TOPICS IN POINT SET TOPOLOGY

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Abstract. These notes supplement Point Set Topology courses taught at the University of Chicago in Winters 2021, 2023 and 2024.

Contents

1. Tychonoff's Theorem

1.1. Statement of the Theorem.

Theorem 1.1 (Tychonoff). *Any product of compact spaces is compact (in the product topology).*

1.2. **Characterization of compactness.** A collection $\mathcal{B}$ of open subsets for X is a *basis* if every open subset of X is a union of elements of $\mathcal{B}$. The elements of $\mathcal{B}$ are called *basic open sets*, and their complements are called *basic closed sets*. Note that a collection $\mathcal{K}$ of closed subsets of X are the basic closed sets for some basis if and only if every closed set of X is the intersection of elements of $\mathcal{K}$.

Definition 1.2 (Finite intersection property). A collection of subsets $\mathcal{A}$ of a set X has the *finite intersection property* if for every finite subset $A_1, \cdots, A_n \in \mathcal{A}$ the intersection $\cap_i A_i$ is nonempty.

Lemma 1.3. *Let X be a space, and fix a basis for X. The following are equivalent.*

(1) X is compact;
(2) every basic open cover has a finite subcover;
(3) if $\mathcal{A}$ is a collection of basic closed sets with the finite intersection property then $\cap_{A\in\mathcal{A}} A$ is nonempty.

Proof. The definition of compactness is that every open cover has a finite subcover; thus (1) implies (2). Conversely, suppose (2) holds, and let $\{U_\alpha\}$ be a finite cover of X. For every $x \in U_\alpha$ there is a basic open set $x \in B_x \subset U_\alpha$ and therefore the open cover $\{B_x\}$ refines $\{U_\alpha\}$. By (2) there is a finite subcover $B_{x_1}, \cdots, B_{x_n}$. Each B_{x_i} is contained in some $U_i \in \{U_\alpha\}$ and therefore $U_1, \cdots, U_n$ is a finite subcover of $\{U_\alpha\}$, so that (2) implies (1).

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Finally we show that (2) is equivalent to (3). Taking contrapositives, (2) is equivalent to the statement that if $\{B_\alpha\}$ is a family of basic open sets with no finite subcover then the entire family does not cover. For a family of sets to fail to cover X is equivalent to the statement that their complements have a nonempty intersection, so $\{B_\alpha\}$ has no finite subcover if and only if $\{X - B_\alpha\}$ has the finite intersection property, and $\cup B_\alpha \neq X$ is equivalent to $\cap(X - B_\alpha) \neq \emptyset$. Thus (2) and (3) are equivalent. $\square$

1.3. **Ultrafilters.**

Definition 1.4 (Filter)**.** Let X be a set. A collection of subsets $\mathcal{F}$ of X is a *filter on* X if

(1) every $A \in \mathcal{F}$ is a *nonempty* subset of X;
(2) if $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$; and
(3) if $A \in \mathcal{F}$ and $A \subset B \subset X$ then $B \in \mathcal{F}$.

Example 1.5*.* Suppose X is infinite. The collection $\mathcal{F}$ of all cofinite subsets of X (i.e. subsets with finite complement) is a filter.

Definition 1.6. Let $\mathcal{A}$ be a collection of subsets of X with the finite intersection property. Define $\langle\mathcal{A}\rangle$ to be the collection of all subsets B of X so that B contains some finite intersection of elements of $\mathcal{A}$. We call $\langle\mathcal{A}\rangle$ the *filter generated by* $\mathcal{A}$.

Lemma 1.7. *For any $\mathcal{A}$ with the finite intersection property $\langle\mathcal{A}\rangle$ is the smallest filter containing $\mathcal{A}$ (in particular it is a filter).*

Proof. Since $\mathcal{A}$ has the finite intersection property, every $A \in \langle\mathcal{A}\rangle$ contains a nonempty subset and is therefore nonempty. Furthermore, if A contains a finite intersection of elements of $\mathcal{A}$ then so does B for any $A \subset B \subset X$. Finally if A contains $A_{n_1} \cap \cdots \cap A_{n_k}$ and B contains $A_{m_1} \cap \cdots \cap A_{m_l}$ then $A \cap B$ contains

$$A_{n_1} \cap \cdots \cap A_{n_k} \cap A_{m_1} \cap \cdots \cap A_{m_l}$$

Thus $\langle\mathcal{A}\rangle$ is a filter containing $\mathcal{A}$.

If $\mathcal{F}$ is any filter containing $\mathcal{A}$ it necessarily contains all finite intersections of elements of $\mathcal{A}$, and all supersets of these, so it contains $\langle\mathcal{A}\rangle$. $\square$

Lemma 1.8. *Let $\{\mathcal{F}_\alpha\}$ be a collection of filters of X, simply ordered by inclusion (i.e. for all $\mathcal{F}_\alpha$, $\mathcal{F}_\beta$ either $\mathcal{F}_\alpha \subset \mathcal{F}_\beta$ or $\mathcal{F}_\beta \subset \mathcal{F}_\alpha$). Then $\mathcal{F} := \cup_\alpha \mathcal{F}_\alpha$ is a filter.*

Proof. Let $A \in \mathcal{F}$. Then $A \in \mathcal{F}_\alpha$ for some α so that A is nonempty. Furthermore, if $A \subset B \subset X$ then $B \in \mathcal{F}_\alpha$ so $B \in \mathcal{F}$. Finally, if $A, B \in \mathcal{F}$ then $A \in \mathcal{F}_\alpha$ and $B \in \mathcal{F}_\beta$ for some α, β where without loss of generality $\mathcal{F}_\alpha \subset \mathcal{F}_\beta$ so that actually $A, B \in \mathcal{F}_\beta$ and therefore $A \cap B \in \mathcal{F}_\beta \subset \mathcal{F}$. $\square$

Definition 1.9 (Ultrafilter)**.** An *ultrafilter* is a filter that is maximal with respect to inclusion; i.e. it is not a proper subset of any filter.

