that e and σ are both faces of a 3–cell τ. Let $\widehat{U}_e$ consist of the union of all 3 cells τ as in the previous sentence. For any such $\tau \subseteq \widehat{U}_e$, the union of τ with all τ_i that $H(\tau_i) = H(\tau)$ as unordered sets forms a $k+1$–torus, and this torus is naturally isomorphic to the product of e and the minimal k–torus in X_g containing σ. Hence, there is a natural isomorphism $\widehat{U}_e \cong e \times U_e$. By construction, the collection

$$\overline{\mathcal{U}} = \{\widehat{U}_e / U_e\}_{e \in \mathcal{E}_1^w}$$

covers the 2–skeleton of $X_g^{w,2}/X_w$. For any cover $* = \mathcal{U}, \widehat{\mathcal{U}}$, or $\overline{\mathcal{U}}$, let $\mathbb{E}^r_{p,q}(*;\mathbb{Q})$ denote the Čech–to–singular spectral sequence corresponding to the cover $*$. Since $\overline{\mathcal{U}}$ covers the 2–skeleton of $X_g^{w,2}/X_g^w$, we have

$$\bigoplus_{p+q=2} \mathbb{E}^\infty_{p,q}(\overline{\mathcal{U}};\mathbb{Q}) \twoheadrightarrow H_2(X_g^{w,2}/X_g^w;\mathbb{Q}).$$

From the long exact sequence in homology for the pair $(X_g^{w,2}, X_g^w)$, we have an inclusion

$$\operatorname{cok}(H_2(X_g^w;\mathbb{Q}) \to H_2(X_g^{w,2};\mathbb{Q})) \hookrightarrow H_2(X_g^{w,2}, X_g^w;\mathbb{Q}).$$

Then we have $H_2(X_g^{w,2}, X_g^w;\mathbb{Q}) = H_2(X_g^{w,2}/X_g^w;\mathbb{Q})$. Since $\mathbb{E}^r_{p,q}(\overline{\mathcal{U}};\mathbb{Q}) = 0$ for $p < 0$ or $q < 0$, the vector space $\mathbb{E}^\infty_{0,2}(\overline{\mathcal{U}};\mathbb{Q})$ is a quotient of $\mathbb{E}^1_{0,2}(\overline{\mathcal{U}};\mathbb{Q})$. Therefore it is enough to prove the following three facts:

1. The image of $\mathbb{E}^1_{0,2}(\widehat{\mathcal{U}};\mathbb{Q}) \to \mathbb{E}^1_{0,2}(\overline{\mathcal{U}},\mathbb{Q})$ is generated by Bestvina–Margalit tori,
2. the vector space $\mathbb{E}^2_{1,1}(\overline{\mathcal{U}};\mathbb{Q})$ is the 0–space, and
3. the vector space $\mathbb{E}^2_{2,0}(\overline{\mathcal{U}};\mathbb{Q})$ is the 0–space.

We prove each of these in turn.

The proof of Fact (1). Since $\widehat{U}_e \cong e \times U_e$, the cokernel $\operatorname{cok}(H_2(U_e;\mathbb{Q}) \to H_2(\widehat{U}_e;\mathbb{Q}))$ is isomorphic to $H_1(U_e;\mathbb{Q}) \otimes H_1(e;\mathbb{Q})$ by the Künneth formula. The tensor product of a class represented by an edge $f \subseteq U_e$ with e is the Bestvina–Margalit torus containing e and f, so Fact (1) holds.

The proof of Fact (2). Let $e_0, \ldots, e_k \in \mathcal{E}^w$. By construction, we have

$$\dim\left(H_1\left(\left(\widehat{U}_{e_0} \cap \ldots \cap \widehat{U}_{e_k}\right) / \left(U_{e_0} \cap \ldots \cap U_{e_k}\right); \mathbb{Q}\right)\right) \leq 1$$

so $\mathbb{E}^1_{*,1}(\overline{\mathcal{U}}; \mathbb{Q})$ is the cellular chain complex of a simplicial complex Z, where the k–cells of Z are sets of $k+1$ edges $e_0, \ldots, e_k \in \mathcal{E}^w_1$ with

$$\dim\left(H_1\left(\widehat{U}_{e_0} \cap \ldots \cap \widehat{U}_{e_k} / U_{e_0} \cap \ldots \cap U_{e_k}; \mathbb{Q}\right)\right) = 1.$$

Now, note that if $U_{e_0}, \ldots, U_{e_k}$ form a k–cell in Z, then there is an edge $f \subseteq X_g$ with $[f] \notin H_1(X^w_g; \mathbb{Q})$ such that f and e_i are two edges of a 2–cell $\sigma_i \subseteq X_g$ with the third edge in X^w_g. Hence if $e \in \mathcal{E}^w_1$, the path component P_e of Z containing $\widehat{U}_e/U_e$ has 2–skeleton canonically identified with the 2–skeleton of $Y(e)$. Therefore $H_1(P_e; \mathbb{Q}) = 0$ by Lemma 5.3.8. Therefore $H_1(Z; \mathbb{Q}) = 0$, so $\mathbb{E}^2_{1,1}(\overline{\mathcal{U}}; \mathbb{Q}) = 0$ as desired.

The proof of Fact (3). For any choice of $e_0, \ldots, e_k \in \mathcal{E}^w$, both $U_{e_0} \cap \ldots \cap U_{e_k}$ and $\widehat{U}_{e_0} \cap \ldots \cap \widehat{U}_{e_k}$ contain the unique vertex of X_g, and hence are connected. Therefore $\mathbb{E}^2_{2,0}(\overline{\mathcal{U}}; \mathbb{Q})$ is trivial, as desired. □

We now prove Lemma 5.3.1.

Proof of Lemma 5.3.1. There is a noncanonical surjection

$$\begin{aligned}\operatorname{cok}(H_2(X^w_g; \mathbb{Q}) \to H_2(X^{w,2}_g; \mathbb{Q})) \bigoplus \operatorname{cok}(H_2(X^{w,2}_g; \mathbb{Q}) \to H_2(X_g; \mathbb{Q}))\\ \twoheadrightarrow \operatorname{cok}(H_2(X^w_g; \mathbb{Q}) \to H_2(X_g; \mathbb{Q})).\end{aligned}$$

Hence the lemma follows by Lemmas 5.3.2 and 5.3.7. □

5.3.3 The proof of Proposition 1.3.1

We now conclude Section 5.3. We first connect the results of Section 5.1 with Lemma 5.3.1.

Lemma 5.3.9. *Let $g \geq 33$ and $a \subseteq S_g$ a nonseparating simple closed curve. Let $w \in \vec{x}^\perp$ be a nonzero primitive element such that w is not in the kernel of the adjoint map $\vec{x}^\perp \to \mathrm{Hom}_{\mathbb{Z}}(\vec{x}^\perp, \mathbb{Z})$. Let φ be the composition*

$$H_2^{\mathrm{ab,bp}}(\mathcal{I}_g; \mathbb{Q}) \to H_2(X_g; \mathbb{Q}) \to H_2(X_g; \mathbb{Q}))/H_2(X_g^w; \mathbb{Q})$$

where the first map is the map in the five term exact sequence for the equivariant homology spectral sequence given by the action of $\mathcal{I}_g$ on $\mathcal{C}_{\vec{x}}(S_g)$. Then the map φ is surjective.

Proof. By Lemma 5.3.1, the quotient space $H_2(X_g; \mathbb{Q})/H_2(X_g^w; \mathbb{Q})$ is generated by the images of fundamental classes of Bestvina–Margalit tori. Hence it suffices to show that any $[\mathrm{BM}_\sigma] \in H_2(X_g; \mathbb{Q})$ is the image of some $[T_{c,c'}, T_{d,d'}] \in H_2^{\mathrm{ab,bp}}(\mathcal{I}_g; \mathbb{Q})$ under the map $H_2^{\mathrm{ab,bp}}(\mathcal{I}_g; \mathbb{Q}) \to H_2(X_g; \mathbb{Q})$. Let BM_σ be a Bestvina–Margalit torus, and let $\widehat{\sigma}$ be a lift of σ to $\mathcal{C}_{\vec{x}}(S_g)$ such that $a \in \widehat{\sigma}$. Let a_1, a_2 be the other two vertices of $\widehat{\sigma}$. Choose a curve $b \subseteq S_g$ such that the geometric intersections $|a \cap b| = |a_1 \cap b| = |a_2 \cap b|$ are all equal to 1. Now, there are curves b_1, b_2 such that $b \cup b_1$ and $b \cup b_2$ are bounding pairs, and the corresponding bounding pair maps T_{b,b_1} and T_{b,b_2} both commute and take a to a_1 and a to a_2 respectively. The construction of such bounding pairs can be seen in Figure 5.5. Let

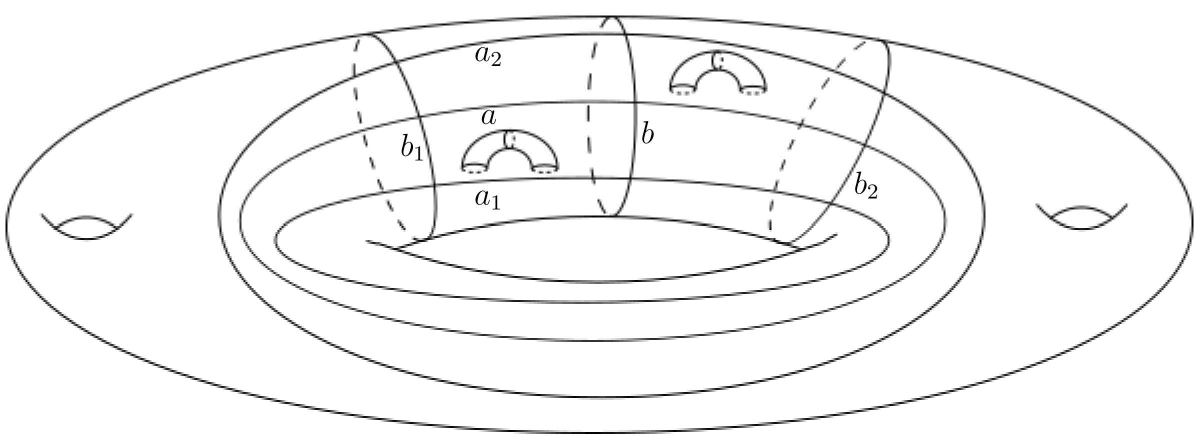

Figure 5.5: The curves a, a_1, a_2 and bounding pairs $b \cup b_1$ and $b \cup b_2$

$\gamma_i \in \pi_1(X_g)$ be the image of the edge in $\mathcal{C}_{\vec{x}}(S_g)$ connecting a to a_i. The bounding pair map T_{b,b_i} is sent to the loop γ_i under the natural map $\mathcal{I}_g \to \pi_1(X_g)$ for $i = 1, 2$. Hence the image of the abelian cycle $[T_{b,b_1}, T_{b,b_2}]$ under the map $H_2(\mathcal{I}_g;\mathbb{Q}) \to H_2(X_g;\mathbb{Q})$ is the abelian cycle $[\gamma_1, \gamma_2] \in H_2(X_g;\mathbb{Q})$, which is $[\mathrm{BM}_\sigma]$. $\square$

We now prove Lemma 5.3.10, which verifies that the first hypothesis of Proposition 4.1.1 holds for $\mathrm{Sp}(\vec{x}^\perp, \mathbb{Z})$, $H_2(X_g;\mathbb{Q})$, and $d = 1$.

Lemma 5.3.10. *Let $g \geq 33$ and let $w \in \vec{x}^\perp$ be a primitive element such that the image of w under the adjoint map $\vec{x}^\perp \to \mathrm{Hom}_\mathbb{Z}(\vec{x}^\perp, \mathbb{Z})$ is nonzero. Then the vector space $\mathrm{cok}(H_2(X_g^w;\mathbb{Q}) \to H_2(X_g;\mathbb{Q}))$ is finite dimensional.*

Proof of 5.3.10. Let d be a representative of w disjoint from a, and let $G = \mathrm{im}(\mathrm{Mod}(S_g \searrow (a \cup d)) \to \mathrm{Sp}(2g, \mathbb{Z}))$. We will show that the G–representation $V = \mathrm{cok}(H_2(X_g^w;\mathbb{Q}) \to H_2(X_g;\mathbb{Q}))$ satisfies the hypotheses of Proposition 4.1.1 for $\delta = 9$, namely:

1. for $M \subseteq S_g \searrow (a \cup d)$ a nonseparating multicurve with $|M| \geq 9$, the map

$$\bigoplus_{c \in M} V^{T_c} \to V$$

   is surjective, and

2. for $M \subseteq S_g \searrow (a \cup d)$ a nonseparating multicurve with $|M| \leq 8$, and $G_M = \mathrm{Stab}_G(M)$, the coinvariants module

$$H_0(G_M; V)$$

   is finite dimensional.

Then V is finite dimensional by Proposition 4.1.1.

Hypothesis (1). Lemma 5.3.9 tells us that the map

$$H_2^{\mathrm{ab,bp}}(\mathcal{I}_g;\mathbb{Q}) \to V$$

is surjective. Now Proposition 5.1.1 tells us that, if $M \subseteq S_g \setminus (a \cup d)$ is a nonseparating multicurve with $|M| \geq 9$, the map

$$\bigoplus_{c \in M} H_2^{\mathrm{ab,bp}}(\mathcal{I}_g; \mathbb{Q})_c^T \to H_2^{\mathrm{ab,bp}}(\mathcal{I}_g; \mathbb{Q})$$

is surjective. Then the image $H_2^{\mathrm{ab,bp}}(\mathcal{I}_g; \mathbb{Q})^{T_c} \to V$ is contained in V^{T_c}, so the map

$$\bigoplus_{c \in M} V^{T_c} \to V$$

is surjective, as desired.

Hypothesis (2). Let $M \subseteq S_g \setminus (a \setminus d)$ be a multicurve with $|M| \leq 8$. The coinvariants module $\mathrm{BM}_2(X_g; \mathbb{Q})_{G_M}$ is finite dimensional by Lemma 5.2.1 applied to $M' = M \cup d$. Now by Lemma 5.3.1, there is a surjection $\mathrm{BM}_2(X_g; \mathbb{Q}) \to V$, so there is a surjection $\mathrm{BM}_2(X_g; \mathbb{Q})_{G_M} \to V_{G_M}$, so in particular V_{G_M} is surjective. $\square$

Proof of Proposition 1.3.1. Let $G = \mathrm{im}(\mathrm{Mod}(S_g \setminus a) \to \mathrm{Sp}(2g, \mathbb{Z}))$. We will use Proposition 4.1.1 applied to the G–representation $H_2(X_g; \mathbb{Q})$ with $d = 1$ to show that $H_2(X_g; \mathbb{Q})$ is finite dimensional. In particular, we will show that:

1. for any nonseparating $c \subseteq S_g \setminus a$, we have $\mathrm{cok}(H_2(X_g; \mathbb{Q})^{T_c} \to H_2(X_g; \mathbb{Q}))$ finite dimensional, and
2. the coinvariants module $H_2(X_g; \mathbb{Q})_G$ is finite dimensional.

