**Spectral Properties of the Transfer Operator:** Let $(X,d)$ be a compact metric space. Let $f : X \to \mathbb{C}$, and define $\|f\|_0 := \sup_{x \in X} |f(x)|$ to be the uniform norm of $f$ in the $C^0$ topology. Let $\theta \in (0,1]$. We define the $\theta$-Hölder norm of $f$ to be

$$\|f\|_\theta = \|f\|_0 + \sup_{x,y \in X} \frac{|f(x) - f(y)|}{d(x,y)^\theta}.$$

$C^\theta(X) := \{f \in C^0(X) : \|f\|_\theta < \infty\}$ is a Banach space. Further, $\{f \in C^0(X) : \|f\|_\theta \leq 1\}$ is compact in the $C^0$ topology induced by the norm $\|\cdot\|_0$. When $X$ is a differentiable manifold, we reserve the notation $C^1(X)$ for the space of $C^1$ maps $f : X \to \mathbb{C}$ with norm

$$\|f\|_{C^1} = \|f\|_0 + \|Df\|_0.$$

Note that this will agree up to a constant factor with the norm $\|f\|_1$. We write $\text{Lip}(X)$ for the space $\{f : \|f\|_1 < \infty\}$. $C^1(X)$ is dense in $\text{Lip}(X)$.

We will compute the operator norm in $C^1(X)$ of $L_\varphi$ in the case that $X = \mathbb{R}/\mathbb{Z}$, $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ is a $C^2$ expanding map of degree greater than 1, and $\varphi = -\log |f'|$, where the absolute value is taken in the standard flat metric on $X = \mathbb{R}/\mathbb{Z}$. Let $u : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ be a $C^1$ map.

$$[L_\varphi u](x) = \sum_{y \in f^{-1}(x)} \frac{u(y)}{|f'(y)|}.$$

We will consider $f$ as a map $f : [0,1] \to [0,1]$ with $f(0) = 0$, $f(1) = 1$ (this can be done since $f$ is a map of degree 1, so has a fixed point on the circle). Let $m$ be the degree of $f$. There will be $m$ points $p_1 < \cdots < p_m = 1$ such that $f(p_i) = 1$, so we can cut $[0,1]$ into $m$ intervals $[p_{i-1},p_i]$, $1 \leq i \leq m$, where $p_0 := 0$, such that on each of these intervals, $f : (p_i,p_{i+1}) \to (0,1)$ is a diffeomorphism with inverse $g_i$. Let $\mathcal{F}^{-1}$ be the set of inverse branches $g_1, \ldots, g_m$ for $f$. Since $f$ is an expanding map, it’s easy to see that there is some $1 > \beta > 0$ such that for each $g_i \in \mathcal{F}^{-1}$, $\|g_i\|_0 \leq \beta$. In fact, we can take $\beta = (\inf_{x \in \mathbb{R}/\mathbb{Z}} |f'(x)|)^{-1}$.

Then we have

$$L_\varphi u = \sum_{\mathcal{F}^{-1}} (u \circ g) \cdot g'$$

$$(L_\varphi u)' = \sum_{\mathcal{F}^{-1}} (u' \circ g) \cdot (g')^2 + (u \circ g) g''$$
and therefore
\[
\|(L_\varphi u)'\|_0 \leq \left\| \sum_{g \in \mathcal{F}^{-1}} (u' \circ g) \cdot (g')^2 \right\|_0 + \left\| \sum_{g \in \mathcal{F}^{-1}} (u \circ g)g'' \right\|_0 \\
\leq \beta \left\| \sum_{g \in \mathcal{F}^{-1}} (u' \circ g) \cdot g' \right\|_0 + \left( \sup_{g \in \mathcal{F}^{-1}} \left| \frac{g''}{g'} \right| \right) \left\| \sum_{g \in \mathcal{F}^{-1}} (u \circ g)g' \right\|_0 \\
= \beta \|L_\varphi(u')\|_0 + \left( \sup_{g \in \mathcal{F}^{-1}} \left| \frac{g''}{g'} \right| \right) \|L_\varphi u\|_0
\]

We can iterate this \( m \) times, and we will obtain after some computation
\[
\|(L_\varphi u)'\|_0 \leq \beta^m \|L_\varphi^m(u')\|_0 + C \|L_\varphi^m u\|_0
\]
where \( C \) is a constant independent of \( m \). This implies that
\[
\|L_\varphi^m u\|_{C^1} \leq C'|\|L_\varphi^m u\|_0 + \beta^m \|L_\varphi^m(u')\|_0
\]
for some constant \( C' \) independent of \( m \). An inequality of this type is called a Lasota-Yorke inequality.

Write \( dx \) for Lebesgue measure on \( \mathbb{R}/\mathbb{Z} \). Recall that \( L_\varphi^*(dx) = dx \) by a computation in a previous lecture. By the Ruelle-Perron-Frobenius theorem, this implied that the dominant eigenvalue of \( L_\varphi \) is 1, this eigenvalue has multiplicity 1, and further that \( \|L_\varphi\|_0 = 1 \).