Zorn's Lemma and Lemma 1.8 together imply that every filter on a set is contained in some ultrafilter.

Example 1.10 (Principal ultrafilter)*.* Let $x \in X$. The collection $\mathcal{F}_x$ of all subsets of X containing x is an ultrafilter, called the *principal ultrafilter* generated by x.

Lemma 1.11. *A filter $\mathcal{F}$ is an ultrafilter if and only if, for every $A \subset X$, exactly one of A and $X - A$ is in $\mathcal{F}$.*

Proof. The sets A and $X - A$ are disjoint and therefore cannot be contained in the same filter. Suppose $\mathcal{F}$ is a filter and neither A nor $X - A$ are in $\mathcal{F}$. If some $B \in \mathcal{F}$ is disjoint from A then $B \subset X - A$ so that $X - A$ is in $\mathcal{F}$; thus A intersects every element of $\mathcal{A}$ and therefore $\mathcal{A} \cup \{A\}$ has the finite intersection property. But then $\langle \mathcal{A} \cup \{A\} \rangle$ is a filter properly containing $\mathcal{F}$, so that $\mathcal{F}$ is not maximal.

Conversely suppose $\mathcal{F}$ is a filter and for every $A \subset X$ exactly one of A and $X - A$ is in $\mathcal{F}$. Let B be a subset of X that is not in $\mathcal{F}$. Then $X - B$ is in $\mathcal{F}$, so there is no filter containing $\mathcal{F}$ and B; thus $\mathcal{F}$ is maximal. □

Lemma 1.12. *Let $\mathcal{F}$ be an ultrafilter and suppose $A \in \mathcal{F}$. If we can write $A = B \cup C$ then at least one of B, C is in $\mathcal{F}$.*

Proof. If neither of B,C are in $\mathcal{F}$ then $X - B$ and $X - C$ are in $\mathcal{F}$ so $(X - B) \cap (X - C)$ is in $\mathcal{F}$. But $(X - B) \cap (X - C) = X - (B \cup C) = X - A$ contrary to the fact that $A \in \mathcal{F}$. □

1.4. **Ultrafilters and Topology.**

Definition 1.13. Let $\mathcal{F}$ be an ultrafilter on a topological space X. We say $\mathcal{F}$ *converges to* $x \in X$ (denoted $\mathcal{F} \to x$) if $\mathcal{F}$ contains every open neighborhood of x.

Lemma 1.14. *Let X be a topological space.*

(1) X is compact if and only if every ultrafilter converges to at least one point; and
(2) X is Hausdorff if and only if every ultrafilter converges to at most one point.

Proof. First we prove (1). Suppose $\mathcal{F}$ is an ultrafilter. If $\mathcal{F}$ fails to converge to any point, then for every point x there is an open neighborhood U_x which is not in $\mathcal{F}$. Thus $X - U_x \in \mathcal{F}$ for all x. Evidently $\{U_x\}$ is an open cover. If it contained a finite subcover $U_{x_1}, \cdots, U_{x_n}$ then $\cap(X - U_{x_i})$ would be empty, despite the fact that it is a finite intersection of elements of $\mathcal{F}$. Thus X is not compact.

Conversely, suppose every ultrafilter on X converges to some point. Let $\{U_\alpha\}$ be an open cover of X. If this open cover failed to have a finite subcover then the collection $\{X - U_\alpha\}$ would have the finite intersection property, and would therefore be contained in some ultrafilter $\mathcal{F}$. By hypothesis $\mathcal{F} \to x$ for some x so $\mathcal{F}$ contains some U_α with $x \in U_\alpha$. But $\mathcal{F}$ also contains $X - U_\alpha$, which is a contradiction. Thus X is compact.

Now we prove (2). Suppose $\mathcal{F}$ converges to two distinct points $x, y \in X$. If X were Hausdorff then we could find disjoint open sets U, V with $x \in U$, $y \in V$. But $\mathcal{F}$ contains both U and V, which is absurd. Thus X is not Hausdorff.

Conversely suppose X is not Hausdorff, so that there are distinct x, y so that every neighborhood of x intersects every neighborhood of y. Thus the collection of all neighborhoods of x and of y satisfies the finite intersection property, and is contained in some ultrafilter. By construction this ultrafilter converges to both x and y. □

1.5. **Pushforward of ultrafilters.**

Let $\mathcal{F}$ be an ultrafilter on X and let $f : X \to Y$ be any map of sets. Define $f_*\mathcal{F}$ to be the collection of all subsets $A \subset Y$ for which $f^{-1}(A) \in \mathcal{F}$.