Hypothesis (1). This is exactly the content of Lemma 5.3.10.

Hypothesis (2). Let $\mathcal{I}(\vec{x}) \subseteq \mathrm{Mod}(S_g)$ denote the partial Torelli group defined by Putman [35], which is the subgroup of $\mathrm{Mod}(S_g)$ acting trivially on the homology class $\vec{x}$. Now, $\mathcal{I}(\vec{x})$ fits into a short exact sequence

$$1 \to \mathcal{I}_g \to \mathcal{I}(\vec{x}) \to G \to 1.$$

For $g \geq 3$, we know that $H_1(\mathcal{I}_g; \mathbb{Q})$ is finite dimensional by the work of Johnson [22]. Therefore by the Lyndon–Hochschild–Serre [8] spectral sequence and the fact that there is a $K(G, 1)$ with finite 3–skeleton [39], we see that the vector space $H_2(\mathcal{I}_g; \mathbb{Q})_G$ is finite dimensional if and only if $H_2(\mathcal{I}(\vec{x}); \mathbb{Q})$ is finite dimensional. We now consider the equivariant homology spectral sequence for the action of $\mathcal{I}(\vec{x})$ on $\mathcal{C}_{\vec{x}}(S_g)$. This action is cocompact on $\mathcal{C}_{\vec{x}}(S_g)$. Furthermore, the stabilizers of vertices and edges are finitely presented and finitely generated respectively, since these are mapping class groups of surfaces, which are finitely presented [12]. Then $\mathcal{C}_{\vec{x}}(S_g)$ is $(g-3)$–acyclic by Theorem B, so $H_2(\mathcal{I}(\vec{x}); \mathbb{Q})$ is finite dimensional, and thus the second hypothesis is satisfied. $\square$

CHAPTER 6

THE PROOF OF THEOREM C

6.1 Finite-dimensionality of coinvariants in $\mathbb{E}^2_{1,1}(\mathcal{I}_g, \mathcal{C}_{\vec{x}}(S_g))$

The main work of this section is to prove the following result.

Lemma 6.1.1. *Let $g \geq 33$ and $a \subseteq S_g$ be a nonseparating simple closed curve. Let $\vec{x} = [a]$. Let $M \subseteq S_g \setminus a$ be a nonseparating multicurve with $|M| = 8$. Let*

$$G = \operatorname{im}(\operatorname{Mod}(S_g \setminus M) \to \operatorname{Sp}(2g, \mathbb{Z})).$$

Let $\mathbb{E}^r_{p,q}$ denote the equivariant homology spectral sequence for the action of $\mathcal{I}_g$ on $\mathcal{C}_{\vec{x}}(S_g)$. Then the vector space

$$H_0(G; \mathbb{E}^2_{1,1})$$

is finite dimensional.

Notation. For the remainder of this section, we will fix $g \geq 33$ and $a \subseteq S_g$ a nonseparating simple closed curve. We will also fix $M \subseteq S_g \setminus a$ as in the statement of Lemma 6.1.1. We will use $\mathbb{E}^r_{p,q}$ to denote the equivariant homology spectral sequence given by the action of $\mathcal{I}_g$ on $\mathcal{C}_{\vec{x}}(S_g)$. Additionally, for the remainder of the section, we will let $\mathcal{V} = \{[c] : c \in M\}$.

The outline of the proof of Lemma 6.1.1. The strategy used is similar to that used in the proof of Lemma 5.2.1. In particular, we will associate to each edge $e \subseteq X_g$ certain invariants that record how the elements of $\mathcal{V}$ project onto the elements of $H(e)$. We will show in Lemma 6.1.2 that these numbers are preserved under the action of G. We then prove Lemma 6.1.4, which describes an intermediate quotient between $\mathbb{E}^1_{1,1}$ and $\mathbb{E}^2_{1,1}$ We then describe in Lemmas 6.1.5 and 6.1.7 how these invariants change under addition of edges. With these results in hand, we will prove Lemma 6.1.1.

Algebraic invariants of edges. Let $[b] \in H_1(S_g;\mathbb{Z})$ be a nonzero primitive class such that $\langle \vec{x}, [b]\rangle = 1$ and $[b]$ intersects any element of $\mathcal{V}$ trivially. Let $x \subseteq X_g$ be an edge and let $\mathcal{H}(x) = \{\mathcal{H}_0^x, \mathcal{H}_1^x\}$. For each $1 \le i \le 8$, $0 \le k \le 1$, let $v_{i,k}^x = \operatorname{proj}_{\vec{x}^\perp \cap [b]^\perp} v_i$. We define the following two invariants of the edge x:

1. $\operatorname{rk}_{i,k}^{\mathcal{V}}(x)$ is the maximal n such that $v_{i,k}^x = nv$ for some $v \in \mathcal{H}_k^x$ and
2. $\theta(\mathcal{V})_{i,j,k}(x) = \langle v_{i,k}^x, v_{j,k}^x\rangle$.

We have the following result about these algebraic invariants, which parallels Lemma 5.2.2.

Lemma 6.1.2. *Let $g \ge 33$ and $a \subseteq S_g$ be a nonseparating simple closed curve. Let $M \subseteq S_g \setminus a$ be a nonseparating multicurve with $|M| = 8$. Let $b \subseteq S_g$ be a nonseparating simple closed curve such that b has geometric intersection number with a, and such that b intersects M trivially. Let $G = \operatorname{im}(\operatorname{Mod}(S_g \setminus M) \to \operatorname{Sp}(2g,\mathbb{Z}))$. Let $\mathcal{V} = \{[c] : c \in M\}$. After possibly relabeling $\mathcal{H}_0^y$ and $\mathcal{H}_1^y$, suppose the following hold:*

1. *$g(\mathcal{H}_0^x) = g(\mathcal{H}_0^y)$,*
2. *$\operatorname{rk}_{i,k}^{\mathcal{V}}(x) = \operatorname{rk}_{i,k}^{\mathcal{V}}(y)$ for all $1 \le i \le 8$ and $0 \le k \le 1$ and*
3. *$\theta(\mathcal{V})_{i,j,k}(x) = \theta(\mathcal{V})_{i,j,k}(y)$ for all $1 \le i, j \le 8$ and $0 \le k \le 1$.*

Then there exists $g \in G$ such that $gx = y$.

This follows by a similar argument to Lemma 5.2.2, except with 2–cells replaced by edges. The $\operatorname{rk}^{\mathcal{V}}$ here is the same as the $\operatorname{rk}^{\mathcal{V}}$ in Lemma 5.2.2, and similarly for $\theta(\mathcal{V})$ and $\theta(\mathcal{V})$.

Before proving Lemmas 6.1.5 and 6.1.7, we will prove Lemma 6.1.4, which describes an intermediate quotient between $\mathbb{E}_{1,1}^1 \to \mathbb{E}_{1,1}^2$. We begin by showing that $\mathbb{E}_{1,1}^2$ is a quotient of $\mathbb{E}_{1,1}^1$.

Lemma 6.1.3. *Let $g \ge 33$ and $a \subseteq S_g$ be a nonseparating simple closed curve. Let $\vec{x} = [a]$. Let $\mathbb{E}_{p,q}^r$ denote the equivariant homology spectral sequence for the action of $\mathcal{I}_g$ on $\mathcal{C}_{\vec{x}}(S_g)$. The inclusion map $\mathbb{E}_{1,1}^2 \hookrightarrow \mathbb{E}_{1,1}^1/d_{2,1}^1(\mathbb{E}_{2,1}^1)$ is an isomorphism.*

Proof. The lemma is equivalent to the statement that the differential $d^1_{1,1}$ is the zero map. Since $g \geq 33 \geq 4$, Theorem B says that we have $H_1(C_{\vec{x}}(S_g); \mathbb{Z}) = 0$. Hence

$$\mathbb{E}^r_{p,q} \Rightarrow H_1(\mathcal{I}_g; \mathbb{Q}) \text{ for } p + q = 1$$

by the properties of the equivariant homology spectral sequence [8, Section VII]. Therefore there is an exact sequence

$$\mathbb{E}^1_{1,1} \xrightarrow{d^1_{1,1}} \mathbb{E}^1_{0,1} \to H_1(\mathcal{I}_g; \mathbb{Q}).$$

Since $\mathbb{E}^1_{0,1} \cong H_1(\mathrm{Stab}_{\mathcal{I}_g}(a); \mathbb{Q})$, the exact sequence is

$$\mathbb{E}^1_{1,1} \xrightarrow{d^1_{1,1}} H_1(\mathrm{Stab}_{\mathcal{I}_g}(a); \mathbb{Q}) \to H_1(\mathcal{I}_g; \mathbb{Q}).$$

A theorem of Putman [37, Theorem B] says that the map $H_1(\mathrm{Stab}_{\mathcal{I}_g}(a); \mathbb{Q}) \to H_1(\mathcal{I}_g; \mathbb{Q})$ is an injection, so $d^1_{1,1} = 0$. □

As in Section 5.1, let $\tau_g : \mathcal{I}_g \to \wedge^3 H_1(S_g; \mathbb{Z})/H_1(S_g; \mathbb{Z})$ denote the Johnson homomorphism [19]. If $\sigma \subseteq X_g$ is a cell, let A_σ denote the vector space

$$\mathrm{im}\left(\bigoplus_{\mathcal{H} \in \mathcal{H}(\sigma)} \wedge^3 \mathcal{H} \to \wedge^3 H_1(S_g; \mathbb{Z})/H_1(S_g; \mathbb{Z}) \right) \otimes \mathbb{Q}.$$

We have the following result about $\mathbb{E}^2_{1,1}$.

Lemma 6.1.4. *Let $g \geq 33$ and $a \subseteq S_g$ be a nonseparating simple closed curve. Let $\vec{x} = [a]$. Let $\mathbb{E}^r_{p,q}$ denote the equivariant homology spectral sequence for the action of $\mathcal{I}_g$ on $\mathcal{C}_{\vec{x}}(S_g)$. The quotient map $\varphi : \mathbb{E}^1_{1,1} \to \mathbb{E}^2_{1,1}$ factors through the natural projection*

$$\rho : \mathbb{E}^1_{1,1} \to \bigoplus_{x \in X_g^{(1)}} A_x.$$

Proof. Let $\varphi : \mathbb{E}^1_{1,1} \to \mathbb{E}^1_{1,1}/d^1_{2,1}(\mathbb{E}^1_{2,1}) = \mathbb{E}^2_{1,1}$ denote the quotient map, which exists by Lemma 6.1.3. For each $x \subseteq X_g$, let $\widehat{x} \subseteq C_{\vec{x}}(S_g)$ be a lift of x such that a is a vertex of $\widehat{x}$. Let

$$A_{\widehat{x}} = \operatorname{im}\left(H_1(\operatorname{Stab}_{\mathcal{I}_g}(\widehat{x}); \mathbb{Q}) \to H_1(\mathcal{I}_g; \mathbb{Q})\right).$$

Let S and S' be the connected components of $S_g \setminus \widehat{x}$. Lemma 5.1.5 implies that

$$\operatorname{im}(H_1(\mathcal{I}(S, S_g); \mathbb{Q}) \to H_1(\mathcal{I}_g; \mathbb{Q}))$$

is sent to $\operatorname{im}(\wedge^3 H_1(S; \mathbb{Q}) \to \operatorname{im}(\tau_g) \otimes \mathbb{Q})$ under the Johnson homomorphism, and similarly for S'. Therefore we have $A_{\widehat{x}} = A_x$ by the definition of A_x. Hence it suffices to show that φ factors through the quotient map

$$\rho : \mathbb{E}^1_{1,1} = \bigoplus_{x \in X_g^{(1)}} H_1(\operatorname{Stab}_{\mathcal{I}_g}(\widehat{x}); \mathbb{Q}) \to \bigoplus_{x \in X_g^{(1)}} A_x.$$

Showing that φ factors through ρ is equivalent to showing that $\ker(\rho) \subseteq \ker(\varphi)$, so it suffices to show that

$$\bigoplus_{x \in X_g^{(1)}} \ker\left(H_1(\operatorname{Stab}_{\mathcal{I}_g}(\widehat{x}); \mathbb{Q}) \to H_1(\mathcal{I}_g; \mathbb{Q})\right) \subseteq \ker(\varphi).$$

The idea of the proof is to rewrite any class in $\ker(\rho)$ as a linear combination of classes, where the edges in these classes have sufficiently large genus. If $x \subseteq X_g$ is an edge and $f \in H_1(\operatorname{Stab}_{\mathcal{I}_g}(\widehat{x}); \mathbb{Q})$ is a class, we use the notation $(x, f) \in \bigoplus_{x \in X_g^{(1)}} H_1(\operatorname{Stab}_{\mathcal{I}_g}(\widehat{x}); \mathbb{Q})$ to denote the class in $\mathbb{E}^2_{1,1}$ equal to f in the index x and equal to zero in every other index. Now, we have

$$\ker(\rho) = \bigoplus_{x \in X_g^{(1)}} \ker\left(H_1(\operatorname{Stab}_{\mathcal{I}_g}(\widehat{x}); \mathbb{Q}) \to H_1(\mathcal{I}_g; \mathbb{Q})\right),$$

and the latter space is spanned by elements of the form (x, f). Hence it suffices to show that $(x, f) \in \ker(\varphi)$ for any $x \in X_g^{(1)}$ and $f \in \ker(H_1(\mathrm{Stab}_{\mathcal{I}_g}(\widehat{x}); \mathbb{Q}) \to H_1(\mathcal{I}_g; \mathbb{Q}))$. Let S', S'' be the connected components of $S_g \setminus \widehat{x}$. We have a surjection $H_1(\mathcal{I}(S', S_g); \mathbb{Q}) \oplus H_1(\mathcal{I}(S'', S_g); \mathbb{Q}) \to H_1(\mathrm{Stab}_{\mathcal{I}_g}(\widehat{x}); \mathbb{Q}$ by applying the Künneth formula to the product $\mathcal{I}(S', S_g) \times \mathcal{I}(S'', S_g)$, so we may assume that f is supported on one connected component $S_g \setminus \widehat{x}$. Without loss of generality, we will assume that $f \in H_1(\mathcal{I}(S', S_g); \mathbb{Q})$. If $g(S') \geq 3$, then Lemma 5.1.5 says that the map $H_1(\mathcal{I}(S', S_g); \mathbb{Q}) \to H_1(\mathcal{I}_g; \mathbb{Q})$ is injective. Since f is in the kernel of the map $H_1(\mathcal{I}(S', S_g); \mathbb{Q}) \to H_1(\mathcal{I}_g; \mathbb{Q})$, this implies that $f = 0$, so $(x, f) = 0 \in \mathbb{E}^1_{1,1}$ and thus $(x, f) \in \ker(\varphi)$. Otherwise, $g(S') \leq 2$, which implies in particular that $g(S'') \geq 4$. Let $\sigma \subseteq C_{\vec{x}}(S_g)$ be a 2–cell such that:

- $\widehat{x} \subseteq \sigma$
- for the other two edges y, z of σ, the connected components S'_y, S'_z of $S_g \setminus y$ and $S_g \setminus z$ respectively that contain S' have $g(S'_y), g(S'_z) \geq 3$.

