RIEMANNIAN GEOMETRY, SPRING 2019, HOMEWORK 7

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Homework is assigned on Thursdays; it is due at the start of class the week after it is assigned. So this homework is due May 30th.

Problem 1. Let G be a Lie group, and H a closed subgroup. Prove that the space G/H is complete in any G-invariant metric (where G acts on the left in the obvious way).

Problem 2 (Bott). Let G be a compact Lie group, with a bi-invariant Riemannian metric. Let p be a point, and let q be conjugate to p along a geodesic γ. Show that the dimension of the space of Jacobi fields along γ vanishing at p and q is even.

Problem 3. For any n let $\mathrm{SL}(n,\mathbb{R})$ denote the group of $n \times n$ real matrices with determinant 1. Embed $\mathrm{SL}(n-1,\mathbb{R})$ as a subgroup of $\mathrm{SL}(n,\mathbb{R})$ by the homomorphism $M \to \left(\begin{smallmatrix} 1 & 0 \\ 0 & M \end{smallmatrix}\right)$. The group $\mathrm{SL}(n,\mathbb{R})$ acts on the homogeneous space $\mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(n-1,\mathbb{R})$ by multiplication on the left. Show that for $n \ge 3$ this homogeneous space admits no (left)-invariant metric.

Problem 4. The Lie group Nil (also sometimes called the *Heisenberg group*) is the group

$$\mathrm{Nil} = \left\{ 3 \times 3 \text{ real matrices of the form } \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \right\}$$

(i): Find a basis for the Lie algebra and compute the adjoint action in those coordinates.
(ii): Give an explicit closed formula for the exponential map and show that it is a diffeomorphism from the Lie algebra to the group.
(iii): Define vector fields in $\mathfrak{X}(\mathbb{R}^3)$ by the formulae

$$X := \frac{\partial}{\partial x} - \frac{1}{2} y \frac{\partial}{\partial z}, \quad Y := \frac{\partial}{\partial y} + \frac{1}{2} x \frac{\partial}{\partial z}, \quad Z := \frac{\partial}{\partial z}$$

Show that $[X,Y] = Z$ and $[X,Z] = [Y,Z] = 0$. Use this to construct an identification of Nil with $\mathbb{R}^3$.
(iv): Let θ denote the 1-form $\theta := dz - \frac{1}{2}(xdy - ydx)$. The 2-plane field $\xi = \ker(\theta)$ is a distribution spanned locally by X and Y. Show that for any points p and q there is a smooth path γ from p to q with $\theta(\gamma') = 0$. In fact, show that for *any* continuous path δ from p to q there is a smooth path γ which is C^0 close to δ (i.e. arbitrarily close in the C^0 topology), satisfies $\theta(\gamma') = 0$, and runs from p to q.
(v): Forgetting the z coordinate defines a projection $\mathbb{R}^3 \to \mathbb{R}^2$, where we think of the image as the x–y plane. Show that if $\gamma : [0,1] \to \mathbb{R}^2$ is a smooth curve in the x–y plane, and $p \in \mathbb{R}^3$ is any point that projects to $\gamma(0)$, there is a unique smooth curve $\tilde{\gamma} : [0,1] \to \mathbb{R}^3$ with $\theta(\tilde{\gamma}') = 0$ which projects to γ and starts at p. If $\gamma(0) = \gamma(1)$ (so that the image is a smooth, immersed circle in $\mathbb{R}^2$) show that the difference $z(\tilde{\gamma}(1)) - z(\tilde{\gamma}(0))$ is equal to half the *algebraic area* enclosed by γ.

Note: for a smooth map $\gamma : S^1 \to \mathbb{R}^2$, the algebraic area enclosed by γ is defined as follows. For each point $p \in \mathbb{R}^2 - \gamma(S^1)$, join p to infinity by a smooth ray δ_p, and define $\mathrm{wind}(\gamma, p)$ to be the algebraic intersection number of δ_p with $\gamma(S^1)$. Then the algebraic area enclosed by γ is just $\int_{\mathbb{R}^2} \mathrm{wind}(\gamma,p) d\mathrm{area}(p)$.

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