Lemma 1.15. *The collection $f_*\mathcal{F}$ is an ultrafilter.*

Proof. If A is any subset of Y then $f^{-1}(A) = X - f^{-1}(Y - A)$ so exactly one of A and $Y - A$ is in $f_*\mathcal{F}$. If $A \in f_*\mathcal{F}$ then $f^{-1}(A) \in \mathcal{F}$ is nonempty, so A is nonempty. Likewise, if $A \subset B$ and $f^{-1}(A) \in \mathcal{F}$ then $f^{-1}(A) \subset f^{-1}(B)$ so $f^{-1}(B) \in \mathcal{F}$. Finally, if A, B are in $f_*\mathcal{F}$ then $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$ so $A \cap B \in F_*\mathcal{F}$. Thus $f_*\mathcal{F}$ is an ultrafilter. □

Lemma 1.16 (Pushforward of limits)**.** *Let $f : X \to Y$ be any function between topological spaces. Then f is continuous if and only if for every ultrafilter $\mathcal{F}$ on X with $\mathcal{F} \to x$ we have $f_*\mathcal{F} \to f(x)$.*

Proof. Suppose X is continuous, and let $\mathcal{F} \to x \in X$. Let U be an open neighborhood of $f(x)$. Then $f^{-1}(U)$ is an open neighborhood of x and is therefore in $\mathcal{F}$, so that $U \in f_*\mathcal{F}$. Thus $f_*\mathcal{F} \to f(x)$.

Conversely suppose for every ultrafilter $\mathcal{F}$ on X with $\mathcal{F} \to x$ we have $f_*\mathcal{F} \to f(x)$. Let $U \subset Y$ be open and suppose $f^{-1}(U)$ is not open, so that $f^{-1}(Y - U)$ is not closed. Let $x \in f^{-1}(U)$ be a point of the closure of $f^{-1}(Y - U)$ so that every open neighborhood V of x intersects $f^{-1}(Y - U)$. The collection of all open neighborhoods V of x together with $f^{-1}(Y - U)$ has the finite intersection property, so it is contained in some ultrafilter $\mathcal{F}$; by construction $\mathcal{F} \to x$ so that $f_*\mathcal{F} \to f(x)$ so that $f_*\mathcal{F}$ contains U and therefore $\mathcal{F}$ contains $f^{-1}(U)$. But $\mathcal{F}$ contains $f^{-1}(Y - U) = X - f^{-1}(U)$ which is a contradiction. So $f^{-1}(U)$ is open after all, and f is continuous. □

1.6. **Proof of the Theorem.** We now give the proof of Theorem 1.1.

Proof. Let X_α be a collection of compact spaces, and define $X := \prod_\alpha X_\alpha$. Let $\mathcal{F}$ be an ultrafilter on X. We shall show that $\mathcal{F}$ converges to some point in X. For all α the pushforward $(\pi_\alpha)_*\mathcal{F}$ converges to some x_α; let x be the point in X such that $\pi_\alpha(x) = x_\alpha$ for all α. Then for every α and every open neighborhood U_α of x_α the set U_α is in $(\pi_\alpha)_*\mathcal{F}$ so that $\pi_\alpha^{-1}(U_\alpha) \in \mathcal{F}$.

It follows that for any finite collection of indices $\alpha_1, \cdots, \alpha_n$ and every finite collection of open sets $x_{\alpha_i} \in U_{\alpha_i}$ that $U := \prod_{i=1}^n U_{\alpha_i} \times \prod_{\beta \neq \alpha_i} X_\beta$ is in $\mathcal{F}$. But U is an arbitrary basic open neighborhood of x; thus $\mathcal{F}$ contains every open neighborhood of x, so that $\mathcal{F} \to x$. Since $\mathcal{F}$ is arbitrary, X is compact by Lemma 1.14. □

2. Alexandroff–Hausdorff Theorem

2.1. Statement of the Theorem.

Definition 2.1 (Cantor Set)**.** Let $\{0,1\}$ be a 2 element space with the discrete topology. The *Cantor set* $\mathcal{C}$ is the space $\{0,1\}^{\mathbb{N}}$ with the product topology.

Theorem 2.2 (Alexandroff–Hausdorff)**.** *The following are equivalent for a Hausdorff space X:*

- *there is a surjective continuous map $f : \mathcal{C} \to X$;*
- *X is compact, nonempty and metrizable.*

2.2. Properties of the Cantor set.

Definition 2.3. Let X be Hausdorff. We say that X is *totally disconnected* if the components are single points. We say that X is *perfect* if no point is open; equivalently, if any neighborhood of any point $x \in X$ contains a point of $X - x$.

Proposition 2.4. *Let X be any countably infinite product of finite, nonempty discrete spaces X_i. Then X is compact, nonempty, metrizable, and totally disconnected. Furthermore, if infinitely many of the X_i have more than one point, X is perfect. In particular, this is true of $X = \mathcal{C}$.*

Proof. Suppose X_i has n_i points. For each i let's choose an identification of each X_i with a subset of $[0,1]$ as follows. If $n_i = 1$ we identify X_i with the point 0 and if $n_i > 1$ we identify X_i with the multiples of $1/(n_i - 1)$.

A finite space is compact, so X is a product of compact spaces and is therefore compact by Tychonoff's Theorem 1.1. It contains (for example) the point (0) all of whose coordinates are 0, so it is nonempty.