Now, let $\overline{\sigma}$ denote the image of σ in X_g and similarly for $\overline{y}$ and $\overline{z}$. We have chosen σ so that f has a representative $F \in \mathrm{Stab}_{\mathcal{I}_g}(\sigma)$. Therefore, after possibly reorienting σ, x and y, we have a relation in $\mathbb{E}^2_{1,1}$ given by $d^1_{2,1}(\overline{\sigma}, [F])$, which tells us in particular that

$$0 = (\overline{y}, [F]) + (\overline{z}, [F]) - (x, f)$$

in $\mathbb{E}^2_{1,1}$. We have F supported on S'_y and S'_z, both of which have genus at least 3. Then by Lemma 5.1.5, the map $H_1(\mathcal{I}(S'_y, S_g); \mathbb{Q}) \to H_1(\mathcal{I}_g; \mathbb{Q})$ is injective, and similarly for S'_z. But now, we have assumed that $f \in \ker(H_1(\mathrm{Stab}_{\mathcal{I}_g}(\widehat{x}); \mathbb{Q}) \to H_1(\mathcal{I}_g; \mathbb{Q}))$. Therefore $[F] \in H_1(\mathcal{I}(S'_y, S_g); \mathbb{Q})$ and $[F] \in H_1(\mathcal{I}(S'_z, S_g); \mathbb{Q})$ are both zero, so $(\overline{y}, [F]) = (\overline{z}, [F]) = 0$. Since $d^1_{2,1}(\overline{\sigma}, [F]) \in \ker(\varphi)$ by the definition of $\mathbb{E}^2_{1,1}$, we have $(x, f) \in \ker(\varphi)$, as desired. □

Notation. For the remainder of this thesis, if $x \subseteq X_g$ is an edge and $f \in A_x$ is a class, we will denote the corresponding element in $\mathbb{E}^2_{1,1}$ under the image of the quotient map $\bigoplus_{x\in X_g^{(1)}} A_x \to \mathbb{E}^2_{1,1}$ by (x, f).

6.1.1 The proof of Lemma 6.1.6

We now continue with the definitions. Let $x \subseteq X_g$ be an edge. Let $y \subseteq X_g$ be another edge such that x, y are two edges of a 2–cell σ, and let z denote the third edge of σ. We say that such a y is *rank $\mathcal{V}$–shrinking relative to x* if, after possibly relabeling $\mathcal{H}(x)$, $\mathcal{H}(y)$ and $\mathcal{H}(z)$, the following hold:

1. $\mathrm{rk}^{\mathcal{V}}_{i,k}(x) \geq \mathrm{rk}^{\mathcal{V}}_{i,k}(y)$ and $\mathrm{rk}^{\mathcal{V}}_{i,k}(z)$ for all $1 \leq i \leq 8$, $0 \leq k \leq 1$, and

2. for at least one choice of $1 \leq i \leq 8$ and $0 \leq k \leq 1$ with

$$\max\{\mathrm{rk}^{\mathcal{V}}_{i,0}(x), \mathrm{rk}^{\mathcal{V}}_{i,1}(x) \geq \max\{\mathrm{rk}^{\mathcal{V}}_{j,k}(x), : 1 \leq j \leq 8, 0 \leq k \leq 1\}$$

   we have $\max\{\mathrm{rk}^{\mathcal{V}}_{i,0}(x), \mathrm{rk}^{\mathcal{V}}_{i,1}(x)\} > \max\{\mathrm{rk}^{\mathcal{V}}_{i,k}(y), \mathrm{rk}^{\mathcal{V}}_{i,k}(z)\}$.

If $\mathcal{H}^x_0 \in \mathcal{H}(x)$, we say that y is *rank $\mathcal{V}$–shrinking relative to $\mathcal{H}^x_0$* if, in addition, there is $\mathcal{H}^y_i \in \mathcal{H}(y)$ with $\mathcal{H}^y_i \subseteq \mathcal{H}^x_0$. Denote the set of rank $\mathcal{V}$–shrinking edges relative to $\mathcal{H}^x_0$ by $\mathrm{rkshrink}_{\mathcal{V}}(\mathcal{H}^x_0)$. We now prove Lemma 6.1.5, which will allow us to rewrite classes in $\mathbb{E}^2_{1,1}$ as linear combinations of classes with lower $\mathrm{rk}^{\mathcal{V}}_{i,k}$.

Lemma 6.1.5. *Let $g \geq 33$ and $a \subseteq S_g$ be a nonseparating curve. Let $M \subseteq S_g \setminus a$ be a nonseparating multicurve with $|M| = 8$. Let $b \subseteq S_g$ be a nonseparating curve such that b has geometric intersection number with a, and such that b intersects M trivially. Let $\mathcal{V} = \{[c] : c \in M\}$. Let $x \subseteq X_g$ be an edge. Suppose that $g(\mathcal{H}^x_0) \geq 12$, and that $\mathrm{rk}^{\mathcal{V}}_{i,1}(x) > 1$ for at least one $1 \leq i \leq 8$. Then the natural map*

$$\varphi : \bigoplus_{y\in \mathrm{rkshrink}_{\mathcal{V}}(\mathcal{H}^x_0)} A_y \cap A_x \to A_x$$

is surjective.

Proof. Let $r_1 \wedge r_2 \wedge r_3 \in A_x$. We will show that $r_1 \wedge r_2 \wedge r_3 \in A_y$ for some $y \in \mathrm{rkshrink}_{\mathcal{V}}(\mathcal{H}_0^x)$. If $r_1, r_2, r_3 \in \mathcal{H}_1^x$ then $r_1 \wedge r_2 \wedge r_3 \in \wedge^3 \mathcal{H}_1^x \subseteq \wedge^3 \mathcal{H}_1^y \subseteq A_y$ for any $y \in \mathrm{rkshrink}_{\mathcal{V}}(\mathcal{H}_0^x)$, so it remains to prove the result in the case that $r_1 \wedge r_2 \wedge r_3 \in \mathcal{H}_0^x$. Suppose without loss of generality that $\mathrm{rk}_{1,1}^{\mathcal{V}}(x) > 1$ and

$$\mathrm{rk}_{1,1}^{\mathcal{V}}(x) = \max\{\max\{\mathrm{rk}_{i,0}^{\mathcal{V}}(x), \mathrm{rk}_{i,1}^{\mathcal{V}}(x)\} \geq \max\{\max\{\mathrm{rk}_{j,0}^{\mathcal{V}}(x), \mathrm{rk}_{j,1}^{\mathcal{V}}(x)\} : 1 \leq j \leq 8\}.$$

For each $1 \leq i \leq 8$, let $w_{i,k}$ denote the nonzero primitive homology class in $H_1(S_g; \mathbb{Z})$ such that

$$\mathrm{proj}_{\mathcal{H}_k^x \cap [b]^\perp}(v_i) = \mathrm{rk}_{i,k}^{\mathcal{V}}(x) w_{i,k}.$$

We claim that there is a nonzero primitive class $u \in \mathcal{H}_0^x$ such that:

- $\langle u, w_{i,0} \rangle = 0$ for all $1 \leq i \leq 8$,
- $\langle u, r_s \rangle = 0$ for all $1 \leq s \leq 3$, and
- u is not in the span of $\{w_{1,0}, \ldots, w_{8,1}, r_1, \ldots, r_3\}$.

Such a u exists since we have assumed that $g(\mathcal{H}_0^x) \geq 12$ and there are only eleven elements in the set $\{w_{1,0}, \ldots, w_{8,0}, r_1, r_2, r_3\}$, so the subspace $w_{1,0}^\perp \cap \ldots w_{8,0}^\perp \cap r_1^\perp \cap r_2^\perp \cap r_3^\perp$ has genus at least one. Let $h = w_{1,0} - u$. Let $y \subseteq X_g$ be an edge such that such that:

- $\mathcal{H}_0^y \subseteq \mathcal{H}_0^x$
- $u \in \mathcal{H}_0^y$,
- $h \in \mathcal{H}_1^y \cap \mathcal{H}_0^x$,
- $w_{i,0} \in \mathcal{H}_0^x \cap \mathcal{H}_1^y$ for all $2 \leq i \leq 8$, and
- $r_s \in \mathcal{H}_1^y \cap \mathcal{H}_0^x$ for all $1 \leq s \leq 3$.

Let σ be a 2–cell containing x and y, and let z be the third edge of σ. Index $\mathcal{H}(z)$ so that $\mathcal{H}_0^z \subseteq \mathcal{H}_0^x$. By construction, the vectors $\mathrm{proj}_{[b]^\perp \cap \mathcal{H}_k^y}(v_i)$ and $\mathrm{proj}_{[b]^\perp \cap \mathcal{H}_k^z}(v_i)$ are given as follows:

1. $\mathrm{proj}_{[b]^\perp \cap \mathcal{H}_0^y}(v_1) = \mathrm{rk}_{1,0}^{\mathcal{V}}(x) u$,

2. $\mathrm{proj}_{[b]^\perp \cap \mathcal{H}_0^y}(v_i) = 0$ for $2 \le i \le 8$,

3. $\mathrm{proj}_{[b]^\perp \cap \mathcal{H}_1^y}(v_1) = \mathrm{rk}^{\mathcal{V}}_{1,1}(x) w_{1,1} + \mathrm{rk}^{\mathcal{V}}_{1,0}\, h$,

4. $\mathrm{proj}_{[b]^\perp \cap \mathcal{H}_1^y}(v_i) = \mathrm{rk}^{\mathcal{V}}_{i,1}(x) w_{i,1} + \mathrm{rk}^{\mathcal{V}}_{i,0}(x) w_{i,0}$ for $2 \le i \le 8$,

5. $\mathrm{proj}_{[b]^\perp \cap \mathcal{H}_0^z}(v_1) = \mathrm{rk}^{\mathcal{V}}_{1,0}(x) h$,

6. $\mathrm{proj}_{[b]^\perp \cap \mathcal{H}_0^z}(v_i) = \mathrm{rk}^{\mathcal{V}}_{i,0}(x) w_{i,0}$ for $2 \le i \le 8$,

7. $\mathrm{proj}_{[b]^\perp \cap \mathcal{H}_1^z}(v_1) = \mathrm{rk}^{\mathcal{V}}_{1,1}(x) w_{1,1} + \mathrm{rk}^{\mathcal{V}}_{1,0}\, u$, and

8. $\mathrm{proj}_{[b]^\perp \cap \mathcal{H}_1^z}(v_i) = \mathrm{rk}^{\mathcal{V}}_{i,1}\, w_{i,1}$ for $2 \le i \le 8$.

By assumption, the homology classes $w_{i,k}$, h and u are all primitive. Hence the numbers $\mathrm{rk}^{\mathcal{V}}_{i,k}(y)$ and $\mathrm{rk}^{\mathcal{V}}_{i,k}(z)$ are given as follows, where each relation in the following list follows from the corresponding relation in the previous list:

(a) $\mathrm{rk}^{\mathcal{V}}_{1,0}(y) = \mathrm{rk}^{\mathcal{V}}_{1,0}(x)$,

(b) $\mathrm{rk}^{\mathcal{V}}_{i,0}(y) = 0$ for $2 \le i \le 8$,

(c) $\mathrm{rk}^{\mathcal{V}}_{1,1}(y) = \gcd(\mathrm{rk}^{\mathcal{V}}_{1,1}(x), \mathrm{rk}^{\mathcal{V}}_{1,0}(x))$,

(d) $\mathrm{rk}^{\mathcal{V}}_{i,1}(y) = \gcd(\mathrm{rk}^{\mathcal{V}}_{i,1}(x), \mathrm{rk}^{\mathcal{V}}_{i,0}(x))$ for $2 \le i \le 8$,

(e) $\mathrm{rk}^{\mathcal{V}}_{1,0}(z) = \mathrm{rk}^{\mathcal{V}}_{1,0}(x)$,

(f) $\mathrm{rk}^{\mathcal{V}}_{i,0}(z) = \mathrm{rk}^{\mathcal{V}}_{i,0}(x)$ for $2 \le i \le 8$,

(g) $\mathrm{rk}^{\mathcal{V}}_{1,1}(z) = \gcd(\mathrm{rk}^{\mathcal{V}}_{1,1}(x), \mathrm{rk}^{\mathcal{V}}_{1,0}(x))$,

(h) $\mathrm{rk}^{\mathcal{V}}_{i,1}(z) = \mathrm{rk}^{\mathcal{V}}_{i,1}(x)$ for $2 \le i \le 8$.

We have assumed that $\mathrm{rk}^{\mathcal{V}}_{1,1}(x) > 1$. Since v_1 is primitive, we have $\gcd(\mathrm{rk}^{\mathcal{V}}_{1,0}(x), \mathrm{rk}^{\mathcal{V}}_{1,1}(x)) = 1$, so relation (c) implies that $\mathrm{rk}^{\mathcal{V}}_{1,1}(y) < \mathrm{rk}^{\mathcal{V}}_{1,1}(x)$. Similarly, relation (g) implies that $\mathrm{rk}^{\mathcal{V}}_{1,1}(z) < \mathrm{rk}^{\mathcal{V}}_{1,1}(x)$. Then relations (a), (b) and (d) imply that $\mathrm{rk}^{\mathcal{V}}_{i,k}(y) \le \mathrm{rk}^{\mathcal{V}}_{i,k}(x)$ for all $1 \le i \le 8$ and $0 \le k \le 1$, and relations (e), (f) and (h) imply that $\mathrm{rk}^{\mathcal{V}}_{i,k}(z) \le \mathrm{rk}^{\mathcal{V}}_{i,k}(y)$ for all $1 \le i \le 8$ and $0 \le k \le 1$, and thus $y \in \mathrm{rkshrink}_{\mathcal{V}}(\mathcal{H}_0^x)$. Then we have chosen y so that $r_1, r_2, r_3 \in \mathcal{H}_0^y$, so $r_1 \wedge r_2 \wedge r_3 \in A_y$, and thus the proof is complete. $\square$

We use Lemma 6.1.5 to prove the following, which is the first step of the proof of Lemma 6.1.1.

