# RIEMANNIAN GEOMETRY, SPRING 2019, HOMEWORK 6

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Homework is assigned on Thursdays; it is due at the start of class the week after it is assigned. So this homework is due May 23rd.

*Problem* 1. Let $G$ be a Lie group. Show that there is some open neighborhood $U$ of the identity element $e$ so that *any* subgroup $\Gamma$ of $G$ (closed or not, discrete or not) contained in $U$ is equal to $e$. (Bonus question: give an example of a locally compact topological group which fails to have this property).

*Problem* 2. Describe precisely the image of the map $X \to e^X$ in the group $\mathrm{SL}(2,\mathbb{R})$.

*Problem* 3. Consider the standard symplectic form $\omega$ on $\mathbb{R}^{2n}$, and let $W$ be the space of polynomial functions on $\mathbb{R}^{2n}$ of degree $\le 2$. The *Poisson bracket* of two smooth functions $f, g$ is defined as follows. First, given a function $f$ on $\mathbb{R}^{2n}$ (actually any symplectic manifold) there is a unique smooth vector field $X_f$ defined by $df(Y) = \omega(X_f, Y)$ for every smooth vector field $Y$. Then the Poisson bracket is defined by

$$\{f, g\} := \omega(X_f, X_g)$$

Show that this bracket makes $W$ into a Lie algebra, and identify this Lie algebra.

*Problem* 4. A *submersion* is a differentiable map between smooth manifolds $\pi : M^{n+k} \to N^n$ such that at each point $d\pi$ has rank $n$ (i.e. it is surjective on tangent spaces). It follows that for each $p \in N$ the preimage $\pi^{-1}(p)$ is a smooth $k$-dimensional submanifold of $M$. Let $V_q$ denote the tangent space to $\pi^{-1}(p)$ at some point $q \in \pi^{-1}(p)$. If $M$ is a Riemannian metric, set $H_q$ to be the subspace of $T_qM$ perpendicular to $V_q$, so that $TM = V \oplus H$.

The map $\pi$ is said to be a *Riemannian submersion* if $d\pi|H$ is an isometry. Suppose $\pi : M \to N$ is a Riemannian submersion.

(i): Show for each vector field $X$ on $N$ there is a unique vector field $\bar{X}$ on $M$ so that $\bar{X}$ is everywhere contained in $H$, and $d\pi(\bar{X}) = X$.

(ii): If $X, Y, Z$ are vector fields on $N$, show that

$$\langle [\bar{X}, \bar{Y}], \bar{Z} \rangle = \langle [X, Y], Z \rangle$$

Conclude that

$$\langle \bar{\nabla}_{\bar{X}} \bar{Y}, \bar{Z} \rangle = \langle \nabla_X Y, Z \rangle$$

where $\bar{\nabla}$ denotes the Levi-Civita connection on $M$, and $\nabla$ the Levi-Civita connection on $N$.

(iii): If $X$ and $Y$ are vector fields on $N$ and $T$ is vertical (i.e. is a section of $V$), show

$$\langle [\bar{X}, T], \bar{Y} \rangle = 0$$

Conclude that

$$\langle \bar{\nabla}_{\bar{X}} \bar{Y}, T \rangle = \frac{1}{2} \langle [\bar{X}, \bar{Y}], T \rangle$$

and therefore deduce $\bar{\nabla}_{\bar{X}} \bar{Y} = \overline{\nabla_X Y} + \frac{1}{2}[\bar{X}, \bar{Y}]^V$. where the superscript $V$ denotes the vertical component of a vector field.

(iv): Deduce that if $\pi : M \to N$ is a Riemannian submersion, and $\gamma : [0,1] \to N$ is a smooth curve and $\bar{\gamma} : [0,1] \to M$ is a horizontal lift (i.e. $\bar{\gamma}'$ is horizontal and satisfies $\pi \circ \bar{\gamma} = \gamma$) then $\gamma$ is a geodesic if and only if $\bar{\gamma}$ is. (Bonus question: give a completely different proof of this fact).

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