CLASSICAL TESSELLATIONS AND 3-MANIFOLDS, SPRING 2014, HOMEWORK 6

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Homework is assigned on Fridays; it is due at the start of class the week after it is assigned. So this homework is due May 23rd.

Problem 1. Suppose O is a closed orbifold with $\chi(O) < 0$. Show that $\chi(O) \le -\frac{1}{84}$. Deduce that if G is a discrete group of symmetries of a closed oriented surface of genus $g \ge 2$ then $|G| \le 168(g-1)$. Can you find an example in which there is equality?

Problem 2. Suppose two non-elliptic orientation-preserving isometries of the hyperbolic plane commute. Show that either they are both hyperbolic with the same axis, or both parabolic and fix the same point at infinity.

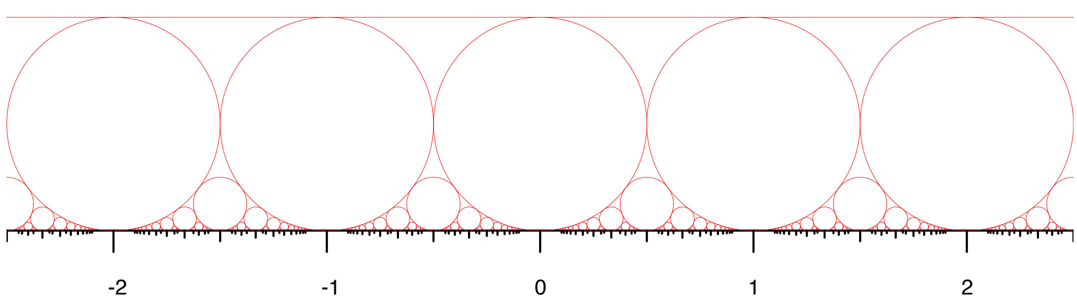
Problem 3. Let f and g be elliptic isometries of the hyperbolic plane (possibly with distinct fixed points), rotating through angles α and β respectively. Suppose their composition fg is elliptic, rotating through angle γ. What are the possible values for γ?

Problem 4. Recall that the group $\mathrm{SL}(2,\mathbb{R})$ of real 2×2 matrices with determinant 1 acts on the upper half-space $\mathbb{H}$ (i.e. the set of complex numbers z with positive imaginary part) by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z \to \frac{az+b}{cz+d}$$

The center $\pm\mathrm{Id}$ acts trivially, so this action descends to the quotient group $\mathrm{PSL}(2,\mathbb{R}) := \mathrm{SL}(2,\mathbb{R})/\pm\mathrm{Id}$. Let $\mathrm{PSL}(2,\mathbb{Z})$ denote the subgroup with integer matrices.

(1) Show that $\mathrm{PSL}(2,\mathbb{Z})$ is discrete as a subgroup of $\mathrm{PSL}(2,\mathbb{R})$.
(2) At each rational number p/q on the real line (thought of as part of the boundary of $\mathbb{H}$) put a round circle (as in the figure), tangent to p/q, of (Euclidean) radius $1/2q^2$, plus an "infinite circle" — i.e. a horizontal straight line — at height 1. Show that this collection of circles does not overlap, but



is packed tightly. Using this, show that for every irrational number α, there are infinitely many rational numbers p_i/q_i so that

$$\left|\alpha - \frac{p_i}{q_i}\right| < \frac{1}{2q_i^2}$$

On the other hand, show that for any $\epsilon > 0$ for "most" irrational numbers α there are only finitely many rational numbers p_i/q_i so that

$$\left|\alpha - \frac{p_i}{q_i}\right| < \frac{1}{2q_i^{2+\epsilon}}$$

(3) Show that $\mathrm{PSL}(2,\mathbb{Z})$ permutes this collection of circles, and that the stabilizer of the "infinite circle" is the parabolic subgroup of matrices of the form $\pm\left(\begin{smallmatrix}1 & n\\ 0 & 1\end{smallmatrix}\right)$. Deduce that the quotient of $\mathbb{H}$ by $\mathrm{PSL}(2,\mathbb{Z})$ has finite area, although it is non-compact. Draw the quotient orbifold.

Problem 5. The group $\mathrm{SL}(2,\mathbb{R})$ acts on a 2-dimensional real vector space V with basis x, y in the "standard" way (i.e. by matrix multiplication), so that for some basis x, y the matrix $A := \left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)$ acts by

$$Ax = ax + cy, \qquad Ay = bx + dy$$

(1) Let S^2V denote the vector space of homogeneous polynomials of degree 2 in the variables x, y; i.e. the 3-dimensional real vector space with basis x^2, xy, y^2. Show that there is a natural action of $\mathrm{SL}(2,\mathbb{R})$ on S^2V by *linear* maps; equivalently, show that the map $\rho : \mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(3,\mathbb{R})$ defined at the level of matrices by

$$\rho : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} a^2 & ab & b^2 \\ 2ac & ad+bc & 2bd \\ c^2 & cd & d^2 \end{pmatrix}$$

is a homomorphism between groups. What is the kernel of this homomorphism?

(2) Show that the group $\rho(\mathrm{SL}(2,\mathbb{R}))$ preserves the form Q given by the symmetric matrix

$$Q := \begin{pmatrix} 0 & 0 & 2 \\ 0 & -1 & 0 \\ 2 & 0 & 0 \end{pmatrix}$$

i.e. show that for every $A \in \rho(\mathrm{SL}(2,\mathbb{R}))$ we have $A^TQA = Q$. Does every 3×3 real matrix preserving Q arise this way?

(3) Find a new basis for S^2V in which the invariant form is diagonal. What does this have to do with the hyperboloid model of the hyperbolic plane?

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