Lemma 6.1.6. *Let $g \geq 33$ and $a \subseteq S_g$ be a nonseparating simple closed curve. Let $\vec{x} = [a]$. Let $M \subseteq S_g \setminus a$ be a nonseparating multicurve with $|M| = 8$. Let $b \subseteq S_g$ be a nonseparating simple closed curve such that b has geometric intersection number with a, and such that b intersects M trivially. Let $\mathcal{V} = \{[c] : c \in M\}$. Let $\mathbb{E}^r_{p,q}$ denote the equivariant homology spectral sequence for the action of $\mathcal{I}_g$ on $\mathcal{C}_{\vec{x}}(S_g)$. Let $(x, f) \in \mathbb{E}^2_{1,1}$ be a class. There is a relation in $\mathbb{E}^2_{1,1}$ given by*

$$(x, f) = \sum_{\ell=1}^{m} \lambda_\ell (y_\ell, f_\ell)$$

such that $\lambda_\ell \in \mathbb{Q}$, $\operatorname{rk}^{\mathcal{V}}_{i,k}(y_\ell) \leq 1$ for all $1 \leq i \leq 8, 0 \leq k \leq 1$, and $1 \leq \ell \leq m$.

Proof. Let $\operatorname{maxrk}(x) = \max_{1 \leq i \leq 8, 0 \leq k \leq 1} \operatorname{rk}^{\mathcal{V}}_{i,k}(x)$. Let $\operatorname{nummaxrk}(x)$ be the number of pairs (i, k) with $1 \leq i \leq 8$ and $0 \leq k \leq 1$ such that $\operatorname{rk}^{\mathcal{V}}_{i,k}(x) = \operatorname{maxrk}(x)$. The proof proceeds by double induction on $\operatorname{maxrk}(x)$ and $\operatorname{nummaxrk}(x)$.

Base case: $\operatorname{maxrk}(x) = 1$. In this, the resulting linear relation is $(x, f) = (x, f)$.

Inductive step: the lemma holds for all $y \subseteq X_g$ that satisfy either $\operatorname{maxrk}(y) < \operatorname{maxrk}(x)$ *or both* $\operatorname{maxrk}(y) \leq \operatorname{maxrk}(x)$ *and* $\operatorname{nummaxrk}(y) < \operatorname{nummaxrk}(x)$. We want to apply Lemma 6.1.5. We begin with the the following claim.

Claim. There is a linear relation

$$(x, f) = \sum_{p=1}^{q} (z_p, f_p)$$

such that for every $1 \leq p \leq q$:

- either $\operatorname{maxrk}(z_p) = \operatorname{maxrk}(x)$ and $\operatorname{nummaxrk}(z_p) \leq \operatorname{nummaxrk}(x)$, or $\operatorname{maxrk}(z_p) < \operatorname{maxrk}(x)$, and
- if $\operatorname{maxrk}(z_p) \neq 1$, there is a pair (i, k) such that $\operatorname{rk}^{\mathcal{V}}_{i,k}(z_p) = \operatorname{maxrk}(z_p)$ and $g(\mathcal{H}^{z_p}_k) \geq 13$.

Proof of claim. By reindexing, we may assume without loss of generality that $\mathrm{rk}^{\mathcal{V}}_{1,0}(x) = \mathrm{maxrk}(x)$. If $g(\mathcal{H}_0^x) \geq 13$ then we are done, so suppose that $g(\mathcal{H}_0^x) \leq 12$. Since $f \in A_x$, we may rewrite f as a $\mathbb{Q}$–linear combination of pure tensors $r_1 \wedge r_2 \wedge r_3 \in A_x$ such that each r_i is primitive. Therefore we may assume without loss of generality that $f = r_1 \wedge r_2 \wedge r_3$ with $r_1, r_2, r_3 \in \mathcal{H}_k^x$ for some $0 \leq k \leq 1$. Now, since $g(\mathcal{H}_0^x) \leq 12$, we have $g(\mathcal{H}_1^x) \geq 33 - 12 \geq 13$. Hence there is a primitive subgroup $\mathcal{H} \subseteq \mathcal{H}_1^x$ such that the following hold:

- $2 * g(\mathcal{H}) + 1 = \mathrm{rk}^{\mathcal{V}}(\mathcal{H})$,
- $g(\mathcal{H}_1^x) - g(\mathcal{H}) \leq 11$,
- $\vec{x} \in \mathcal{H}$,
- $r_1, r_2, r_3, v_{1,1}, v_{2,1}, \ldots, v_{8,1} \in \mathcal{H}^{\perp}$.

Let $z_1 \subseteq X_g$ be the unique edge with $\mathcal{H} \in \mathcal{H}(z_1)$. By construction, z_1 and x share a 2–cell $\sigma \subseteq X_g$ that satisfies $\mathcal{H}(\sigma) = \{\mathcal{H}_0^x, \mathcal{H}, \mathcal{H}_1^x \cap \mathcal{H}^{\perp}\}$. Let z_2 be the third edge of σ. We have $\mathcal{H}(z_1) = \{\mathcal{H}, \mathcal{H}^{\perp}\}$ and $\mathcal{H}(z_2) = \{\mathcal{H}^{\perp} \cap \mathcal{H}_1^x, \mathcal{H}_0^x + \mathcal{H}\}$. Since r_1, r_2, r_3 are all in either $\mathcal{H}_0^x$ or $\mathcal{H}_1^x$ and are in $\mathcal{H}^{\perp}$ by hypothesis, the fact that $\mathcal{H}(\sigma) = \{\mathcal{H}_0^x, \mathcal{H}, \mathcal{H}^{\perp} \cap \mathcal{H}_1^x\}$ implies that $f \in A_\sigma$. Therefore in $\mathbb{E}^2_{1,1}$, the image of $d^1_{2,1}(\sigma, f)$ (after possibly reorienting x, z_1 and z_2) yields a relation

$$(x, f) = (z_1, f) + (z_2, f).$$

Assume that $\mathcal{H}(z_1)$ is indexed so that $\mathcal{H} = \mathcal{H}_0^{z_1}$. By our choice of $\mathcal{H}$, the projections of each v_i to the elements of $\mathcal{H}(z_1)$ are given as follows:

- $v_{i,1}^{z_1} = v_i$, and
- $v_{i,0}^{z_1} = 0$.

Since each v_i is primitive by assumption, we have $\mathrm{rk}^{\mathcal{V}}_{i,k}(z_1) = 1$ or 0 for every $1 \leq i \leq 8$ and $0 \leq k \leq 1$. Hence (z_1, f) satisfies the desired properties of z_i in the claim, since:

- $\mathrm{maxrk}(z_1) = 1 < \mathrm{maxrk}(x)$, and

- z_1 does not satisfy the hypothesis in the second condition.

For the class (z_2, f), after indexing $\mathcal{H}(z_2)$ so that $\mathcal{H}_0^x \subseteq \mathcal{H}_0^{z_2}$, the projections $v_{i,k}^{z_2}$ satisfy $v_{i,k}^{z_2} = v_{i,k}^x$. Hence we have $\mathrm{rk}_{i,k}^{\mathcal{V}}(z_2) = \mathrm{rk}_{i,k}^{\mathcal{V}}(x)$ for $1 \leq i \leq 8$ and $0 \leq k \leq 1$. Now, by construction we have $g(\mathcal{H}_0^{z_2}) = g(\mathcal{H}_0^x) + g(\mathcal{H})$. Since we have chosen $\mathcal{H}$ so that $g(\mathcal{H}_1^x) - g(\mathcal{H}) \leq 11$, we have $g(\mathcal{H}) \geq g(\mathcal{H}_1^x) - 11$. Then we have

$$g(\mathcal{H}_0^{z_2}) = g(\mathcal{H}_0^x) + g(\mathcal{H}_1^x) \geq g(\mathcal{H}_0^x) + g(\mathcal{H}_1^x) - 11 \geq 33 - 1 - 11 \geq 13,$$

since we have assumed that $g = g(\mathcal{H}_0^x) + g(\mathcal{H}_1^x) + 1 \geq 33$. Therefore z_2 satisfies the properties in the claim, since:

- $\mathrm{maxrk}(z_2) = \mathrm{maxrk}(x)$ and $\mathrm{nummaxrk}(z_2) = \mathrm{nummaxrk}(x)$, and
- any pair $(i, 0)$ that satisfies $\mathrm{rk}_{i,0}^{\mathcal{V}}(x) = \mathrm{maxrk}(x)$ also satisfies $\mathrm{rk}_{i,0}^{\mathcal{V}}(z_2) = \mathrm{maxrk}(z_2)$, and $g(\mathcal{H}_0^{z_2}) \geq 13$,

so the claim holds.

We now continue with the inductive step of the proof. By the claim, we may rewrite the class (x, f) as a sum so that each summand (z, f') satisfies $\mathrm{maxrk}(z) \leq \mathrm{maxrk}(x)$, and $\mathrm{nummaxrk}(z) \leq \mathrm{nummaxrk}(x)$ if $\mathrm{maxrk}(z) = \mathrm{maxrk}(x)$, and, if (z, f) does not already satisfy $\mathrm{maxrk}(z) \leq 1$, we have $g(\mathcal{H}_0^z) \geq 13$ with $\mathrm{rk}_{i,0}^{\mathcal{V}}(z) = \mathrm{maxrk}(z)$ for some i. Hence we may assume without loss of generality that $g(\mathcal{H}_0^x) \geq 13$ and $\mathrm{rk}_{i,0}^{\mathcal{V}}(x) = \mathrm{maxrk}(x)$ for some $1 \leq i \leq 8$. Then by Lemma 6.1.5, there is a collection of edges $y_\ell \in \mathrm{rkshrink}_{\mathcal{V}}(\mathcal{H}_0^x)$ and a choice of $f_\ell \in A_{y_\ell} \cap A_x$ such that

$$f = \sum_{\ell=1}^{m} f_\ell.$$

For each y_ℓ, there is a 2–cell $\sigma_\ell \subseteq X_g$ with $x_\ell, y_\ell \in X_g$ by the definition of $\mathrm{rkshrink}_{\mathcal{V}}(\mathcal{H}_0^x)$. For each of these σ_ℓ, let $z_\ell \subseteq \sigma_\ell$ denote the third edge besides x and y_ℓ. Then $f_\ell \in A_{y_\ell} \cap A_x$

and $A_x \cap A_{y_\ell} = A_{\sigma_\ell}$ by the definition of A_τ for any $\tau \subseteq X_g$, so we have $f_\ell \in A_\tau$. Therefore we have the following relation in $\mathbb{E}^2_{1,1}$:

$$d^1_{2,1}\left(\sum_{\ell=1}^{m}(\sigma_\ell, f_\ell)\right) = \sum_{\ell=1}^{m}(x, f_\ell) + (y_\ell, f_\ell) - (z_\ell, f_\ell).$$

But now, by rearranging terms, we have a relation

$$\sum_{\ell=1}^{m}(x, f_\ell) = \sum_{\ell=1}^{m}(y_\ell, f_\ell) - (z_\ell, f_\ell).$$

Then since $\sum_{\ell=1}^{m} f_\ell = f$, we have a relation

$$(x, f) = \sum_{\ell=1}^{m}(z_\ell, f_\ell) - (y_\ell, f_\ell).$$

Since $y_\ell \in \mathrm{rkshrink}_{\mathcal{V}}(\mathcal{H}^x_0)$, for any $1 \leq \ell \leq m$, we have either $\mathrm{maxrk}(y_\ell) < \mathrm{maxrk}(x)$, or $\mathrm{maxrk}(y_\ell) = \mathrm{maxrk}(x)$ and $\mathrm{nummaxrk}(y_\ell) < \mathrm{nummaxrk}(x)$, and similarly for z_ℓ. Therefore by the inductive hypothesis, y_ℓ and z_ℓ are linear combinations of elements as in the statement of the lemma, so the lemma holds for (x, f) as well. □

6.1.2 The proof of Lemma 6.1.8

We now proceed to the second step of the proof of Lemma 6.1.1. Let $x \subseteq X_g$ be an edge such that $\mathrm{rk}^{\mathcal{V}}_{i,k}(x) \leq 1$ for all $1 \leq i \leq 8$ and $0 \leq k \leq 1$. Let $y \subseteq X_g$ be another edge such that y and x are two edges of a 2–cell σ, and let z be the third edge of σ. We say that y is *algebraically $\mathcal{V}$–shrinking relative to x* if, after possibly reindexing $\mathcal{H}(x)$, $\mathcal{H}(y)$, and $\mathcal{H}(z)$, the following hold:

1. $\mathrm{rk}^{\mathcal{V}}_{i,k}(y), \mathrm{rk}^{\mathcal{V}}_{i,k}(z) \leq 1$ for $1 \leq i \leq 8$ and $0 \leq k \leq 1$,
2. $\max\{|\theta(\mathcal{V})_{i,j,1}(x)|, 1\} \geq |\theta(\mathcal{V})_{i,j,1}(y)|, |\theta(\mathcal{V})_{i,j,1}(z)|$ for all $1 \leq i, j \leq 8$, and

3. at least one pair $1 \le i < j \le 8$ and $0 \le k \le 1$ with

$$|\theta(\mathcal{V})_{i,j,k}(x)| = \max\{|\theta(\mathcal{V})_{i',j',k}(x)| : 1 \le i', j' \le 8, 0 \le k \le 1\}$$

has $|\theta(\mathcal{V})_{i,j,k}(x)| > |\theta(\mathcal{V})_{i,j,k}(y)|, |\theta(\mathcal{V})_{i,j,k}(z)|$.

If x is an edge, let $\text{algshrink}_{\mathcal{V}}(x) \subseteq X_g^{(1)}$ denote the set of algebraically $\mathcal{V}$–shrinking relative to x edges in X_g. We will prove the following result about $\text{algshrink}_{\mathcal{V}}(x)$, which completes the second step of the proof of Lemma 6.1.1.

Lemma 6.1.7. *Let $g \ge 33$ and $a \subseteq S_g$ be a nonseparating simple closed curve. Let $\vec{x} = [a]$. Let $M \subseteq S_g \setminus a$ be a nonseparating multicurve with $|M| = 8$. Let $b \subseteq S_g$ be a nonseparating simple closed curve such that $|a \cap b| = 1$ and such that b intersects M trivially. Let $\mathcal{V} = \{[c] : c \in M\}$. Let $\mathbb{E}^r_{p,q}$ denote the equivariant homology spectral sequence for the action of $\mathcal{I}_g$ on $\mathcal{C}_{\vec{x}}(S_g)$. Let $x \subseteq X_g$ be an edge such that $\text{rk}^{\mathcal{V}}_{i,k}(x) \le 1$ for all $1 \le i \le 8$ and $0 \le k \le 1$. Suppose that there is a pair $1 \le i < j \le 8$ and $0 \le k \le 1$ such that $|\theta(\mathcal{V})_{i,j,k}(x)| > 1$. Then the natural map*

$$\varphi : \bigoplus_{y \in \text{algshrink}_{\mathcal{V}}(x)} A_y \cap A_x \to A_x$$

is surjective.