If we give $[0,1]$ the usual Euclidean metric d we may metrize $[0,1]^{\mathbb{N}}$ by

$$d((x),(y)) := \sup_n \left(\frac{d(x_n, y_n)}{n} \right)$$

and as a subspace, X is therefore metrizable. If (x) and (y) are different points of X they must differ in at least one coordinate (say the nth one) and without loss of generality $x_n < y_n$. Then the clopen subsets $\pi_n^{-1}([0, x_n])$ and $\pi_n^{-1}((x_n, 1])$ give a separation of X for which (x) and (y) are in different sets. Thus X is totally disconnected.

Finally let's suppose infinitely many of the X_i have more than one point. If (x) is arbitrary and $\epsilon > 0$ there is an n so that $\epsilon > 1/n$. Then any (y) whose first $n-1$ coordinates agree with (x) and differs in some mth coordinate with $m > n$ is within the ϵ-neighborhood of x for the metric above but is not equal to (x). Thus X is perfect. □

2.3. **Images of compact metric spaces.**

Proposition 2.5. *Let $f : A \to B$ be surjective. If A is a compact and metrizable, and B is Hausdorff, then B is compact and metrizable.*

Proof. The image of any compact space is compact, since an open cover of the range pulls back to an open cover of the domain which has a finite refinement. So it suffices to show B is metrizable.

We use the Urysohn metrization Theorem, which says that any normal space (for instance, any compact Hausdorff space) with a countable basis is metrizable. Every compact metric space has a countable basis (for example, take the union of any collection of finite covers by open balls of radius $1/n$ for all n). Let $\{U_i\}$ be a countable basis for A, and let $\{U'_i\}$ be another countable basis for A whose elements are all finite unions of elements of $\{U_i\}$ (note that this is also countable). Now define $V_i \subset B$ by $V_i := B - f(A - U'_i)$. We claim $\{V_i\}$ is a countable basis for B.

Since A is compact, so is $A - U'_i$ and therefore also $f(A - U'_i)$. Since B is Hausdorff, this set is therefore closed, so V_i is open. Let $b \in B$ be arbitrary, and let V be an open neighborhood of b in B. The set $f^{-1}(b)$ is compact and contained in $f^{-1}(V)$ and therefore there are finitely many basis elements of $\{U_i\}$ whose union contains $f^{-1}(b)$ and is contained in $f^{-1}(V)$. In other words, there is a single basis element U'_i that contains $f^{-1}(b)$ and is contained in $f^{-1}(V)$.

Then taking complements,

$$A - f^{-1}(V) \subset A - U'_i \subset A - f^{-1}(b)$$

and therefore composing with f,

$$f(A - f^{-1}(V)) \subset f(A - U_i') \subset f(A - f^{-1}(b))$$

so taking complements again,

$$B - f(A - f^{-1}(b)) \subset V_i \subset B - f(A - f^{-1}(V))$$

Since f is surjective, for *any* subset $C \subset B$ we have $f(A - f^{-1}(C)) = B - C$. Thus $b \subset V_i \subset V$ so that $\{V_i\}$ is a basis for B. □

2.4. **Components and Quasicomponents.**

Definition 2.6. Let X be a topological space and $x \in X$. The *component* of x, is the maximal connected subset C of X containing x. The *quasicomponent* of x is the maximal subset Q containing x so that for every separation of X into disjoint clopen sets $X = U \sqcup V$ the set Q is contained in either U or V.

Lemma 2.7. *Let X be an arbitrary topological space.*

1. *Every point of x is contained in a (unique) component and quasicomponent.*
2. *Every component is a subset of a quasicomponent.*
3. *Every component and every quasicomponent is closed.*

Proof. The component C_x of x is the union of all connected subsets of X containing x. The quasicomponent Q_x of x is the intersection of all clopen subsets of X containing x. This proves (1).

If C_x is a component then it is connected so for any clopen subset U of X containing x we must have $C_x \subset U$. But then $C_x \subset Q_x$ proving (2).

If C is connected then so is $\overline{C}$; for, if $U \sqcup V$ is a nontrival separation of $\overline{C}$, each of U and V must meet $\overline{C}$ and therefore (by the definition of closure) must meet C and therefore induce a nontrivial separation of C. Likewise, every quasicomponent is an intersection of clopen sets and is therefore closed. This proves (3). □

Proposition 2.8. *If X is compact and Hausdorff, every component is a quasicomponent.*

Proof. Let x be arbitrary, and let $y \in Q_x$ the quasicomponent of x. Let $\{K_\alpha\}$ be the collection of all closed subsets of X containing $x \cup y$ so that there is no separation $K_\alpha = A \sqcup B$, A, B both clopen in K_α, with $x \in A$ and $y \in B$.

By the definition of quasicomponent, $X \in \{K_\alpha\}$ so this collection of sets is nonempty. Let $\{K_{\alpha'}\}$ be a maximal subcollection ordered by inclusion and let $K := \cap K_\alpha'$. Note that K is an intersection of closed sets and therefore closed. We claim $K \in \{K_\alpha\}$. For if not, we can find $K = A \sqcup B$ with $x \in A$ and $y \in B$, and A, B both clopen in K and hence compact in X. Since X is compact and Hausdorff, it is normal, and therefore there are disjoint open $U, V \subset X$ with $U \cap K = A$ and $V \cap K = B$.

