# RIEMANNIAN GEOMETRY, SPRING 2019, HOMEWORK 3

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Homework is assigned on Thursdays; it is due at the start of class the week after it is assigned. So this homework is due April 25th.

*Problem* 1. Give an example of a Riemannian metric on $\mathbb{R}^2$ which is complete but has *finite* total area.

*Problem* 2. Suppose $s_i$ are local sections of a smooth bundle $E$, and $\nabla$ is a connection on $E$ for which we can write (in terms of these coordinates) $\nabla = d + \omega$ where $\omega$ is a matrix of 1-forms (with components $\omega_{ij}$). Express $R$ in the same coordinates as a matrix of 2-forms $\Omega$, and show that

$$\Omega = d\omega - \omega \wedge \omega$$

How does $\Omega$ transform if we change coordinates on $E$ locally to $s_i' := \sum g_{ij} s_j$? What does this have to do with $R$ being a tensor?

*Problem* 3. Show directly that the Riemann curvature tensor (for the Levi-Civita connection on $TM$) can be recovered from the values of the sectional curvature, by giving an explicit formula for $\langle R(X,Y)Z, W\rangle$ in terms of $K$.

*Problem* 4. Let $C$ be the circle in the $x$–$z$ plane defined by the equation $(x-3)^2 + z^2 = 1$, and let $T$ be the surface in $\mathbb{E}^3$ obtained by revolving $C$ around the $z$-axis. For each point on $T$ give a formula for the size and the directions of the principal curvatures, and the sectional curvature. What is the integral of the sectional curvature over $T$?

*Problem* 5. Show that the symmetry $\langle R(X,Y)Z, W\rangle = \langle R(Z,W)X, Y\rangle$ follows from the Bianchi identity, antisymmetry of $R(X,Y)$ in $X$ and $Y$, and antisymmetry of $\langle R(X,Y)Z, W\rangle$ in $Z$ and $W$.

Once you've done this, look up the picture on page 54 of Milnor's "Morse Theory" and explain how you could have come up with the idea of this picture.

*Problem* 6. Show that on a 3-manifold you can recover the full curvature tensor $R$ just from the Ricci curvature Ric.

*Problem* 7 (harder). Let $M$ be a 4-manifold. Let's let $R$ denote the symmetric section of $\Lambda^2 T^*M \otimes \Lambda^2 T^*M$ defined by $R(X,Y,Z,W) := \langle R(X,Y)Z, W\rangle$. Using the metric we can think of $R$ as a section of $\text{End}(\Lambda^2 T^*M)$.

If we let $e_1, e_2, e_3, e_4$ be an oriented orthonormal frame for $T^*M$ then we get a basis for $\Lambda^2 T^*M$ as follows. Define

$$\Lambda^{\pm} := \text{span of } \{e^1 \wedge e^2 \pm e^3 \wedge e^4, \quad e^2 \wedge e^3 \pm e^1 \wedge e^4, \quad e^3 \wedge e^1 \pm e^2 \wedge e^4\}$$

Using this basis for $\Lambda^2 T^*M = \Lambda^+ \oplus \Lambda^-$ write $R$ as a $6 \times 6$ matrix, which decomposes into four $3 \times 3$ blocks, and identify the blocks with components of $R$. Deduce that the Weyl curvature tensor $C$ decomposes into two pieces $C^{\pm}$ coming from this decomposition.

Explain this in terms of the representation theory of $\text{SO}(4)$.

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