Proof. Choose three primitive classes $r_1, r_2, r_3 \in \mathcal{H}_0^x \cup \mathcal{H}_1^x$ such that $r_1 \wedge r_2 \wedge r_3 \in A_x$. We will show that $r_1 \wedge r_2 \wedge r_3 \in \text{im}(\varphi)$. Since $|\theta(\mathcal{V})_{i,j,0}(x)| = |\theta(\mathcal{V})_{i,j,1}(x)|$ for all $1 \le i, j \le 8$ because we have assumed that $\langle v_i, v_j \rangle = 0$, we may assume without loss of generality that $g(\mathcal{H}_0^x) \ge \frac{33}{2}$, so in particular $g(\mathcal{H}_0^x) \ge 13$. If $r_1, r_2, r_3 \in \mathcal{H}_1^x$ then we are done since $\text{algshrink}_{\mathcal{V}}(x)$ is nonempty, so assume that $r_1, r_2, r_3 \in \mathcal{H}_0^x$. We will show that there is an edge $y \in \text{algshrink}_{\mathcal{V}}(x)$ such that $r_1 \wedge r_2 \wedge r_3 \in A_y$. For each $v_i \in \mathcal{V}$, let $w_{i,k}$ be the projection $w_{i,k} = \text{proj}_{[b]^\perp \cap \mathcal{H}_k^x} v_i$. By hypothesis each $w_{i,k}$ is either primitive or zero. Assume without loss of generality that $|\theta(\mathcal{V})_{1,2,0}|$ is maximal over all $|\theta(\mathcal{V})_{i,j,0}|$.

By possibly replacing v_1 with $-v_1$, we may also assume without loss of generality that $\theta(\mathcal{V})_{1,2,0}(x) \geq 0$. Since $g(\mathcal{H}_0^x) \geq 13$ and the set $\{w_{1,0}, \ldots, w_{8,0}, r_1, r_2, r_3\}$ has eleven elements, we may choose two primitive classes $u_1, u_2 \in \mathcal{H}_0^x$ such that the following hold:

1. $\langle u_t, w_{i,0} \rangle = 0$ for all $1 \leq t \leq 2, 1 \leq i \leq 8$,

2. $\langle u_t, r_s \rangle = 0$ for all $1 \leq t \leq 2, 1 \leq s \leq 3$,

3. $\langle u_1, u_2 \rangle = 1$,

Consider the following classes h_i for $1 \leq i \leq 8$:

1. $h_1 = w_{1,0} - u_1$,

2. $h_2 = w_{2,0} - u_2$ and

3. $h_i = w_{i,0}$ for $3 \leq i \leq 8$.

Now, let y be an edge that shares a 2–cell with x such that the following properties hold:

1. $\mathcal{H}_0^y \subseteq \mathcal{H}_0^x$,

2. $h_i \in \mathcal{H}_0^x \cap \mathcal{H}_0^y$ for $1 \leq i \leq 8$,

3. $r_s \in \mathcal{H}_0^x \cap \mathcal{H}_1^y$ for $1 \leq s \leq 3$, and

4. $u_t \in \mathcal{H}_0^x \cap \mathcal{H}_1^y$ for $1 \leq t \leq 2$.

We will show that:

1. $y \in \operatorname{algshrink}_{\mathcal{V}}(x)$ and

2. $r_1 \wedge r_2 \wedge r_3 \in A_y$.

This completes the proof, since then $r_1 \wedge r_2 \wedge r_3 \in \operatorname{im}(\varphi)$. The latter property follows by construction, so it suffices to prove the former. Let z denote the third edge of a 2–cell σ with $x \subseteq \sigma$, $y \subseteq \sigma$. Label $\mathcal{H}(z) = \{\mathcal{H}_0^z, \mathcal{H}_1^z\}$ such that $\mathcal{H}_0^z \subseteq \mathcal{H}_0^x$. Therefore we have $\mathcal{H}_0^z = \mathcal{H}_0^x \cap \mathcal{H}_1^y$ and $\mathcal{H}_1^z = \mathcal{H}_0^y + \mathcal{H}_1^x$. For notational convenience, let $u_i = 0$ for $3 \leq i \leq 8$. By construction, the vectors $\operatorname{proj}_{[b]^\perp \cap \mathcal{H}_k^w}(v_i)$ for each choice of $1 \leq i \leq 8$, $0 \leq k \leq 1$ and $w = x, y, z$ are given as follows:

1. $\mathrm{proj}_{[b]^{\perp}\cap\mathcal{H}_0^y}(v_i) = h_i$ for $1 \leq i \leq 8$,
2. $\mathrm{proj}_{[b]^{\perp}\cap\mathcal{H}_1^y}(v_i) = u_i + w_{i,1}$ for $1 \leq i \leq 8$,
3. $\mathrm{proj}_{[b]^{\perp}\cap\mathcal{H}_0^z}(v_i) = u_i$ for $1 \leq i \leq 8$, and
4. $\mathrm{proj}_{[b]^{\perp}\cap\mathcal{H}_1^z}(v_i) = h_i + w_{i,1}$ for $1 \leq i \leq 8$.

Now, given these projections, we can compute ranks and intersection numbers as follows.

1. We have $\mathrm{rk}^{\mathcal{V}}_{i,k}(y), \mathrm{rk}^{\mathcal{V}}_{i,k}(z) \leq 1$ for $1 \leq i \leq 8$ and $0 \leq k \leq 1$, since by assumption and construction each projection is primitive.
2. We have $|\theta(\mathcal{V})_{i,j,0}(y)| = |\langle h_i, h_j\rangle| = |\theta(\mathcal{V})_{i,j,0}(x)| - \mathbb{1}_{(i,j)=(1,2)}$.
3. We have $|\theta(\mathcal{V})_{i,j,0}(z)| = |\langle u_i, u_j\rangle| = \mathbb{1}_{(i,j)=(1,2)}$.

Furthermore, since $\langle v_i, v_j\rangle = 0$ by assumption, we have $|\theta(\mathcal{V})_{i,j,0}(y)| = |\theta(\mathcal{V})_{i,j,1}(y)|$, and similarly for z. Therefore we have $|\theta(\mathcal{V})_{i,j,k}(x)| \geq |\theta(\mathcal{V})_{i,j,k}(y)|, |\theta(\mathcal{V})_{i,j,k}(z)|$ for all $1 \leq i, j \leq 8$ and $0 \leq k \leq 1$. Additionally, we have assumed that $|\theta(\mathcal{V})_{i,j,0}(x)| > 1$, so we have $|\theta(\mathcal{V})_{1,2,0}(x)| > |\theta(\mathcal{V})_{1,2,0}(y)|, |\theta(\mathcal{V})_{1,2,0}(z)|$. Hence we have $y \in \mathrm{algshrink}_{\mathcal{V}}(x)$, so the proof is complete. □

We now complete the second step of the proof of Lemma 6.1.1.

Lemma 6.1.8. *Let $g \geq 33$ and $a \subseteq S_g$ be a nonseparating simple closed curve. Let $M \subseteq S_g \setminus a$ be a nonseparating multicurve with $|M| = 8$. Let $b \subseteq S_g$ be a nonseparating simple closed curve such that b has geometric intersection number with a, and such that b intersects M trivially. Let $\mathcal{V} = \{[c] : c \in M\}$. Let $\mathbb{E}^r_{p,q}$ denote the equivariant homology spectral sequence for the action of $\mathcal{I}_g$ on $\mathcal{C}_{\vec{x}}(S_g)$. Let $(x, f) \in \mathbb{E}^2_{1,1}$ be a class such that $\mathrm{rk}^{\mathcal{V}}_{i,k}(x) \leq 1$ for all $1 \leq i \leq 8$ and $0 \leq k \leq 1$. Then there is a relation in $\mathbb{E}^2_{1,1}$ given by*

$$(x, f) = \sum_{\ell=1}^{m} \lambda_\ell (y_\ell, f_\ell)$$

such that $\lambda_\ell \in \mathbb{Q}$, $\mathrm{rk}^{\mathcal{V}}_{i,k}(y_\ell) \leq 1$, *and* $\theta(\mathcal{V})_{i,j,k}(y_\ell) \leq 1$ *for all* $1 \leq \ell \leq m$, $1 \leq i, j \leq 8$, *and* $0 \leq k \leq 1$.

Proof. Let $\mathrm{maxalg}(x) = \max_{1 \leq i,j \leq 8} |\theta(\mathcal{V})_{i,j,0}(x)|$. Since $\theta(\mathcal{V})_{i,j,0}(x) = -\theta(\mathcal{V})_{i,j,1}(x)$, we only need take the maximum for $k = 0$. Let $\mathrm{nummaxalg}(x)$ denote the number of pairs $1 \leq i < j \leq 8$ such that $|\theta(\mathcal{V})_{i,j,0}| = \mathrm{maxalg}(x)$. The proof proceeds by double induction on $\mathrm{maxalg}(x)$ and $\mathrm{nummaxalg}(x)$.

Base case: $\mathrm{maxalg}(x) \leq 1$. In this case, the relation in the lemma is the trivial relation $(x, f) = (x, f)$, so the lemma holds.

Inductive step: the lemma holds for all $y \subseteq X_g$ *with either* $\mathrm{maxalg}(y) < \mathrm{maxalg}(x)$ *or with both* $\mathrm{maxalg}(y) \leq \mathrm{maxalg}(x)$ *and* $\mathrm{nummaxalg}(y) < \mathrm{nummaxalg}(x)$. By Lemma 6.1.7, there is a linear combination

$$f = \sum_{\ell=1}^{m} f_\ell$$

such that each $f_\ell \in A_{y_\ell}$, where $y_\ell \in \mathrm{algshrink}_{\mathcal{V}}(x)$. By definition, each y_ℓ shares a 2–cell σ_ℓ with x, and $f_\ell \in A_{\sigma_\ell}$. Let z_ℓ denote the third edge of σ_ℓ. Then there is a relation in $\mathbb{E}^2_{1,1}$ given by

$$\sum_{\ell=1}^{m} d^1_{2,1}(\sigma_\ell, f_\ell) = \sum_{\ell=1}^{m} (x, f_\ell) + (y_\ell, f_\ell) - (z_\ell, f_\ell) = 0.$$

By rearranging terms and applying the fact that $f = \sum_{\ell=1}^{m} f_\ell$, we have

$$(x, f) = \sum_{\ell=1}^{m} (z_\ell, f_\ell) - (y_\ell, f_\ell).$$

But then $y_\ell \in \mathrm{algshrink}_{\mathcal{V}}(x)$ for all $1 \leq \ell \leq m$, so for all $1 \leq \ell \leq m$, either $\mathrm{maxalg}(y_\ell) < \mathrm{maxalg}(x)$ or $\mathrm{maxalg}(y_\ell) = \mathrm{maxalg}(x)$ and $\mathrm{nummaxalg}(y_\ell) < \mathrm{nummaxalg}(x_\ell)$, and similarly for z_ℓ. Therefore the classes (y_ℓ, f_ℓ) and (z_ℓ, f_ℓ) are linear combinations of classes as in the statement of the lemma by the inductive hypothesis, so the lemma holds for (x, f) as well. □

We are now ready to conclude Section 6.1.

Proof of Lemma 6.1.1. Let $W \subseteq \mathbb{E}^2_{1,1}$ denote the subspace of $\mathbb{E}^2_{1,1}$ spanned by elements (x, f) where x satisfies $\mathrm{rk}^{\mathcal{V}}_{i,k}(x), \theta(\mathcal{V})_{i,j,k}(x) \leq 1$ for all $1 \leq i < j \leq 8$, $0 \leq k \leq 1$. We will show that $W = \mathbb{E}^2_{1,1}$. By Lemma 6.1.4, it suffices to show that $(x, f) \in W$ for any $x \subseteq X_g$ and $f \in A_x$. This follows from Lemmas 6.1.6 and 6.1.8. Now, as a consequence of Lemma 6.1.2, there is a finite set of edges $y_1, \ldots, y_n \subseteq X_g$ given by all possible combinations of genera $g(\mathcal{H}(y_\ell))$ and choices of $\mathrm{rk}^{\mathcal{V}}_{i,k}$ and $|\theta(\mathcal{V})_{i,j,k}|$ less than or equal to one, such that any $x \subseteq X_g$ with $\mathrm{rk}^{\mathcal{V}}_{i,k}(x), \theta(\mathcal{V})_{i,j,k}(x) \leq 1$ for all $1 \leq i < j \leq 8$, $0 \leq k \leq 1$ is in the same G–orbit as some y_ℓ, so the natural map

$$\bigoplus_{1 \leq \ell \leq n} \mathrm{Ind}^G_{\mathrm{Stab}_G(y_\ell)} A_{y_\ell} \to W$$

is surjective. Therefore the map

$$H_0\left(G; \bigoplus_{1 \leq \ell \leq n} \mathrm{Ind}^G_{\mathrm{Stab}_G(y_\ell)} A_{y_\ell}\right) \to H_0(G; W)$$

is surjective since $H_0(G, -)$ is left exact. Then Shapiro's lemma says that

$$H_0\left(G; \bigoplus_{1 \leq \ell \leq n} \mathrm{Ind}^G_{\mathrm{Stab}_G(y_\ell)} A_{y_i}\right) \cong \bigoplus_{1 \leq \ell \leq n} H_0(\mathrm{Stab}_G(y_\ell); A_{y_\ell}).$$

But then A_{y_ℓ} is contained in $H_1(\mathrm{Stab}_{\mathcal{I}_g}(a); \mathbb{Q})$ for all y_ℓ, so A_{y_ℓ} is finite dimensional for all y_ℓ. Therefore $H_0(\mathrm{Stab}_G(y_\ell); A_{y_\ell})$ is finite dimensional for any $1 \leq \ell \leq n$, so the proof is complete. □

6.2 The Proof of Theorem C

In this section, we will complete the proof of Proposition 1.3.2, which, along with Proposition 1.3.1, completes the proof of Theorem C. For the remainder of this section, unless otherwise specified, fix a $g \geq 33$ and $a \subseteq S_g$ a nonseparating curve, and set $\vec{x} = [a]$. We will also let $\mathbb{E}^r_{p,q}$ denote the equivariant homology spectral sequence for the action of $\mathcal{I}_g$ on

$C_{\vec{x}}(S_g)$.

