CLASSICAL TESSELLATIONS AND 3-MANIFOLDS, SPRING 2014, HOMEWORK 7

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Homework is assigned on Fridays; it is due at the start of class the week after it is assigned. So this homework is due May 30th.

Problem 1. Suppose Δ is a hyperbolic triangle, and x is a point on one side. Show that there is some point y on one of the other two sides such that $\text{dist}(x, y) \le \log(1 + \sqrt{2})$; one says that triangles in the hyperbolic plane are *uniformly thin*.

Problem 2. Suppose Henri Poincaré goes for a "random walk" in the hyperbolic plane. His random walk is determined as follows:

(1) He starts at the origin facing along the x axis.
(2) At each stage he walks hyperbolic distance 1 in the direction he is facing, then tosses a coin and changes his heading by $30°$ left or right, according to whether the coin comes up heads or tails.

Show that after infinite time, in the Poincaré disk model, Henri's walk "converges" to some definite point on the circle at infinity, and determine the set of possible points at infinity that he might land on.

(Bonus problem: what happens if Henri does a similar random walk in the Euclidean plane?)

Problem 3. Let O, O' be two closed orbifolds with negative Euler characteristic. We know that there are discrete cocompact groups Γ, Γ' of isometries of the hyperbolic plane for which the quotients $\mathbb{H}/\Gamma$ and $\mathbb{H}/\Gamma'$ are O and O' respectively. Show that there are finite index subgroups $\hat{\Gamma}$ of Γ and $\hat{\Gamma}'$ of Γ' which are isomorphic *as abstract groups* (though they are *not* necessarily conjugate in the group of isometries of the hyperbolic plane).

Problem 4. Suppose G is a finitely generated linear group (i.e. a group of $n \times n$ matrices with real entries, for some n). Show that every surjective homomorphism $\phi : G \to G$ is an isomorphism (hint: use Malcev's theorem that finitely generated linear groups are residually finite). A group with this property is said to be *Hopfian*.

Problem 5. Let x be a point in the hyperbolic plane.

(1) Show that the function $\text{dist}(\cdot, x)$ is *convex*; i.e. for any hyperbolic geodesic ℓ parameterized by hyperbolic length t (i.e. so that $\text{dist}(\ell(s), \ell(t)) = |t - s|$) the function $t \to \text{dist}(\ell(t), c)$ is convex.
(2) Show that for any finite set of points x_i and positive integers m_i the function $\sum_i m_i \text{dist}^2(\cdot, x_i)$ is *strictly convex* and *proper*, and therefore achieves its minimum at a *unique* point y.
(3) If G is a finite subgroup of isometries of the hyperbolic plane, show that G has a global fixed point; i.e. there is some point p which is fixed by every element of G.
(4) Classify the finite subgroups of the group of isometries of the hyperbolic plane.

Bonus Problem 6. Consider the tiling of the hyperbolic plane by regular right angled pentagons. Let Γ be the 1-skeleton of this tiling — i.e. the union of vertices and edges. Construct a *nonconstant* function f from the vertices of Γ and taking values in the interval $[0, 1]$ which is *harmonic*, in the sense that the value of f on each vertex is equal to the average value of f on the four adjacent vertices.

Can this be done for the tiling of the Euclidean plane by regular right angled squares?

(Note: it is OK, and maybe instructive and fun, to just construct such a function f numerically out to some depth, both in the hyperbolic and Euclidean planes.)

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