If any $K_{\alpha'} \subset U \cup V$ we would get a separation

$$K_{\alpha'} = (K_{\alpha'} \cap U) \sqcup (K_{\alpha'} \cap V)$$

with x and y in different components, contrary to the definition of $K_{\alpha'}$. Thus $L_{\alpha'} := K_\alpha' \cap (X - U \cup V)$ is nonempty and compact for each α' and because they are nested, $L := \cap_{\alpha'} L_{\alpha'}$ is also nonempty. But $L = K \cap (X - U \cup V) = \emptyset$, a contradiction. It follows that $K \in \{K_\alpha\}$ after all, and is (by construction) a *minimal* element.

If K is not connected, there is a nontrivial separation $K = K_1 \sqcup K_2$ and by definition, x, y must be contained in one of these, say $x, y \in K_1$. But then $K_1 \in \{K_\alpha\}$ contradicting the minimality of K. It follows that y is in the component C_x of x, and since $y \in Q_x$ was arbitrary it follows that $Q_x = C_x$. □

Proposition 2.9. *Let X be compact, Hausdorff and totally disconnected. Then X has a basis of clopen sets.*

Proof. For every $x \in X$ and every open neighborhood U of x we must find a clopen set contained in U and containing x. Since x is totally disconnected, by Proposition 2.8 every point is a quasicomponent, and therefore x is the intersection of all clopen sets containing x. Conversely $X - x$ is a union of clopen sets disjoint from x and in particular $X - U$ is covered by clopen sets disjoint from x. But $X - U$ is compact and therefore it is covered by finitely many clopen sets $U_1, \cdots, U_n$ all disjoint from x. Then $W := \cup U_i$ is clopen and contains U, and $V := X - W$ is clopen and satisfies $x \in V \subset U$. □

2.5. **Inverse Limits.**

Definition 2.10. Let X_i for $i \geq 0$ be a countable family of Hausdorff topological spaces, and for each $i > 0$ let $f_i : X_i \to X_{i-1}$ be a continuous map. The *inverse limit*, denoted $\varprojlim X_i$ is the subspace of the product $\prod_i X_i$ consisting of points (x) with $f_i(x_i) = x_{i-1}$ for all $i > 0$.

Lemma 2.11. *The inverse limit is a closed subset of the infinite product. If all the X_i are compact and nonempty, then the inverse limit is compact and nonempty.*

Proof. For any n define $Y_n \subset \prod X_i$ to be the set of points (x) for which $f_i(x_i) = x_{i-1}$ for all $i \leq n$. Suppose (y) is not in Y_n so that there is an index i with $f_i(y_i) \neq y_{i-1}$. Since each X_i is Hausdorff we may choose disjoint open subsets $U_{i-1}, U'_{i-1} \subset X_{i-1}$ containing y_{i-1} and $f_i(y_i)$ respectively and an open neighborhood $y_i \in V_i \subset X_i$ such that $f_i(V_i) \subset U'_{i-1}$. Let $U \subset \prod_n X_n$ be the open subset whose ith coordinate is in V_i and whose $(i-1)$st coordinate is in U_{i-1}. Then U is open, contains (y), and is disjoint from Y_n; thus Y_n is closed. The inverse limit is by definition equal to $\cap_n Y_n$ and is therefore also closed.

If every X_i is compact, then the projection map from Y_n to $\prod_{i \geq n} X_i$ is a continuous bijection and therefore a homeomorphism. Thus each Y_n is compact and nonempty and since the Y_n are nested, their intersection is also compact and nonempty. □

Proposition 2.12. *Let X be compact, nonempty, metrizable and totally disconnected. Then X is the inverse limit of a sequence of finite spaces.*

Proof. Since X is a compact metric space, it has a finite open cover by sets of diameter < 1. By Proposition 2.9 we may find a finite subordinate cover by clopen sets $U_1, \cdots, U_n$ of diameter < 1. For all i let $V_i = U_i - \cup_{j<i} U_j$. Then $\mathcal{U}_1 := \{V_i\}$ is a finite cover by nonempty disjoint clopen sets, each of diameter < 1.

By the same argument, each V_i is the disjoint union of finitely many clopen sets, all of diameter $< 1/2$, and therefore we may obtain a cover $\mathcal{U}_2$ by finitely many nonempty disjoint clopen sets all of diameter $< 1/2$ so that each element of $\mathcal{U}_1$ is a finite disjoint union of elements of $\mathcal{U}_2$. Proceed by induction to produce $\mathcal{U}_n$, a cover by finitely many nonempty disjoint clopen sets all of diameter $< 1/n$ so that each element of $\mathcal{U}_{n-1}$ is a finite disjoint union of elements of $\mathcal{U}_n$.

For each n let X_n be a finite discrete space whose elements are in bijection with the sets of $\mathcal{U}_n$. Define $f_n : X_n \to X_{n-1}$ as follows: if $x_i \in X_i$ corresponds to the set $U_{i,n_i} \in \mathcal{U}_n$, there is a unique $U_{i-1,n_{i-1}} \in \mathcal{U}_{n-1}$ for which $U_{i,n_i} \subset U_{i-1,n_{i-1}}$, corresponding to an element $x_{i-1} \in X_{i-1}$; then define $f_i(x_i) = x_{i-1}$.

