

CLASSICAL TESSELLATIONS AND 3-MANIFOLDS, SPRING 2014, HOMEWORK 5

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Homework is assigned on Fridays; it is due at the start of class the week after it is assigned. So this homework is due May 16th.

Problem 1. Prove the “second” hyperbolic law of cosines: if abc is a hyperbolic triangle with side lengths A, B, C and opposite angles α, β, γ , show that

$$\cos \alpha = -\cos \beta \cos \gamma + \sin \beta \sin \gamma \cosh A$$

Problem 2. Suppose P is a hyperbolic hexagon with all right angles. Show that if the side lengths are ℓ_1, \dots, ℓ_6 (in order) then

$$\frac{\sinh \ell_1}{\sinh \ell_4} = \frac{\sinh \ell_3}{\sinh \ell_6} = \frac{\sinh \ell_5}{\sinh \ell_2}$$

Problem 3. In the Klein model, the hyperbolic plane is the interior of the unit disk, and (hyperbolic) straight lines are the restriction of (Euclidean) straight lines to this disk.

- (1) Let ℓ be a (hyperbolic) straight line with two points p and q at infinity, and suppose that ℓ does not go through 0. Show that the two tangent lines to the circle at p and q meet at a unique point ℓ' in the complement of the disk. What if ℓ goes through 0?
- (2) Let ϕ be a hyperbolic isometry which takes the line ℓ to itself, and acts as a “translation” along it. Let Φ be the linear map of \mathbb{R}^3 to itself fixing the origin, and acting as ϕ where we identify the hyperbolic plane with the (positive sheet of the) hyperboloid of vectors with norm squared equal to -1 . Show that Φ has an eigenvector with real eigenvalue 1, and that the eigenspace it spans projects to the point ℓ' , under the linear projection which takes the hyperboloid to the unit disk in the Klein model.
- (3) show that three lines ℓ_1, ℓ_2, ℓ_3 in the hyperbolic plane pass through a common point if and only if the points $\ell'_1, \ell'_2, \ell'_3$ are contained in a straight line (interpret this suitably if the ℓ_i all pass through 0).

Problem 4. Let A be a 3×3 matrix with columns u, v, w . Show that A is an isometry of \mathbb{R}^3 in the usual Euclidean metric if and only if the vectors u, v, w all have norm 1 and are mutually perpendicular.

Similarly, show that A is an isometry of \mathbb{R}^3 in the Minkowski metric if and only if the vector w is on the hyperboloid, and if u, v are in the tangent space w^\perp with norm 1 and are mutually perpendicular.

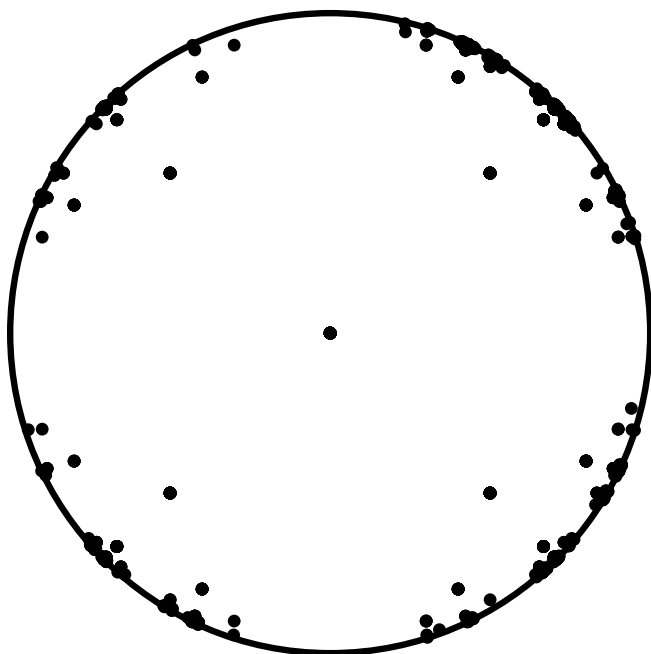
Problem 5. Let’s use hyperbolic geometry to tackle a problem in number theory (this problem is quite involved). Suppose we want to find triples of integers a, b, c such that $a^2 + b^2 - c^2 = n$ for some fixed integer n ; for instance, taking $n = 0$ is the problem of finding Pythagorean triples — integer sides for a right-angled Euclidean triangle. We can encode a solution a, b, c as the entries of a vector, and then denote by V_n the set of vectors corresponding to solutions for some fixed n .

- (1) Let $O(2, 1; \mathbb{Z})$ denote the set of 3×3 matrices with integer entries which preserve the Minkowski inner product. Show that $O(2, 1; \mathbb{Z})$ is a group.
- (2) Show for any n that $O(2, 1; \mathbb{Z})$ takes V_n to itself and permutes it (as a set).
- (3) Show that the matrices

$$R := \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad T := \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{pmatrix}$$

are in $O(2, 1; \mathbb{Z})$ and explain how someone could find these matrices by hand.

- (4) Let G be the group generated by R, T . Show that G is discrete, and that the quotient of the hyperbolic plane by G has finite area (although it is *not* compact!)
- (5) For any negative integer n , show that the (Minkowski) inner product of any two distinct vectors in V_n is nonzero, and integral. Deduce that the action of G on V_n has finitely many orbits for n negative (hint: interpret this inner product in terms of the hyperbolic distance between their projections to the hyperboloid). Use this to show that there is an algorithm to find a complete set of orbit representatives for each negative n ; implement this for $n = -1, -2$.



0	0	1
2	2	3
10	10	17
-4	-4	3
-12	-12	17
-10	-10	99
0	0	1
-22	-22	31
-12	-12	17
68	68	57
-30	-30	99
-208	-208	27
4	4	9
2	2	3
50	50	35
46	46	51
886	886	297
12	12	17
84	84	31
162	162	289
17	17	297
218	218	297
405	405	363
2378	2378	363
5	5	17
8	8	31
60	60	17
5	5	3
-12	-12	17
2	2	3
18	18	35
154	154	201
22	22	51
136	136	297
740	740	1791
34	34	51
0	0	1
98	98	51
146	146	201
218	218	289
1078	1078	1693
70	70	89
12	12	17
206	206	289
1178	1178	1693
405	405	1791
1228	1228	363
2378	2378	363
13680	13680	19651
-2	-2	3
-12	-12	17
-10	-10	99
-8	-8	19
-128	-128	297
5	5	17
-2	-2	3
2	2	3
8	8	31
12	12	17
70	70	89
-18	-18	19
-4	-4	3
0	0	1
-172	-172	169
-1050	-1050	1171
49	49	57
832	832	9
-740	-740	89
-268	-268	297
415	415	507
-1550	-1550	1728
-6184	-6184	12989
-12	-12	17
-10	-10	99
58	58	57
-216	-216	289
4	4	9
0	0	1
-34	-34	51
-156	-156	201
-192	-192	89
-192	-192	1195
-642	-642	819

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