RIEMANNIAN GEOMETRY, SPRING 2019, HOMEWORK 5

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Homework is assigned on Thursdays; it is due at the start of class the week after it is assigned. So this homework is due May 16th.

Problem 1. Let M be a manifold with strictly negative sectional curvature $K \le C < 0$ everywhere for some constant C. Let γ be a finite *closed* geodesic (i.e. a nonconstant map $\gamma : S^1 \to M$ with $\nabla_{\gamma'}\gamma' = 0$). Show that γ admits no normal Jacobi fields.

Problem 2. Suppose M is compact, orientable, even dimensional and satisfies $K \ge C > 0$ for some constant C, where K is the sectional curvature.
(i): Let γ be a closed geodesic in M, and let ν be the normal bundle of γ. Fix $p \in \gamma$ and let $P : \nu(p) \to \nu(p)$ be the result of parallel transport around γ. Show that P fixes a nonzero vector v.
(ii): Let V be the parallel vector field along γ with $V(p) = v$. Show (by using the second variation formula or otherwise) that if γ_s is a smooth variation of γ with $\partial_s \gamma_s = V$ then $\frac{d}{ds^2}\text{length}(\gamma_s)|_{s=0} < 0$. Deduce that γ is not a local minimum for length in its free homotopy class.
(iii): Show (e.g. by using the Arzela-Ascoli theorem) that if M is a compact manifold (with no assumptions on the curvature), every nontrivial conjugacy class in $\pi_1(M)$ contains a distance-minimizing geodesic. Deduce *Synge's Theorem*, which says that a compact, orientable, even dimensional manifold with $K \ge C > 0$ for some constant C is simply-connected.

Problem 3. Given an example of a compact manifold M with strictly positive scalar curvature $s \ge C > 0$ everywhere, but for which $\pi_1(M)$ is infinite.

Problem 4. Let M be a complete simply-connected Riemannian manifold with non-positive sectional curvature, and consider a geodesic triangle in M whose side lengths are a, b, c with opposite angles A, B, C respectively. Then $a^2 + b^2 - 2ab\cos C \le c^2$ and $A + B + C \le \pi$.

Problem 5 (Challenging). Throughout this problem assume that M is a compact, connected manifold.
(i): Let h be a smooth symmetric 2-form (i.e. a section of S^2T^*M). Let g be a Riemannian metric on M (so that g is a smooth symmetric 2-form which is positive definite everywhere). Show that $g + th$ defines a Riemannian metric g_t for all sufficiently small t.
(ii): Let $\text{vol}_{g_t}(M)$ denote the volume of M with respect to the g_t metric for $g_t = g + th$ as above. Show that $\frac{d}{dt}\text{vol}_{g_t}(M)|_{t=0} = 0$ if and only if $\int_M \text{tr}(h) d\text{vol}_g = 0$.
(iii): Recall that the scalar curvature s is the trace of the Ricci curvature of a Riemannian manifold. If we want to emphasize how s depends on the metric g we write s_g. The *total scalar curvature* of the metric g, denoted $\mathbb{S}(g)$, is the integral

$$\mathbb{S}(g) := \int_M s_g d\text{vol}_g$$

Show that

$$\frac{d}{dt}\mathbb{S}(g + th)|_{t=0} = \int_M \langle (s_g/2)g - \text{Ric}_g, h\rangle_g d\text{vol}_g$$

where $\langle \cdot, \cdot \rangle_g$ denotes the inner product on $S^2T^*_pM$ for each p induced by the Riemannian metric g.
(iv): (Hilbert) Suppose that M is of dimension at least 3. Let $\mathcal{M}_1$ denote the space of smooth metrics g on M for which $\text{vol}_g(M) = 1$. Deduce that (M, g) is a critical point for $\mathbb{S}(\cdot)$ in $\mathcal{M}_1$ if and only if it is Einstein (i.e. if and only if $\text{Ric}_g = \lambda g$ for some constant λ).

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