Outline of Section 6.2. We will devote the bulk of the section to proving Lemma 6.2.1, which is done in Section 6.2.1. The statement of Lemma 6.2.1 requires some notation, so we defer the statement for a moment. We then use Lemma 6.1.1 and 6.2.1 to prove Proposition 1.3.2 in Section 6.2.2, and then use Proposition 1.3.2 and Proposition 1.3.1 to prove Theorem C.

6.2.1 The proof of Lemma 6.2.1

Let $\mathcal{V} = \{v_1, \ldots, v_k\} \subseteq \vec{x}^\perp \subseteq H_1(S_g;\mathbb{Z})$ be a set of primitive elements. Let $x \subseteq X_g$ be an edge with $\mathcal{H}(x) = \{\mathcal{H}_0^x, \mathcal{H}_1^x\}$. Let $A_x^{\mathcal{V}}$ denote the subspace of $\operatorname{im}(\tau_g) = \wedge^3 H_1(S_g;\mathbb{Q})/H_1(S_g;\mathbb{Q})$ given by

$$\left(\operatorname{im}\left(\wedge^3\mathcal{H}_0^x \oplus \wedge^3\mathcal{H}_1^x \to \operatorname{im}(\tau_g)\right) \cap \operatorname{im}\left(\wedge^3\mathcal{V}^\perp \to \operatorname{im}(\tau_g)\right)\right) \otimes \mathbb{Q}.$$

Let $X_g^{\mathcal{V}} \subseteq X_g$ denote the subcomplex consisting of cells σ such that $\mathcal{H}(\sigma)$ is *compatible with* $\mathcal{V}$, i.e., every $v \in \mathcal{V}$ satisfies $v \in \mathcal{H}$ for some $\mathcal{H} \in \mathcal{H}(\sigma)$. Let $\mathbb{E}_{1,1}^{2,\mathcal{V}}$ denote the image of the composition

$$\bigoplus_{x \in \left(X_g^{\mathcal{V}}\right)^{(1)}} A_x^{\mathcal{V}} \to \bigoplus_{x \in X_g^{(1)}} A_x \to \mathbb{E}_{1,1}^2$$

where A_x is as in Section 6.1. If $\mathcal{V} = \{v\}$ is a singleton, we will denote $\mathbb{E}_{1,1}^{2,\mathcal{V}}$ and $A_x^{\mathcal{V}}$ by $\mathbb{E}_{1,1}^{2,v}$ and A_x^v respectively. We are now ready to state Lemma 6.2.1.

Lemma 6.2.1. *Let $\mathcal{V} = \{v_1, \ldots, v_9\} \subseteq \vec{x}^\perp$ be a set of primitive elements such that there is a nonseparating multicurve $M \subseteq S_g \setminus a$ with $\mathcal{V} = \{[c] : c \in M\}$. Then the natural map*

$$\xi : \bigoplus_{v_i \in \mathcal{V}} \mathbb{E}_{1,1}^{2,v_i} \to \mathbb{E}_{1,1}^2$$

is surjective.

The outline of the proof of Lemma 6.2.1. We will prove Lemmas 6.2.2, 6.2.3 and 6.2.6. The first two lemmas are statements about generating sets for vector spaces equipped with alternating forms, while the last is a statement about a generating set for the first rational homology of a certain subcomplex of X_g. In order to prove Lemma 6.2.6, we will prove Lemma 6.2.4 and Lemma 6.2.5, which are auxiliary results about rational abelianizations of subgroups and quotients of the Torelli group. We use Lemma 6.2.2, Lemma 6.2.3, and Lemma 6.2.6 to prove Lemma 6.2.1.

Lemma 6.2.2. *Let V be a finite dimensional $\mathbb{Q}$–vector space equipped with an alternating form $\langle \cdot, \cdot \rangle$. Let v_1, v_2, v_3 be elements in V such that the image of the set $\{v_1, v_2, v_3\}$ under the adjoint map $V \to \mathrm{Hom}_{\mathbb{Q}}(V, \mathbb{Q})$ is linearly independent. Then the natural map*

$$\psi : \bigoplus_{i \in \{1,2,3\}} \wedge^2 v_i^{\perp} \to \wedge^2 V$$

is surjective.

Proof. Let $\mathcal{B} = \{a_1, \ldots, a_n\}$ be a basis for V such that $\langle v_i, a_j \rangle = \delta_{ij}$. Let

$$\mathcal{B}_i = \{a_1, \ldots, \widehat{a_i}, \ldots, a_n\},$$

which is a basis for $v_i^{\perp}$ for any $i \in \{1, 2, 3\}$. The vector space $\wedge^2 V$ has a basis consisting of pairs of elements in $\mathcal{B}$. Each pair of these elements is contained in at least one $\mathcal{B}_i$. Therefore $\mathrm{im}(\psi)$ contains a basis for $\wedge^2 V$, so ψ is surjective. □

We now extend Lemma 6.2.2 as follows.

Lemma 6.2.3. *Let V be a finite dimensional $\mathbb{Q}$–vector space equipped with an alternating form $\langle \cdot, \cdot \rangle$. Let $\mathcal{V} = \{v_1, \ldots, v_n\}$ be a set of elements in V with $|\mathcal{V}| \geq 3$ such that the image of $\mathcal{V}$ under the adjoint map $V \to \mathrm{Hom}_{\mathbb{Z}}(V, \mathbb{Z})$ is linearly independent. Let $m \leq |\mathcal{V}| - 2$ be* *a natural number. Then the natural map*

$$\psi_m : \bigoplus_{\mathcal{V}' \subseteq \mathcal{V} : |\mathcal{V}'| = m} \wedge^2 (\mathcal{V}')^{\perp} \to \wedge^2 V$$

is surjective.

Proof. We proceed by induction on m.

Base case: $m = 1$. This is exactly the content of Lemma 6.2.2.

Inductive step: the lemma holds for $m' < m$. Let ψ_m be as in the lemma. By the inductive hypothesis, the map ψ_{m-1} is surjective. Hence it suffices to show that $\wedge^2 \left(\mathcal{W}^{\perp}\right) \subseteq \mathrm{im}(\psi_m)$ for any $\mathcal{W} \subseteq \mathcal{V}$ with $|\mathcal{W}| = m - 1$. Let $\mathcal{W} \subseteq \mathcal{V}$ be a subset with $|\mathcal{W}| = m - 1$. Since we have chosen m with $m \leq |\mathcal{V}| - 2$, we have $|\mathcal{V} \setminus \mathcal{W}| \geq 3$. Hence Lemma 6.2.2 applied to $L = \mathcal{W}^{\perp}$ and the set $\mathcal{V} \setminus \mathcal{W}$ says that the map

$$\bigoplus_{v \in \mathcal{V} \setminus \mathcal{W}} \wedge^2 (v^{\perp} \cap \mathcal{W}^{\perp}) \to \wedge^2 \mathcal{W}^{\perp}$$

is surjective. Since $\wedge^2 (v^{\perp} \cap \mathcal{W}^{\perp}) \subseteq \mathrm{im}(\varphi_m)$ for any $v \in \mathcal{V} \setminus \mathcal{W}$ because $|\{v\} \cup \mathcal{W}| = m$, we have $\wedge^2 \mathcal{W}^{\perp} \subseteq \mathrm{im}(\psi_m)$. Hence $\mathrm{im}(\psi_m) = \mathrm{im}(\psi_{m-1})$ and the inductive hypothesis says that $\mathrm{im}(\psi_{m-1})$ is surjective, so $\mathrm{im}(\psi_m)$ is surjective as well. $\square$

We will now give an explicit description of the vector space $H_1(X_g; \mathbb{Q})$.

The Johnson homomorphism, alternate description. Let $\pi_1^{(k)}(S_g)$ denote the kth term of the lower central series of the fundamental group of $\pi_1(S_g)$ (we suppress the basepoint in the notation, since the choice of basepoint does not affect the construction). The *Johnson homomorphism* is a map

$$\tau_g : \mathcal{I}_g \to \mathrm{Hom}_{\mathbb{Z}} \left(\pi_1(S_g)/\pi_1^{(1)}(S_g), \pi_1^{(1)}(S_g)/\pi_1^{(2)}(S_g) \right).$$

If $\gamma \in \pi_1(S_g)$ is a loop and $f \in \mathcal{I}_g$ is a mapping class, then $\tau_g(f)(\gamma) = \gamma^{-1} f(\gamma)$, where the element $\gamma^{-1}f(\gamma)$ is only defined up to conjugation by $\pi_1(S_g)$. Johnson showed that this is well defined as a map $\pi_1(S_g)/\pi_1^{(1)}(S_g) \to \pi_1^{(1)}(S_g)/\pi_1^{(2)}(S_g)$. Now, if $\omega' \in \wedge^2 H_1(S_g;\mathbb{Z})$ is given by $a_1 \wedge b_1 + \ldots + a_g \wedge b_g$ for $\{a_i, b_i\}_{1\leq i\leq g}$ a symplectic basis for $H_1(S_g;\mathbb{Z})$, then the Johnson homomorphism can be rewritten as a map

$$\tau_g : \mathcal{I}_g \to \operatorname{Hom}_{\mathbb{Z}}\left(H_1(S_g;\mathbb{Z}), \wedge^2 H_1(S_g;\mathbb{Z})/\mathbb{Z}\omega'\right).$$

This description of the Johnson homomorphism allows us to prove Lemma 6.2.4. If $\mathcal{W} \subseteq H_1(S_g;\mathbb{Z})$ is a set of elements, recall that $X_g^{\mathcal{W}} \subseteq X_g$ denotes the subcomplex of X_g generated by elements σ such that $\mathcal{H}(\sigma)$ is compatible with $\mathcal{W}$, i.e., for each $w \in \mathcal{W}$ there there is an $\mathcal{H} \in \mathcal{H}(\sigma)$ such that $w \in \mathcal{H}$. If $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g$ are a symplectic basis for $H_1(S_g;\mathbb{Z})$ with $\vec{x} = \alpha_1$, let $\omega'_a = \alpha_2 \wedge \beta_2 + \ldots + \alpha_g \wedge \beta_g$. If $V \subseteq H_1(S_g;\mathbb{Q})$ is a subspace, let X_g^V denote the subcomplex of X_g consisting of cells x such that $V \subseteq \mathcal{H} \otimes \mathbb{Q}$ for some $\mathcal{H} \in \mathcal{H}(\sigma)$.

Lemma 6.2.4. *Let $g \geq 4$ and let $a \subseteq S_g$ be a nonseparating simple closed curve. Let $\vec{x} = [a]$. Then there is an isomorphism $\overline{\tau_g} : H_1(X_g;\mathbb{Q}) \cong \wedge^2\vec{x}^\perp \otimes \mathbb{Q}/\mathbb{Q}\omega'_a$. Furthermore, this isomorphism is functorial in the following sense. Let $\mathcal{W} \subseteq \vec{x}^\perp$ be a set of elements such that $g(\mathcal{W}^\perp \cap \vec{x}^\perp) \geq 1$. Then $\operatorname{im}(H_1(X_g^{\mathcal{W}};\mathbb{Q}) \to H_1(X_g;\mathbb{Q}) \to \wedge^2\vec{x}^\perp \otimes \mathbb{Q}/\mathbb{Q}\omega'_a)$ contains $\operatorname{im}(\wedge^2(\mathcal{W}^\perp \cap \vec{x}^\perp) \otimes \mathbb{Q} \to \wedge^2\vec{x}^\perp \otimes \mathbb{Q}/\mathbb{Q}\omega'_a)$.*

Proof. Since $g \geq 4$, the complex $\mathcal{C}_{\vec{x}}(S_g)$ is 1–acyclic by Theorem B. Therefore the last three terms of the five term exact sequence associated to the equivariant homology spectral sequence for the action of $\mathcal{I}_g$ on $\mathcal{C}_{\vec{x}}(S_g)$ form a right exact sequence

$$H_1(\operatorname{Stab}_{\mathcal{I}_g}(a);\mathbb{Q}) \to H_1(\mathcal{I}_g;\mathbb{Q}) \to H_1(X_g;\mathbb{Q}) \to 0.$$

We have the following claim.

Claim. We have $\mathrm{im}(H_1(\mathrm{Stab}_{\mathcal{I}_g}(a);\mathbb{Q}) \to H_1(\mathcal{I}_g;\mathbb{Q})) = \ker(\mathrm{eval}_{\vec{x}} \circ \tau_g)$, where

$$\mathrm{eval}_{\vec{x}} : \mathrm{Hom}_{\mathbb{Z}}(H_1(S_g;\mathbb{Z}), \wedge^2 H_1(S_g;\mathbb{Z})/\mathbb{Z}\omega') \to \wedge^2 H_1(S_g;\mathbb{Z})/\mathbb{Z}\omega'$$

is the linear map given by evaluation on $\vec{x}$.

Proof of claim. We prove each containment in turn. If $f \in \mathrm{Stab}_{\mathcal{I}_g}(a)$, we have $\tau_g(f)(\vec{x}) = 0$, so $f \in \ker(\mathrm{eval}_{\vec{x}})$ and the $\subseteq$ containment holds. For the $\supseteq$ containment, let $f \in \mathcal{I}_g$ be a mapping class such that $\mathrm{eval}_{\vec{x}}(\tau_g(f)) = 0$. Let $\gamma \in \pi_1(X_g)$ be a loop such that γ is homotopic to a as an unbased loop. Since the choice of representative of $\vec{x}$ is arbitrary, we must have $\gamma^{-1}f(\gamma) \in \pi_1^{(2)}(S_g)$ since $\tau_g(f)(a) = 0$ by hypothesis. A theorem of Church [9, Theorem 1.1] tells us that there is some $h \in \mathcal{K}_g$, where $\mathcal{K}_g$ is the Johnson kernel [12, Section 6.6], such that $ha = f(a)$, so $h^{-1}f(a) = a$, and therefore $h^{-1}f \in \mathrm{Stab}_{\mathcal{I}_g}(a)$. Johnson [19, Lemma 4A] showed that $h \in \ker(\tau_g)$, so $[h^{-1}f] = [f] \in H_1(\mathcal{I}_g;\mathbb{Q})$, since $\mathrm{im}(\tau_g) \otimes \mathbb{Q} \cong H_1(\mathcal{I}_g;\mathbb{Q})$ [22]. Therefore $[f] \in \mathrm{im}(H_1(\mathrm{Stab}_{\mathcal{I}_g}(a);\mathbb{Q}) \to H_1(\mathcal{I}_g;\mathbb{Q}))$, so the claim holds.

