# Thermodynamic formalism

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**Spectral Properties of the Transfer Operator**: Let $(X, d)$ be a compact metric space. Let $f : X \to \mathbb{C}$, and define $\|f\|_0 := \sup_{x \in X} |f(x)|$ to be the uniform norm of $f$ in the $C^0$ topology. Let $\theta \in (0, 1]$. We define the $\theta$-Hölder norm of $f$ to be

$$\|f\|_\theta = \|f\|_0 + \sup_{x,y \in X} \frac{|f(x) - f(y)|}{d(x, y)^\theta}$$

$C^\theta(X) := \{f \in C^0(X) : \ \|f\|_\theta < \infty\}$ is a Banach space. Further, $\{f \in C^0(X) : \ \|f\|_\theta \leq 1\}$ is *compact* in the $C^0$ topology induced by the norm $\|\cdot\|_0$. When $X$ is a differentiable manifold, we reserve the notation $C^1(X)$ for the space of $C^1$ maps $f : X \to \mathbb{C}$ with norm

$$\|f\|_{C^1} = \|f\|_0 + \|Df\|_0$$

Note that this will agree up to a constant factor with the norm $\|f\|_1$. We write $\mathrm{Lip}(X)$ for the space $\{f : \|f\|_1 < \infty\}$. $C^1(X)$ is dense in $\mathrm{Lip}(X)$.

We will compute the operator norm in $C^1(X)$ of $L_\varphi$ in the case that $X = \mathbb{R}/\mathbb{Z}$, $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ is a $C^2$ expanding map of degree greater than 1, and $\varphi = -\log |f'|$, where the absolute value is taken in the standard flat metric on $X = \mathbb{R}/\mathbb{Z}$. Let $u : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$ be a $C^1$ map.

$$[L_\varphi u](x) = \sum_{y \in f^{-1}(x)} \frac{u(y)}{|f'(y)|}$$

We will consider $f$ as a map $f : [0, 1] \to [0, 1]$ with $f(0) = 0$, $f(1) = 1$ (this can be done since $f$ is a map of degree ¿ 1, so has a fixed point on the circle). Let $m$ be the degree of $f$. There will be $m$ points $p_1 < \cdots < p_m = 1$ such that $f(p_i) = 1$, so we can cut $[0, 1]$ into $m$ intervals $[p_{i-1}, p_i]$, $1 \leq i \leq m$, where $p_0 := 0$, such that on each of these intervals, $f : (p_i, p_{i+1}) \to (0, 1)$ is a diffeomorphism with inverse $g_i$. Let $\mathcal{F}^{-1}$ be the set of inverse branches $g_1, \ldots, g_m$ for $f$. Since $f$ is an expanding map, it's easy to see that there is some $1 > \beta > 0$ such that for each $g_i \in \mathcal{F}^{-1}$, $\|g_i'\|_0 \leq \beta$. In fact, we can take $\beta = (\inf_{x \in \mathbb{R}/\mathbb{Z}} |f'(x)|)^{-1}$.

Then we have

$$L_\varphi u = \sum_{\mathcal{F}^{-1}} (u \circ g) \cdot g'$$

$$(L_\varphi u)' = \sum_{\mathcal{F}^{-1}} (u' \circ g) \cdot (g')^2 + (u \circ g) g''$$

and therefore

$$\begin{aligned}
\|(L_\varphi u)'\|_0 &\le \left\| \sum_{g\in\mathcal{F}^{-1}} (u'\circ g)\cdot (g')^2 \right\|_0 + \left\| \sum_{g\in\mathcal{F}^{-1}} (u\circ g) g'' \right\|_0 \\
&\le \beta \left\| \sum_{g\in\mathcal{F}^{-1}} (u'\circ g)\cdot g' \right\|_0 + \left( \sup_{g\in\mathcal{F}^{-1}} \left\| \frac{g''}{g'} \right\|_0 \right) \left\| \sum_{g\in\mathcal{F}^{-1}} (u\circ g) g' \right\|_0 \\
&= \beta \|L_\varphi(u')\|_0 + \left( \sup_{g\in\mathcal{F}^{-1}} \left\| \frac{g''}{g'} \right\|_0 \right) \|L_\varphi u\|_0
\end{aligned}$$

We can iterate this $m$ times, and we will obtain after some computation

$$\|(L_\varphi u)'\|_0 \le \beta^m \|L_\varphi^m(u')\|_0 + C\|L_\varphi^m u\|_0$$

where $C$ is a constant independent of $m$. This implies that

$$\|L_\varphi^m u\|_{C^1} \le C'\|L_\varphi^m u\|_0 + \beta^m \|L_\varphi^m(u')\|_0$$

for some constant $C'$ independent of $m$. An inequality of this type is called a Lasota-Yorke inequality.

Write $dx$ for Lebesgue measure on $\mathbb{R}/\mathbb{Z}$. Recall that $L_\varphi^*(dx) = dx$ by a computation in a previous lecture. By the Ruelle-Perron-Frobenius theorem, this implied that the dominant eigenvalue of $L_\varphi$ is 1, this eigenvalue has multiplicity 1, and further that $\|L_\varphi\|_0 = 1$.

