RIEMANNIAN GEOMETRY, SPRING 2019, HOMEWORK 4

DANNY CALEGARI

Homework is assigned on Thursdays; it is due at the start of class the week after it is assigned. So this homework is due May 2nd.

Problem 1. Let M be a Riemannian manifold, and suppose that the sectional curvature K is *constant* (i.e. it takes the same value on every 2-plane through every point). Show that there is a formula

$$\langle R(X,Y)Z,W\rangle = -K\cdot(\langle X,Z\rangle\langle Y,W\rangle - \langle Y,Z\rangle\langle X,W\rangle)$$

Deduce that (under the assumption that M has constant sectional curvature K), if $\gamma(t)$ is a geodesic and $e_i(t)$ are parallel orthonormal vector fields along γ giving a basis for the normal bundle $\nu|_\gamma$, every Jacobi field V along γ with $\langle V(0),\gamma'(0)\rangle = 0$ and $\langle V'(0),\gamma'(0)\rangle = 0$ can be written uniquely in the form

- $V(t) = \sum_i (a_i \sin(t\sqrt{K}) + b_i\cos(t\sqrt{K}))e_i(t)$ if $K > 0$;
- $V(t) = \sum_i (a_i t + b_i)e_i(t)$ if $K = 0$; and
- $V(t) = \sum_i (a_i \sinh(t\sqrt{-K}) + b_i\cosh(t\sqrt{-K}))e_i(t)$ if $K < 0$

for suitable constants a_i, b_i.

Problem 2. If we think of S^3 as the unit sphere in $\mathbb{C}^2$ (with its standard Hermitian metric), multiplication of the coordinates by $e^{i\theta}$ exhibits S^3 as a principal S^1 bundle over S^2 (this is usually known as the *Hopf fibration*). Let ξ be the 1-dimensional (real) subbundle of TS^3 tangent to the S^1 fibers, and let $\xi^\perp$ denote the orthogonal complement, so that $TS^3 = \xi\oplus\xi^\perp$. If g denotes the round metric on S^3, define a 1-parameter family of Riemannian metrics g_t by

$$g_t := g|_{\xi^\perp} \oplus t^2 g|_\xi$$

In other words, the length of vectors tangent to ξ are scaled by t (relative to the g metric), while the length of vectors perpendicular to ξ is the same as in the g metric. Compute the sectional curvature as a function of t. How does the sectional curvature behave in the limit as $t\to 0$ or $t\to\infty$?

(Note: a 3-sphere with one of the metrics g_t is sometimes called a *Berger sphere*)

Problem 3. Let M be a Riemannian manifold and let X be a vector field with the property that for any two vector fields Y and Z we have

$$\langle\nabla_Y X, Z\rangle + \langle\nabla_Z X, Y\rangle = 0$$

Show that the restriction of X to *every* geodesic is a Jacobi field.

(Bonus question: write down a nontrivial example of such an X for M the round 2-sphere.)

Problem 4. A *surface of revolution* is a smooth surface in $\mathbb{E}^3$ obtained by rotating a smooth curve (called the *generatrix*) in the x–z plane around the z axis. The generatrix, and the other curves on S obtained by rotating it, are called the *meridians*. Let S be a surface of revolution.

(i): (Clairaut's theorem) Let $\gamma(t)$ be a geodesic on S. Show that the angular momentum of $\gamma(t)$ about the z axis is constant; i.e. if r is the distance to the z-axis, and $\theta(t)$ is the angle between $\gamma'(t)$ and the meridian through $\gamma(t)$, then $r\sin(\theta)$ is constant as a function of t.

(ii): For S the torus obtained by rotating the curve $(x-3)^2 + z^2 = 1$ about the z-axis, give an explicit formula for the geodesics.

(Bonus question: if you want to solve an ODE, why is it helpful to find a conserved quantity — i.e. a function of the dependent variables that is constant on each solution?)

Problem 5. Find a complete noncompact surface properly embedded in $\mathbb{R}^3$ whose sectional curvature is strictly positive everywhere. Is there an example which is complete and noncompact and where $K \geq C > 0$ everywhere for some positive constant C?

Department of Mathematics, University of Chicago, Chicago, Illinois, 60637
E-mail address: dannyc@math.uchicago.edu