We construct a homeomorphism $h : \varprojlim X_i \to X$ as follows. A point (x) in $\varprojlim X_i$ corresponds to a sequence of points $x_i \in X_i$ satisfying $f_i(x_i) = x_{i-1}$ for all i and therefore a sequence $U_{i,n_i} \in \mathcal{U}_i$ of clopen sets of diameter $< 1/i$ satisfying $U_{i,n_i} \subset U_{i-1,n_{i-1}}$. The intersection $\cap U_{i,n_i}$ is a nested intersection of compact sets and therefore nonempty. On the other hand, it has diameter $< 1/i$ for all i so it consists of a single point p. Set $h((x)) = p$.

The map h is evidently continuous, injective and surjective and is therefore a homeomorphism. □

Corollary 2.13. *A space is compact, nonempty, metrizable and totally disconnected if and only if it is the inverse limit of a sequence of finite spaces.*

Proof. One direction is Proposition 2.12. Conversely, an inverse limit of finite spaces is compact and nonempty by Lemma 2.11, and metrizable and totally disconnected by Proposition 2.4. □

2.6. **Characterization of the Cantor Set.**

Lemma 2.14. *Let X be compact, nonempty, metrizable, totally disconnected and perfect. Let U be clopen and nonempty. Then for any positive integer n we may write U as the disjoint union of n clopen nonempty sets.*

Proof. Let $x \in U$. Since X is perfect, there is $y \in U$ with $x \neq y$. By Proposition 2.9 X has a basis of clopen sets, so there is V clopen with $x \in V \subset U - y$. Then $U = V \sqcup U - V$ is a partition of U into two disjoint clopen sets. The result follows by induction. □

Proposition 2.15. *Let X and Y be compact, nonempty, metrizable, totally disconnected and perfect. Then X and Y are homeomorphic (and therefore both spaces are homeomorphic to the Cantor set $\mathcal{C}$).*

Proof. As in the proof of Proposition 2.12 let $\mathcal{U}_n$ and $\mathcal{V}_n$ be a sequence of decompositions of X and Y respectively into finitely many disjoint clopen nonempty sets of diameter $< 1/n$.

Without loss of generality $\mathcal{U}_1$ has at least as many elements as $\mathcal{V}_1$. If it has more elements, subdivide some element of $\mathcal{V}_1$ using Lemma 2.14 so they have the same cardinality. Define $\mathcal{U}_1' := \mathcal{U}_1$ and $\mathcal{V}_1'$ the result of subdividing an element of $\mathcal{V}_1$ in this way. Choose a bijection h_1 between the finite sets $h_1 : \mathcal{U}_1' \to \mathcal{V}_1'$.

Now, $\mathcal{U}_1'$ and $\mathcal{V}_1'$ are finite covers of compact metric space, so they have positive Lebesgue numbers. It follows that there is an n so that every element of $\mathcal{V}_n$ is contained in some element of $\mathcal{V}_1'$ and every element of $\mathcal{U}_n$ is contained in some element of $\mathcal{U}_1'$. Let's suppose we reorder indices so that $h_1(U_i) = V_i$ for all i. Let $\mathcal{U}_{n,i}$ be the subcover of $\mathcal{U}_n$ refining U_i and $\mathcal{V}_{n,i}$ the subcover of $\mathcal{V}_n$ refining V_i. If one of these sets has smaller cardinality than the other, subdivide one of its elements into clopen sets using Lemma 2.14 so they have the same cardinality. Perform this subdivision for all i. In this way we obtain refinements

$\mathcal{U}'_2$ of $\mathcal{U}_n$ and $\mathcal{V}'_2$ of $\mathcal{V}_n$ that are also refinements of $\mathcal{U}'_1$ and $\mathcal{V}'_1$ respectively, so that there is a bijection $h_2 : \mathcal{U}'_2 \to \mathcal{V}'_2$ compatible with h_1.

Continue inductively. We obtain two sequences of finite spaces X_i, f_i and Y_i, g_i and bijections $h_i : X_i \to Y_i$ so that $h_i f_{i+1} = g_i h_{i+1}$ for all i. Thus the h_i induce a homeomorphism between the inverse limits $h : \varprojlim X_i \to \varprojlim Y_i$. As in the proof of Proposition 2.12 these spaces are homeomorphic to X and Y respectively, and we are done. □

Example 2.16 (Middle third Cantor Set). The *middle third Cantor set* is the subset of $[0, 1]$ consisting of points with a base 3 expansion containing only the digits 0 and 2. It satisfies the properties of Proposition 2.15 and is therefore homeomorphic to $\mathcal{C}$.

2.7. Proof of the Theorem.

We now give the proof of Theorem 2.2.

Proof. Suppose X is compact, metrizable and nonempty. Then as in the proof of Urysohn metrization we may embed X as a nonempty closed subset of $[0, 1]^{\mathbb{N}}$.