Recall that for a Banach space $V$ and a bounded linear operator $L : V \to V$, the spectrum of $L$ is defined as $\mathrm{Spec}(L, V) = \{\lambda \in \mathbb{C} : \ L - \lambda I \text{ is not invertible}\}$, where $I : V \to V$ is the identity. Then

$$r(L, V) := \sup\{|\lambda| : \ \lambda \in \mathrm{Spec}(L, V)\}$$

is the *spectral radius* of $L$. $r(L, V)$ can be computed as

$$r(L, V) = \lim_{m\to\infty} \|L^m\|^{1/m} = \inf_m \|L^m\|^{1/m}$$

Since $L_\varphi^*(dx) = dx$, $r(L_\varphi, \mathrm{Lip}(\mathbb{R}/Z)) = 1$. We will show that $r(L_\varphi, \mathrm{Lip}(\mathbb{R}/Z)/\langle \mathbb{C}h\rangle) < 1$, where $\langle \mathbb{C}h\rangle$ is the subspace generated by $h$, which we recall is the positive continuous function $h : \mathbb{R}/Z \to \mathbb{C}$ such that $L_\varphi(h) = h$ and $\int h\, dx = 1$.

Note first that

$$\|L_\varphi^m u\|_{C^1} \le C\|u\|_0 + \beta^m \|u'\|_0 \le (C' + \beta^m)\|u\|_{C^1}$$

for some constant $C'$. In particular, $L_\varphi$ is bounded in the $C^1$ norm (and thus in the Lipschitz norm). We showed in previous lectures that for a $C^1$ function $u : \mathbb{R}/\mathbb{Z} \to \mathbb{C}$, if $\int u\, dx = 0$, then $\lim_{m\to\infty} \|L_\varphi^m u\|_0 = 0$. The Lasota-Yorke inequality then implies that

$$\|L_\varphi^m u\|_{C^1} \le C'\|L_\varphi^m u\|_0 + \beta^m \|L_\varphi^m(u')\|_0 \to 0 \text{ as } m \to \infty$$

and therefore for each $C^1$ function $u$ with $\int u\, dx = 0$, there is some $m$ such that $\|L_\varphi^m u\|_{C^1} < \frac{1}{2}\|u\|_{C^1}$. This inequality extends immediately to the Lipschitz norm. Then in $\mathrm{Lip}(\mathbb{R}/\mathbb{Z})$, there is an $m \in \mathbb{N}$ which works for *every* $u$ with $\|u\|_{\mathrm{Lip}} \le 1$. This is because the unit ball in $\mathrm{Lip}(\mathbb{R}/\mathbb{Z})$ is compact in the $C^0$ norm.

More generally, if $\Sigma_A^+$ is a mixing subshift of finite type, $\varphi$ is $\theta$-Hölder, then there is some $C > 0$ and $M > 0$ such that for every $u \in C^\theta(\Sigma_A^+)$,

$$\|L_\varphi^M u\|_\theta \le C'\|u\|_0 + \beta^m \|u'\|_\theta$$

(where $\varphi$ has been adjusted by an additive constant so that 1 is the Perron-Frobenius eigenvalue of $L_\varphi$. Then by the compactness of $C^\theta(\Sigma_A^+)$ in the $C^0$ norm, there is some $m$ such that for every $u \in C^\theta(\Sigma_A^+)$ with $\|u\|_\theta \le 1$ and $\nu(u) = 0$, we have $\|L_\varphi^m u\|_{C^1} < \frac{1}{2}\|u\|_\theta$.

This gives a spectral gap for the transfer operator $L_\varphi$. There are numerous consequences of such a gap. Suppose $\varphi$ has been adjusted so that the Perron-Frobenius eigenvalue of $L_\varphi$ is 1. We have shown that $\|L_\varphi\|_\theta = 1$, and that if we let $\nu^\perp = \{u \in C^\theta |\ \nu(u) = 0\}$, then there is some $m > 0$ such that $\|L^m|_{\nu^\perp}\|_\theta < \frac{1}{2}$. This immediately implies that $r(L_\varphi, C^\theta(\Sigma_A^+)) = 1$ and $r(L_\varphi, \nu^\perp) < 1$.

Some of the consequences of a spectral gap include exponential decay of correlations and analytic dependence of $\lambda$, $\nu$, and $h$ on $\varphi$. The analytic dependence comes from the observation that if the spectrum of $L_\varphi$ breaks into two parts in $\mathbb{C}$, one of which is the Perron-Frobenius eigenvalue 1 and the other is contained in a disk centered at the origin of radius ¡ 1, then if we take a small circle $\Gamma$ centered at 1 and oriented counterclockwise (small enough that it does not intersect the rest of the spectrum) we obtain a spectral projection operator $u \to \frac{1}{2\pi i}\int_\Gamma (\zeta I - L_\varphi)(u)\, d\zeta$