Given the claim, we have an exact sequence

$$H_1(\mathrm{Stab}_{\mathcal{I}_g}(a);\mathbb{Q}) \to H_1(\mathcal{I}_g;\mathbb{Q}) \xrightarrow{\mathrm{eval}_{\vec{x}}} \wedge^2 H_1(S_g;\mathbb{Q})/\mathbb{Q}\omega'.$$

Therefore, it suffices to show that $\mathrm{im}\,(\mathrm{eval}_{\vec{x}}) = \wedge^2 \vec{x}^{\perp}/\mathbb{Q}\omega'_a$. This is a consequence of Johnson's computation of the image of the Johnson homomorphism [19, Theorem 1].

We now use Lemma 5.1.4 to prove the second part of the lemma. In particular, it suffices to show that any element $\gamma \wedge \delta \in \wedge^2 \mathcal{W}^{\perp} \cap \wedge^2 \vec{x}^{\perp}$ with γ, δ primitive and $\langle \gamma, \delta \rangle = 1$ lies in the image of the composition

$$H_1(X_g^{\mathcal{W}};\mathbb{Q}) \to H_1(X_g;\mathbb{Q}) \to \wedge^2 \vec{x}^{\perp} \otimes \mathbb{Q}/Q\omega'_a.$$

Let $x \subseteq X_g$ be an edge, $\widehat{x}$ a lift of x to $\mathcal{C}_{\vec{x}}(S_g)$ with one vertex equal to a, and $f \in \mathcal{I}_g$

a mapping class taking a to the other endpoint of $\widehat{x}$. By the construction of the equivariant homology spectral sequence, the class $[x] \in H_1(X_g;\mathbb{Q})$ is the image under the map $H_1(\mathcal{I}_g;\mathbb{Q}) \to H_1(X_g;\mathbb{Q})$ of the class $[f] \in H_1(\mathcal{I}_g;\mathbb{Q})$. Then by alternate definition of τ_g given before the lemma, we have $\mathrm{eval}_{\vec{x}}(f) = \omega_{\mathcal{H}}$, where $\omega_{\mathcal{H}} \in \wedge^2 H_1(S_g;\mathbb{Q})$ is the characteristic element for some $\mathcal{H} \in \mathcal{H}(x)$. Then for any $\gamma \wedge \delta$ with γ, δ primitive and $\langle \gamma, \delta \rangle = 1$, there is an $x \subseteq X_g$ and $\mathcal{H} \in \mathcal{H}(x)$ with $\omega_{\mathcal{H}} = \gamma \wedge \delta$. The set of such $\gamma \wedge \delta$ spans $\wedge^2 \vec{x}^{\perp} \otimes \mathbb{Q}$ by Lemma 5.1.4, so the image of the composition

$$H_1(X_g^{\mathcal{W}};\mathbb{Q}) \to H_1(X_g;\mathbb{Q}) \to \wedge^2 \vec{x}^{\perp} \otimes \mathbb{Q}/Q\omega'_a.$$

contains $\mathrm{im}(\wedge^2(\mathcal{W}^{\perp} \cap \vec{x}^{\perp}) \otimes \mathbb{Q} \to \wedge^2 \vec{x}^{\perp} \otimes \mathbb{Q}/\mathbb{Q}\omega'_a)$, so the proof of the lemma is complete.

□

We will need another auxiliary lemma about the rational abelianizations of the Torelli groups of surfaces.

Lemma 6.2.5. *Let $g \geq 4$, and let $\mathcal{T}_0, \mathcal{T}_1$ be two surfaces each equipped with an embedding $\iota^i : \mathcal{T}_i \hookrightarrow S_g$. Assume that the following hold:*

- *each embedding ι^i is clean,*
- *$S_g \searrow \iota^i(\mathcal{T}_i)$ is connected,*
- *$g(\mathcal{T}_i) \geq 3$ for $i = 0, 1$,*
- *the pullback S of the maps ι^0 and ι^1 is a connected, smooth manifold,*
- *$S_g \searrow (\iota^0(\mathcal{T}_0) \cap \iota^1(\mathcal{T}_1))$ is connected, and*
- *$g(S) \geq 3$.*

The following commutative square

$$\begin{array}{ccc} H_1(\mathcal{I}(S,S_g);\mathbb{Q}) & \longrightarrow & H_1(\mathcal{I}(\mathcal{T}_1,S_g);\mathbb{Q}) \\ \downarrow & & \downarrow \iota^1_* \\ H_1(\mathcal{I}(\mathcal{T}_2,S_g);\mathbb{Q}) & \xrightarrow{\iota^2_*} & H_1(\mathcal{I}_g;\mathbb{Q}), \end{array}$$

is a pullback square.

Proof. For each $\mathcal{T}_i$, let κ^i denote the map $S \to \mathcal{T}_i$, and let ι denote the composition $\iota^i \circ \kappa^i$. Note that since S is a pullback, we have $\iota^1 \circ \kappa^1 = \iota^0 \circ \kappa^0$, so the definition of ι does not depend on i. Now, since we have assumed that each embedding ι^i is clean, we have ι^i_* injective by a theorem of Putman [37, Theorem B]. The map ι is also clean by hypothesis, and therefore the pushforward ι_* is injective by the same theorem of Putman, so it suffices to show that $\mathrm{im}(\iota_*) = \mathrm{im}(\iota^0_*) \cap \mathrm{im}(\iota^1_*)$. Lemma 5.1.5 says that that $\mathrm{im}(\mathcal{I}(\iota^i)_*) = \mathrm{im}(\wedge^3 H_1(\mathcal{T}_i;\mathbb{Q}) \to \wedge^3 H_1(S_g;\mathbb{Q}))$ and $\mathrm{im}(\mathcal{I}(\iota)_*) = \mathrm{im}(\wedge^3 H_1(S;\mathbb{Q}) \to \wedge^3 H_1(S_g;\mathbb{Q}))$. Since the functor $\wedge^3$ from $\mathbb{Q}$–vector spaces to $\mathbb{Q}$–vector spaces sends pullbacks of monomorphisms to pullbacks of monomorphisms, it is enough to show that

$$\mathrm{im}(H_1(\mathcal{T}_0;\mathbb{Q}) \to H_1(S_g;\mathbb{Q})) \cap \mathrm{im}(H_1(\mathcal{T}_1;\mathbb{Q}) \to H_1(S_g;\mathbb{Q}))$$

is equal to

$$\mathrm{im}(H_1(S;\mathbb{Q}) \to H_1(S_g;\mathbb{Q})).$$

This follows from our hypotheses that $S_g \setminus \iota(S)$ and $S_g \setminus \iota^i(\mathcal{T}_i)$ are connected, so embedding S and $\mathcal{T}_i$ into S_g does not introduce any new relations in $H_1(S;\mathbb{Q})$ or $H_1(\mathcal{T}_i;\mathbb{Q})$. $\square$

We now prove the following.

Lemma 6.2.6. *Let $L \subseteq \vec{x}^\perp$ be a free abelian subgroup with $g(L) = 2$ and $\dim(L \otimes \mathbb{Q}) = 4$. Let $V = L \otimes \mathbb{Q}$. Let $\mathcal{V}$ be as in Lemma 6.2.1. Then the natural map*

$$\psi : \bigoplus_{\mathcal{V}' \subseteq \mathcal{V} : |\mathcal{V}'| = 4} H_1(X_g^V \cap X_g^{\mathcal{V}'};\mathbb{Q}) \to H_1(X_g^V;\mathbb{Q})$$

is surjective.

Proof. We begin with the following claim.

Claim. The pushforward $H_1(X_g^V;\mathbb{Q}) \to H_1(X_g;\mathbb{Q})$ is an injection, and the image of this map is sent to $\wedge^2 V^\perp \cap \wedge^2 \vec{x}^\perp$ under the isomorphism $H_1(X_g;\mathbb{Q}) \cong \wedge^2\vec{x}^\perp/\mathbb{Q}\omega_\alpha$ from Lemma 6.2.4.

Proof of claim. Let $S_2^1 \subseteq S_g$ be a compact subsurface such that

$$\operatorname{im}\big(H_1(S_2^1;\mathbb{Z}) \hookrightarrow H_1(S_g;\mathbb{Z})\big) = V.$$

Let $\mathcal{I}_{g-2}^1$ denote the subgroup of $\mathcal{I}_g$ generated by elements that fix ∂S_2^1 and restrict to the identity on S_2^1. Let $\mathcal{C}_{\vec{x}}(S_g, S_2^1)$ denote the subcomplex of $\mathcal{C}_{\vec{x}}(S_g)$ generated by curves c such that c is disjoint from S_2^1. By a result of Kent, Leininger and Schleimer [24, Theorem 7.2], the fibers of the natural map $\mathcal{C}_{\vec{x}}(S_g, S_2^1) \to \mathcal{C}_{\vec{x}}(S_{g-2})$ are all trees, so $\mathcal{C}_{\vec{x}}(S_g, S_2^1)$ is homotopy equivalent to $\mathcal{C}_{\vec{x}}(S_{g-2})$. Then by Theorem B, $\mathcal{C}_{\vec{x}}(S_{g-2})$ is at least 1–acyclic, and thus $\mathcal{C}_{\vec{x}}(S_g, S_2^1)$ is at least 1–acyclic as well. Therefore the equivariant homology spectral sequence $\mathbb{E}^r_{p,q}(\mathcal{I}^1_{g-2}, \mathcal{C}_{\vec{x}}(S_g, S_2^1);\mathbb{Q})$ converges to $H_1(\mathcal{I}^1_{g-2};\mathbb{Q})$. Hence there is a right exact sequence

$$H_1(\operatorname{Stab}_{\mathcal{I}^1_{g-2}}(a);\mathbb{Q}) \to H_1(\mathcal{I}^1_{g-2};\mathbb{Q}) \to H_1(X_g^V;\mathbb{Q}) \to 0.$$

Since $g \geq 33$, we have $g\left(S^1_{g-2} \setminus a\right) \geq 3$. Then the inclusion $S^1_{g-2} \setminus a \hookrightarrow S^1_{g-2}$ is a clean embedding (as in Section 4.2), so a theorem of Putman [37, Theorem B] says that the pushforward map $H_1(\operatorname{Stab}_{\mathcal{I}^1_{g-2}}(a);\mathbb{Q}) \to H_1(\mathcal{I}^1_{g-2};\mathbb{Q})$ is an injection. Hence the above right exact sequence is in fact exact:

$$0 \to H_1(\operatorname{Stab}_{\mathcal{I}^1_{g-2}}(a);\mathbb{Q}) \to H_1(\mathcal{I}^1_{g-2};\mathbb{Q}) \to H_1(X_g^V;\mathbb{Q}) \to 0.$$

Furthemore, there is a morphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_1(\mathrm{Stab}_{\mathcal{I}^1_{g-2}}(a);\mathbb{Q}) & \longrightarrow & H_1(\mathcal{I}^1_{g-2};\mathbb{Q}) & \longrightarrow & H_1(X_g^V;\mathbb{Q}) & \longrightarrow & 0 \\ & & \downarrow{\rho_a} & & \downarrow{\rho} & & \downarrow{\overline{\rho}} & & \\ 0 & \longrightarrow & H_1(\mathrm{Stab}_{\mathcal{I}_g}(a);\mathbb{Q}) & \longrightarrow & H_1(\mathcal{I}_g;\mathbb{Q}) & \longrightarrow & H_1(X_g;\mathbb{Q}) & \longrightarrow & 0, \end{array}$$

where ρ_a and ρ are the natural pushforward maps. Lemma 6.2.5 says that left square is a pullback. Furthermore, each map in this square is an injection by the aforementioned theorem of Putman, so the map $\mathrm{cok}(\rho_a) \to \mathrm{cok}(\rho)$ is injective. Furthermore, since ρ is injective, we have $\ker(\rho) = 0$. Then the snake lemma says that there is an exact sequence

$$\ker(\rho) \to \ker(\overline{\rho}) \to \mathrm{cok}(\rho_a) \to \mathrm{cok}(\rho).$$

The map $\mathrm{cok}(\rho_a) \to \mathrm{cok}(\rho)$ is injective and $\ker(\rho) = 0$, so $\ker(\overline{\rho}) = 0$, and in particular we see that $\overline{\rho} : H_1(X_g^V;\mathbb{Q}) \to H_1(X_g;\mathbb{Q})$ is an injection. The second part of the claim follows from the fact that for any edge $x \subseteq X_g^V$, the characteristic element $\omega_{\mathcal{H}}$ for any $\mathcal{H} \in \mathcal{H}(x)$ lies in $\wedge^2 V^\perp \cap \wedge^2 \vec{x}^\perp \otimes \mathbb{Q}$. This implies that $\mathrm{im}\left(H_1(X_g^V;\mathbb{Q}) \to H_1(X_g;\mathbb{Q})\right)$ is contained in $\wedge^2 V^\perp \cap \wedge^2 \vec{x}^\perp$. Then the second part of Lemma 6.2.4 gives the containment in the other direction, so we have equality.

Given the claim, we now continue with the proof of the lemma. The claim says that $H_1(X_g^V;\mathbb{Q}) \cong \wedge^2[V]^\perp \cap \vec{x}^\perp \otimes \mathbb{Q}$. Since $\dim(V) = 4$ and the elements of $\mathcal{V}$ have pairwise trivial algebraic intersection, the set $\mathrm{proj}_{V^\perp}(\mathcal{V})$ contains at least six linearly independent elements. Let $\mathcal{B} \subseteq \mathcal{V}$ be a subset of maximal size with $\mathcal{B}' = \mathrm{proj}_{V^\perp}(\mathcal{B})$ linearly independent. Since we have assumed that each element $v \in \mathcal{V}$ has trivial algebraic intersection with $\vec{x}$, we have $\mathcal{B} \subseteq \vec{x}^\perp \cap V^\perp$. Then since $|\mathcal{B}'| = 6$, Lemma 6.2.3 applied to the set $\mathcal{B}'$ and the vector space $V^\perp \cap \vec{x}^\perp$ with $m = 4$ says that the natural map

$$\psi : \bigoplus_{\mathcal{V}' \subseteq \mathcal{B}' : |\mathcal{V}'| = 4} \wedge^2(V^\perp \cap (\mathcal{V}')^\perp \cap \vec{x}^\perp) \otimes \mathbb{Q} \to \wedge^2(V^\perp \cap \vec{x}^\perp) \otimes \mathbb{Q}$$

is a surjection. Then, for any $v \in \mathcal{B}$, we have $v^\perp \cap V^\perp = w^\perp \cap V^\perp$, where $w = \mathrm{proj}_V(v)$, so the natural map

$$\psi : \bigoplus_{\mathcal{V}' \subseteq \mathcal{B} : |\mathcal{V}'| = 4} \wedge^2(V^\perp \cap (\mathcal{V}')^\perp) \otimes \mathbb{Q} \to \wedge^2(V^\perp \cap \vec{x}^\perp) \otimes \mathbb{Q}$$

is a surjection. We have $H_1(X_g^V) \cong \wedge^2(V^\perp \cap \vec{x}^\perp) \otimes \mathbb{Q}$ by the claim. Furthermore, by Lemma 6.2.4, we have $\mathrm{im}(H_1(X_g^V \cap X_g^v); \mathbb{Q}) \to H_1(X_g; \mathbb{Q})) \supseteq \wedge^2(V^\perp \cap v^\perp)$ for any $v \in \mathcal{B}$. Then by substituting in each of these homology groups in the previous equality, we see that the natural map

$$\psi : \bigoplus_{\mathcal{V}' \subseteq \mathcal{B} : |\mathcal{V}'| = 4} H_1(X_g^V \cap X_g^{\mathcal{V}'}; \mathbb{Q}) \to H_1(X_g^V; \mathbb{Q})$$

is surjective, so the lemma is complete. □

We are now almost ready to conclude Section 6.2.1. We will first prove the following auxiliary result, and then we will prove Lemma 6.2.1.