There is a continuous surjective map $\mathcal{C} \to [0, 1]$ that takes a point (x) to the number whose base 2 expansion is $0.x_1x_2x_3\cdots$. Thus there is a continuous surjective map $\mathcal{C}^{\mathbb{N}} \to [0, 1]^{\mathbb{N}}$. But $\mathcal{C}^{\mathbb{N}}$ is compact, nonempty, metrizable, totally disconnected and perfect so by Proposition 2.15 there is a homeomorphism $h : \mathcal{C} \to \mathcal{C}^{\mathbb{N}}$. The preimage $Y := h^{-1}(X)$ is closed, and therefore (since it is a subset of $\mathcal{C}$) it is compact, nonempty, metrizable and totally disconnected. Then $Y \times \mathcal{C}$ has all these properties and is furthermore perfect, so it is homeomorphic to $\mathcal{C}$ and we have a chain of surjective maps

$$\mathcal{C} \to Y \times \mathcal{C} \to Y \to X$$

This proves one direction of the theorem.

Conversely suppose X is Hausdorff, and there is a surjective map $f : \mathcal{C} \to X$. Since f is continuous, X is compact. Since $\mathcal{C}$ is nonempty, so is X. By Proposition 2.5 X is metrizable. □

3. Hahn–Mazurkiewicz Theorem

3.1. Statement of the Theorem.

Definition 3.1 (Locally Connected). A space X is *locally connected at a point* x if for all open neighborhoods U of x there is a connected open neighborhood V of x contained in U. A space is *locally connected* if it is locally connected at every point.

Theorem 3.2 (Hahn–Mazurkiewicz). *The following are equivalent for a Hausdorff space* X*:*

- *there is a surjective continuous map* $f : [0, 1] \to X$*;*
- X *is compact, metrizable, connected, and locally connected.*

3.2. Constructing paths.

Definition 3.3 (Chain). Let X be a set and $x, y \in X$. A *chain* from x to y is a finite collection of subsets $U_1, \cdots, U_n$ so that $x \in U_1$, $y \in U_n$, and $U_i \cap U_{i+1}$ is nonempty for all i.

Lemma 3.4 (Chain exists). *Suppose* X *is connected, and* $\{U_\alpha\}$ *is an open cover of* X*. Then any two points of* X *may be connected by a chain of elements of the cover.*

Proof. Fix $x \in X$ and let Y denote the subset of X consisting of points that may be connected to x by a chain. Note that $x \in U_\alpha$ for some α so that $x \in Y$ and therefore Y is nonempty. We claim $Y = X$.

First, Y is open. For, if $y \in Y$, there is a chain $U_1, \cdots, U_n$ with $x \in U_1$ and $y \in U_n$. But then this is also a chain from x to any point in U_n, so $U_n \subset Y$.

Second, Y is closed. For if $y \in \overline{Y}$ and $y \in U_\beta$ then U_β contains some point y' of Y. Let $U_1, \cdots, U_n$ be a chain from x to y. Then $U_1, \cdots, U_n, U_\beta$ is a chain from x to y.

Since X is connected, any clopen nonempty set (e.g. Y) must be all of X. □

Proposition 3.5. *Let X be a locally compact, connected, locally connected metric space. Then X is path-connected.*

Proof. We shall construct a path between arbitrary $a, b \in X$. Without loss of generality we may assume $a \neq b$.

Fix $\epsilon > 0$. Since X is locally connected and locally compact, there is an open cover of X by connected open sets of diameter $< \epsilon/2$ whose closures are compact. Therefore by Lemma 3.4 there is a chain $U_1, \cdots, U_n$ of such sets from a to b. If we let K denote the union of the closures of the U_i, then K is compact and therefore complete.

Define $x_0 := a$, $x_n := b$ and for all $0 < i < n$ pick a point $x_i \in U_{i-1} \cap U_i$; without loss of generality we may assume the x_i are all distinct. Let $P_1 \subset [0,1]$ be the finite set $\{0, 1/n, 2/n, \cdots, 1\}$ and define $f_1 : P_1 \to K$ by $f_1(i/n) = x_i$.

At the next stage we will find a new finite subset $P_1 \subset P_2 \subset [0,1]$ and a map $f_2 : P_2 \to K$ such that $f_2|P_1 = f_1$. Here is how we do the construction. For each $0 \le i < n$ we may cover the open ball U_i by connected open sets of diameter $< \epsilon/2^2$. Therefore, again by Lemma 3.4 there is a chain $V_1, \cdots, V_m$ of such sets, all contained in U_i, from x_i to x_{i+1}. Define $y_0 := x_i$, $y_m := x_{i+1}$ and for all $0 < j < m_i$ pick a point $y_j \in V_{j-1} \cap V_j$; without loss of generality we may assume the y_j are all distinct. Do this for each i and let $P_2 \subset [0,1]$ be the finite set whose intersection with each $[i/n, (i+1)/n]$ is $\{i/n, i/n+1/mn, i/n+2/mn, \cdots, (i+1)/n\}$ as above, and define $f_2 : P_2 \cap [i/n, (i+1)/n]$ by $f_2(i/n + j/mn) = y_j$.

Continue inductively. We get a sequence of finite subsets $P_1 \subset P_2 \subset \cdots$ of $[0,1]$ and functions $f_n : P_n \to K$ with $f_n|P_m = f_m$ for all $n \ge m$. Furthermore, by construction, if p, q are consecutive points of P_n then $f_n(p) \neq f_n(q)$, and $f_m(P_m \cap [p,q])$ has diameter $< \epsilon/2^n$ for all $m \ge n$.