Recall that for a Banach space \( V \) and a bounded linear operator \( L : V \to V \), the spectrum of \( L \) is defined as \( \text{Spec}(L, V) = \{ \lambda \in \mathbb{C} : L - \lambda I \text{ is not invertible} \} \), where \( I : V \to V \) is the identity. Then
\[
r(L, V) := \sup\{ |\lambda| : \lambda \in \text{Spec}(L, V) \}
\]
is the spectral radius of \( L \). \( r(L, V) \) can be computed as
\[
r(L, V) = \lim_{m \to \infty} \|L^m\|^{1/m} = \inf_{m} \|L^m\|^{1/m}
\]
Since \( L_\varphi^*(dx) = dx \), \( r(L_\varphi, \text{Lip}(\mathbb{R}/\mathbb{Z})) = 1 \). We will show that \( r(L_\varphi, \text{Lip}(\mathbb{R}/\mathbb{Z})/(\langle Ch \rangle)) < 1 \), where \( \langle Ch \rangle \) is the subspace generated by \( h \), which we recall is the positive continuous function \( h : \mathbb{R}/\mathbb{Z} \to \mathbb{C} \) such that \( L_\varphi(h) = h \) and \( \int h \, dx = 1 \).

Note first that
\[
\|L_\varphi^m u\|_{C^1} \leq C\|u\|_0 + \beta^m \|u'\|_0 \leq (C' + \beta^m)\|u\|_{C^1}
\]
for some constant \( C' \). In particular, \( L_\varphi \) is bounded in the \( C^1 \) norm (and thus in the Lipschitz norm). We showed in previous lectures that for a \( C^1 \) function \( u : \mathbb{R}/\mathbb{Z} \to \mathbb{C} \), if \( \int u \, dx = 0 \), then \( \lim_{m \to \infty} \|L_\varphi^m u\|_0 = 0 \).

The Lasota-Yorke inequality then implies that
\[
\|L_\varphi^m u\|_{C^1} \leq C'|\|L_\varphi^m u\|_0 + \beta^m \|L_\varphi^m(u')\|_0 \to 0 \text{ as } m \to \infty
\]
and therefore for each \( C^1 \) function \( u \) with \( \int u \, dx = 0 \), there is some \( m \) such that \( \|L_\varphi^m u\|_{C^1} < \frac{1}{2} \|u\|_{C^1} \). This inequality extends immediately to the Lipschitz norm. Then in \( \text{Lip}(\mathbb{R}/\mathbb{Z}) \), there is an \( m \in \mathbb{N} \) which works for every \( u \) with \( \|u\|_{\text{Lip}} \leq 1 \). This is because the unit ball in \( \text{Lip}(\mathbb{R}/\mathbb{Z}) \) is compact in the \( C^0 \) norm.

More generally, if \( \Sigma_A^+ \) is a mixing subshift of finite type, \( \varphi \) is \( \theta \)-Hölder, then there is some \( C > 0 \) and \( M > 0 \) such that for every \( u \in C^M(\Sigma_A^+) \),
\[
\|L_\varphi^M u\|_\theta \leq C'|\|u\|_0 + \beta^m ||u'||_\theta
\]
(where \( \varphi \) has been adjusted by an additive constant so that 1 is the Perron-Frobenius eigenvalue of \( L_\varphi \)).

Then by the compactness of \( C^M(\Sigma_A^+) \) in the \( C^0 \) norm, there is some \( m \) such that for every \( u \in C^M(\Sigma_A^+) \) with \( ||u||_\theta \leq 1 \) and \( \nu(u) = 0 \), we have \( ||L_\varphi^m u||_{C^1} < \frac{1}{2} ||u||_\theta \).
This gives a spectral gap for the transfer operator $L_\varphi$. There are numerous consequences of such a gap. Suppose $\varphi$ has been adjusted so that the Perron-Frobenius eigenvalue of $L_\varphi$ is 1. We have shown that $\|L_\varphi\|_\theta = 1$, and that if we let $\nu^\perp = \{u \in C^\theta | \nu(u) = 0\}$, then there is some $m > 0$ such that $\|L^m|_{\nu^\perp}\|_\theta < \frac{1}{2}$. This immediately implies that $r(L_\varphi,C^\theta(\Sigma^A_\theta)) = 1$ and $r(L_\varphi,\nu^\perp) < 1$.

Some of the consequences of a spectral gap include exponential decay of correlations and analytic dependence of $\lambda$, $\nu$, and $h$ on $\varphi$. The analytic dependence comes from the observation that if the spectrum of $L_\varphi$ breaks into two parts in $\mathbb{C}$, one of which is the Perron-Frobenius eigenvalue 1 and the other is contained in a disk centered at the origin of radius $|1$, then if we take a small circle $\Gamma$ centered at 1 and oriented counterclockwise (small enough that it does not intersect the rest of the spectrum) we obtain a spectral projection operator $u \rightarrow \frac{1}{2\pi}\int_{\Gamma}(\zeta I - L_\varphi)(u) d\zeta$. 