RIEMANNIAN GEOMETRY, SPRING 2019, MIDTERM

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This midterm exam was posted online on Thursday, May 2nd, and is due before class on Thursday, May 9th.

Problem 1. The *helicoid* is the surface in $\mathbb{E}^3$ given in parametric form by the equations

$$x = \rho\cos(\theta),\ y = \rho\sin(\theta),\ z = \theta$$

for $\rho, \theta \in (-\infty, \infty)$. Show that the second fundamental form has zero trace at every point.

Problem 2. Let $M = S^2 \times S^2$ with the product metric $g := g_1 \oplus g_2$ where g_1 and g_2 are metrics on the two sphere factors making them isometric to the round spheres in $\mathbb{E}^3$ of radius 1 and 2 respectively. Let $T \subset M$ be a torus, obtained by taking the product of the equators in the two spheres. Show that T (with its intrinsic metric) is flat, and that it is totally geodesic in M (i.e. that geodesics on T are geodesics in M).

Problem 3. Let B be the open unit disk in $\mathbb{R}^2$. Define a Riemannian metric g on B by setting

$$g(X, Y)_x := \frac{4}{(1 - |x|^2)^2}\langle X, Y\rangle$$

for each point $x \in B$, where $X, Y \in T_xB$, and $\langle X, Y\rangle$ denotes the usual inner product (i.e. the dot product) for vectors in $\mathbb{R}^2$.

(1) Show that g is preserved by every conformal self-map of B.
(2) Show that the geodesics in the metric are straight lines through the origin, and arcs of circles perpendicular to ∂B.
(3) Show that the sectional curvature is equal to -1 everywhere.

This is known as the *hyperbolic metric* on B, and with this metric, B becomes isometric to the *hyperbolic plane.*

Problem 4. Prove the *second Bianchi identity* (or: the *differential* Bianchi identity): for any three vector fields X, Y, Z we have

$$(\nabla_X R)(Y, Z) + (\nabla_Y R)(Z, X) + (\nabla_Z R)(X, Y) = 0$$

where $\nabla_X R$ is the covariant derivative in the direction of X of the tensor field R, thought of as a section of the bundle of two-forms with values in endomorphisms of the tangent bundle (hint: compute in geodesic normal coordinates at a point).

Problem 5. A Riemannian manifold is said to be an *Einstein manifold* if the Ricci curvature is proportional to the metric at every point; i.e. if there is a smooth function f on M so that $\mathrm{Ric} = f(p)g$ at every point p.

Show that if the dimension of M is at least 3 and M is connected, the function f as above must be constant. (hint: use the second Bianchi identity, whether you were able to prove it or not).

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