It follows that for any two consecutive points p, q of P_n there is an $m > n$ so that p, q are not consecutive in P_m (or else the distance from p to q would be less than $\epsilon/2^m$ for all m, violating $f_n(p) \neq f_n(q)$). Thus in particular, $P := \cup P_n$ is dense in $[0,1]$.

Let $f : P \to K$ agree with f_n on each P_n. Then for each $r \in [0,1]$ and each monotone sequence $p_i \in P$ with $p_i \to r$ we claim that the images $f(p_i)$ form a Cauchy sequence in K. To see this, observe that for any n, all but finitely many p_i are contained between consecutive elements of P_n and therefore $d(f(p_i), f(p_j)) < \epsilon/2^n$ for all but finitely many i, j. Since K is compact it is complete, and we may define $f(r)$ to be the limit of $f(p_i)$.

To see that f is continuous, pick any $r \in [0,1]$ and let U be an open neighborhood of $f(r)$. There is n so that the closed ball of radius $\epsilon/2^n$ about $f(r)$ is contained in U. Choose points $p < r < q$ (or on only one side of r if r is 0 or 1) so that $p, q \in P_n$ and there are no elements of P_n strictly between p and r or between r and q. Then $f([p,r])$ and

$f([r,q])$ both have diameter at most $\epsilon/2^n$, and are therefore contained in U. This proves continuity. □

Definition 3.6 (Uniformly Locally Path-Connected)**.** A metric space X is *uniformly locally path-connected* if for all $\epsilon > 0$ there is a $\delta > 0$ so that if $p, q \in X$ satisfy $d(p,q) < \delta$, then p and q may be joined by a path of diameter at most ϵ.

Proposition 3.7. *Suppose X is a compact, connected, locally connected metric space. Then X is uniformly locally path-connected.*

Proof. Since X is locally connected, it admits a cover by open connected subsets of diameter $< \epsilon$. Any open subset of a locally connected space is locally connected, and any open subset of a compact space is locally compact, so each set in the cover satisfies the hypothesis of Proposition 3.5 and is therefore path-connected.

Let $\delta > 0$ be the Lesbesgue number of the covering (this exists because X is compact). Then any two points at distance $< \delta$ are contained in a path-connected subset of diameter $< \epsilon$. □

3.3. **Proof of the Theorem.** We now give the proof of Theorem 3.2.

Proof. Let X be compact, metrizable, connected and locally connected. By Theorem **??** there is a surjective map $f : \mathcal{C} \to X$ where $\mathcal{C} \subset [0,1]$ is the middle third Cantor set. By Lemma 3.7 X is uniformly locally path connected. Let $\epsilon_n \to 0$ be a sequence of positive numbers, and $\delta_n \to 0$ with δ_0 equal to the diameter of X such that any two points in X with distance $< \delta_n$ may be joined by a path of diameter $< \epsilon_n$.

For each maximal connected open interval $J \subset [0,1] - \mathcal{C}$ let $J^\pm \in \mathcal{C}$ denote the endpoints of the closure $\overline{J}$. Since $\mathcal{C}$ is compact and f is continuous, for all n there are only finitely many such J so that the $d(f(J^+), f(J^-)) > \delta_n$.

For each complementary interval J with $\delta_n > d(f(J^+), f(J^-)) > \delta_{n+1}$, extend f over J to a path of diameter $< \epsilon_n$. In this way we extend f to $f : [0,1] \to X$. Since $f|\mathcal{C}$ was already surjective, the same is true of $f|[0,1]$. It remains to check that f is continuous. Evidently f is continuous on each J. It remains to check that f is continuous at each point of $\mathcal{C}$. Let $p \in \mathcal{C}$ be arbitrary. If $p \in \overline{J}$ on one side, f is continuous on this side, so we just need to check continuity on a side for which p is not in the closure of a complementary interval. But then for any n there is a one-sided neighborhood U of p so that $f(\mathcal{C} \cap U)$ has diameter $< \delta_n$ and therefore $f([0,1] \cap U)$ has diameter $< \epsilon_n + \delta_n$ which may be taken to be as small as we like, and continuity is proved.

Conversely, suppose X is Hausdorff and there is a surjective map $f : I \to X$. Since I is compact and connected, the same is true of X. Since I is metrizable, the same is true of X by Proposition 2.5. Finally we show that X is locally connected.

Fix $x \in U$; we must find an open subset $x \in V \subset U$ so that V is connected. Let V be the union of all paths in U starting at x. This set is path-connected, hence connected; we claim it is open. Let $y \in V$ be arbitrary, so that there is some path α from x to y. Suppose to the contrary that V does not contain any open neighborhood of y. Then there is a sequence of points $y_n \to y$ so that $y_n \in U - V$. In particular, there is no path β from y to y_n contained in U, or else we could concatenate β with α to get a path in U from x to y_n, showing that $y_n \in V$ after all.

But each $y_n = f(p_n)$ for some p_n, and by compactness of $[0,1]$ the p_n contains a Cauchy subsequence converging to some $p \in [0,1]$ with $f(p) = y$. It follows by continuity of f that there is an n so that $f([p_n, p]) \subset U$, so that there is a path in U from y to y_n after all. Thus V contains an open neighborhood of any $y \in V$ so that V is open and connected, and X is locally connected. □

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