Lemma 6.2.7. *Let $(x, f) \in \bigoplus_{x \in X_g^{(1)}} A_x$ be a class. There is a linear combination in $\mathbb{E}^2_{1,1}$ given by*

$$(x, f) = \sum_{i=1}^{m} (x_i, f_i)$$

such that, for each $1 \leq i \leq m$, each x_i has a representative $\widehat{x}_i \subseteq \mathcal{C}_{\vec{x}}(S_g)$ and f_i has a representative $F_i \in \mathrm{Stab}_{\mathcal{I}_g}(\widehat{x}_i)$. These representatives have the property that there is a subsurface $\mathcal{T}_i \subseteq S_g$ with $\mathcal{T}_i \cong S_2^1$, F_i supported on $\mathcal{T}_i$, and $\widehat{x}_i$ disjoint from $\mathcal{T}_i$.

Proof. Let $\widehat{x} \subseteq \mathcal{C}_{\vec{x}}(S_g)$ be a representative for x. Let S', S'' be the connected components of $S_g \setminus \widehat{x}$. We have a surjection $\mathcal{I}(S', S_g) \times \mathcal{I}(S'', S_g) \to \mathrm{Stab}_{\mathcal{I}_g}(\widehat{x})$, and hence by the Künneth formula we have a surjection

$$H_1(\mathcal{I}(S', S_g); \mathbb{Q}) \oplus H_1(\mathcal{I}(S'', S_g); \mathbb{Q}) \to H_1(\mathrm{Stab}_{\mathcal{I}_g}(\widehat{x}); \mathbb{Q}).$$

Hence we may assume that, without loss of generality, f is represented by a mapping class

$F \in \mathcal{I}(S', S_g)$. Furthermore, we may assume that $g(S') \geq 4$. Indeed, if $g(S') \leq 3$, then $g \geq 33$ implies that there is a 2–cell $\widehat{\sigma} \subseteq \mathcal{C}_{\vec{x}}(S_g)$ such that $\widehat{x} \subseteq \widehat{\sigma}$, and such that for $\widehat{y}, \widehat{z} \subseteq \widehat{\sigma}$ the other two edges, the connected component of $S_g \searrow \widehat{y}$ that contains S' has genus at least 3, and similarly for $\widehat{z}$. Then since $g(S') \geq 4 \geq 3$, Lemma 5.1.3 says that $\mathcal{I}(S', S_g)$ is generated by bounding pair maps. Then bounding pair maps supported on separating curves vanish in $H_1(\mathcal{I}_g; \mathbb{Q})$ since Dehn twists along separating curves vanish under τ_g. Then Lemma 5.1.5 says that if $\iota : S' \to S_g$ is the inclusion map, the pushforward $\mathcal{I}(\iota)_*$ in H_1 is injective, so bounding pair maps supported on separating curves are trivial in $H_1(\mathcal{I}(S', S_g); \mathbb{Q})$. Hence the class f is a linear combination of classes represented by bounding pair maps supported on nonseparating curves contained in S', so we may assume that f has a representative $T_{c,c'} \in \mathcal{I}(S', S_g)$ for $c \cup c'$ a bounding pair with $[c] \neq 0$. We now have two cases.

Case 1: $[c] \neq \vec{x}$. We first show that we can assume that no connected component of $S' \searrow (c \cup c')$ has genus zero. Suppose otherwise, so one connected component of

$$S' \searrow (c \cup c')$$

is a subsurface P with $P \cong S_0^4$. Since we have assumed that $g(S') \geq 4 \geq 3$, there is another curve c'' disjoint from $c \cup c'$ and not equal to c or c' with $[c''] = [c]$ and such that both connected components of $S' \searrow (c \cup c'')$ have positive genus. Now, we have $T_{c,c'} = T_{c,c''} T_{c'',c'}$, and both $c \cup c''$ and $c'' \cup c'$ satisfied the desired condition on the genera of connected components. Now, assuming that both connected components of $S' \searrow (c \cup c')$ have positive genus, we can rewrite $T_{c,c'}$ as a product of bounding pair maps $T_{c_0,c_1} T_{c_1,c_2} \ldots T_{c_{n-1},c_n}$ such that:

- $c_0 = c$, $c_n = c'$, and
- at least one connected component of $S' \searrow (c_i \cup c_{i+1})$ has genus one.

Hence we may assume without loss of generality that $c \cup c'$ is supported on a surface of genus one with two boundary components. Then there is a surface $\mathcal{T}$ that contains $c \cup c'$ and is disjoint from $\widehat{x}$ such that $\mathcal{T} \cong S_2^1$, as desired.

Case 2: $[c] = \vec{x}$. We will reduce to Case 1, by showing that $[T_{c,c'}] \in H_1(\mathcal{I}(S', S_g); \mathbb{Q})$ is a sum $[T_{d,d'}] + [T_{e,e'}]$ with $d \cup d'$ and $e \cup e'$ bounding pairs such that $[d], [e] \neq \vec{x}$. As in Case 1, we may assume that the connected component of $S' \setminus (c \cup c')$ that does not contain $\widehat{x}$ has genus one. Then by Lemma 5.1.5, we have $H_1(\mathcal{I}(S', S_g); \mathbb{Q}) \cong \wedge^3 H_1(S'; \mathbb{Q})$. By Lemma 5.1.2, if the connected component S'' of $S' \setminus (c \cup c')$ that does not contain $\widehat{x}$ has $a, b \in H_1(S''; \mathbb{Z})$ a pair of primitive elements with $\langle a, b \rangle = 1$, then the image of $T_{c,c'}$ in $\wedge^3 H_1(S'; \mathbb{Q})$ is $\vec{x} \wedge [b] \wedge [c]$. Now, choose nonzero primitive $[d], [e] \in H_1(S'; \mathbb{Z})$ such that $[d] + [e] = [c]$ and $[d], [e] \in \vec{x}^{\perp} \cap [b]^{\perp}$. Then we take $T_{d,d'}$ and $T_{e,e'}$ bounding pair maps in $\mathcal{I}(S', S_g)$ such that $[T_{d,d'}] = \vec{x} \wedge [b] \wedge [d]$ and $[T_{e,e'}] = \vec{x} \wedge [b] \wedge [e]$. □

We are now ready to conclude Section 6.2.1.

Proof of Lemma 6.2.1. Let $(x, f) \in \bigoplus_{x \in X_g^{(1)}} A_x$ be a class, where $x \subseteq X_g$ is an edge and $f \in A_x$. Let $\widehat{x}$ be a lift of x to $\mathcal{C}_{\vec{x}}(S_g)$ such that a is a vertex of $\widehat{x}$. We will show that the image of (x, f) in $\mathbb{E}^2_{1,1}$ is contained in $\operatorname{im}(\xi)$. By Lemma 6.2.7, it suffices to prove the result in the case that f is represented by a bounding pair map $F \in \operatorname{Stab}_{\mathcal{I}_g}(\widehat{x})$ and that there is an inclusion $\iota : S_2^1 \hookrightarrow S_g$ such that $\widehat{x}$ is disjoint from $\operatorname{im}(\iota)$ and F is supported on $\operatorname{im}(\iota)$. Let $V = \operatorname{im}(\iota_*)$. By Lemma 6.2.6, there is a linear combination in $H_1(X_g^V; \mathbb{Q})$ given by

$$[x] = \sum_{i=0}^{n} \lambda_i [x_i]$$

such that each x_i is contained in $X_g^{\mathcal{V}'} \cap X_g^V$ for some $\mathcal{V}' \subseteq \mathcal{V}$ with $|\mathcal{V}'| = 4$. Since each $x_i \subseteq X_g^V$, we have $f \in A_{x_i}$ for each x_i, and thus there is a linear combination in $\mathbb{E}^2_{1,1}$ given by

$$(x, f) = \sum_{i=0}^{n} (x_i, f).$$

Hence it is enough to prove the result in the case that $x \in X_g^{\mathcal{V}'}$ for some $\mathcal{V}' \subseteq \mathcal{V}$ with

$|\mathcal{V}'| = 4$. Let $\mathcal{H}(x) = \{\mathcal{H}_0^x, \mathcal{H}_1^x\}$ and, without loss of generality, assume $V \subseteq \mathcal{H}_0^x$. Now, if $\mathcal{V}' \not\subseteq \mathcal{H}_0^x$ for $i = \{0, 1\}$, we have $(x, f) \in \mathbb{E}_{1,1}^{2,v}$ for any $v \in \mathcal{V}'$. Otherwise, the vector space $\mathcal{H}_0^x \otimes \mathbb{Q}$ and the set $\mathcal{V}'$ satisfy the hypotheses of Lemma 5.1.8. Hence there is a linear combination

$$f = \sum_{j=1}^{m} \lambda_j f_j$$

where each $f_j \in \wedge^3(\mathcal{H}_0^x \cap v_i^\perp) \otimes \mathbb{Q}$ for some $v_i \in \mathcal{V}'$. Hence in $\mathbb{E}_{1,1}^2$, we have

$$(x, f) = \sum_{j=1}^{m} \lambda_j (x, f_j)$$

with each $(x, f_j) \in \mathbb{E}_{1,1}^{2,v}$ for some $v \in \mathcal{V}' \subseteq \mathcal{V}$. Therefore we have $(x, f) \in \mathrm{im}(\xi)$ as desired. □

6.2.2 The proof of Theorem C

We begin by proving Proposition 1.3.2, which says that $\mathbb{E}_{1,1}^2$ is finite dimensional.

Proof of Proposition 1.3.2. Let $G = \mathrm{im}(\mathrm{Mod}(S_g \setminus a) \to \mathrm{Sp}(2g, \mathbb{Z}))$. We will show that the hypotheses of Proposition 4.1.1 are satisfied for the G–representation $\mathbb{E}_{1,1}^2$ with $d = 9$. The first hypothesis is the content of Lemma 6.2.1 and the second is the content of Lemma 6.1.1, so $\mathbb{E}_{1,1}^2$ is finite dimensional by Proposition 4.1.1. □

We are now ready to complete Section 6.2.

Proof of Theorem C. Let $g \geq 33$ and $\mathbb{E}_{*,*}^*$ denote the equivariant homology spectral sequence for the action of $\mathcal{I}_g$ on $C_{\vec{x}}(S_g)$. As a consequence of Theorem B, $\mathbb{E}_{p,q}^r$ converges to $H_2(\mathcal{I}_g; \mathbb{Q})$ for $p + q = 2$. Hence it suffices to show that $\mathbb{E}_{2,0}^2$ and $\mathbb{E}_{1,1}^2$ are finite dimensional. The vector space $\mathbb{E}_{2,0}^2$ is finite dimensional by Proposition 1.3.1 and $\mathbb{E}_{1,1}^2$ is finite dimensional by Proposition 1.3.2. □

CHAPTER 7

THE PROOF OF THEOREM A

7.1 The proof of Theorem A

We now prove Theorem A, which we recall says that $H_2(\mathcal{I}_g;\mathbb{Q})$ is finite dimensional for $g \geq 33$. We will begin by proving Lemma 7.1.1, which is an alternate proof of a corollary of a result of Kassabov and Putman result [23, Theorem A]. We will then use Theorem C along with Proposition 4.1.1 to prove Theorem A.

Lemma 7.1.1. *Let $g \geq 3$. The vector space $H_0(\mathrm{Sp}(2g,\mathbb{Z}); H_2(\mathcal{I}_g;\mathbb{Q}))$ is finite dimensional.*

Proof. Let $\mathbb{E}^r_{p,q}$ be the Leray–Serre spectral sequence associated to the short exact sequence

$$1 \to \mathcal{I}_g \to \mathrm{Mod}(S_g) \to \mathrm{Sp}(2g,\mathbb{Z}) \to 1.$$

The vector space $H_2(\mathrm{Mod}(S_g);\mathbb{Q})$ is finite dimensional [12, Section 5.4]. Then since $H_1(\mathcal{I}_g;\mathbb{Q})$ is finite dimensional [20], the modules $\mathbb{E}^2_{p,q} = H_p(\mathrm{Sp}(2g,\mathbb{Z}); H_q(\mathcal{I}_g;\mathbb{Q}))$ are all finite dimensional for $q \leq 1$ [39, Corollary 3]. Hence both the image and the kernel of the pushforward map

$$H_0(\mathrm{Sp}(2g,\mathbb{Z}); H_2(\mathcal{I}_g;\mathbb{Q})) \to H_2(\mathrm{Mod}(S_g);\mathbb{Q})$$

are finite dimensional, so $H_0(\mathrm{Sp}(2g,\mathbb{Z}); H_2(\mathcal{I}_g;\mathbb{Q}))$ is finite dimensional. □

We now prove the main result of the thesis.

Proof of Theorem A. Let $G = \mathrm{Sp}(2g,\mathbb{Z})$ and $V = H_2(\mathcal{I}_g;\mathbb{Q})$. We will show that the G–representation V satisfies the hypotheses of Proposition 4.1.1 for $d = 1$, and hence is finite dimensional. The first hypothesis is that for any primitive $v \in H_1(S_g;\mathbb{Z})$, the cokernel of

$$H_2(\mathcal{I}_g;\mathbb{Q})^{T_v} \to H_2(\mathcal{I}_g;\mathbb{Q})$$

is finite dimensional, which is the content of Theorem C. The second is that $H_2(\mathcal{I}_g;\mathbb{Q})^G$ is finite dimensional, which is the content of Lemma 7.1.1. Hence Proposition 4.1.1 says that $H_2(\mathcal{I}_g;\mathbb{Q})$ is finite dimensional for $g \geq 33$, as desired. □

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