RIEMANNIAN GEOMETRY, SPRING 2019, HOMEWORK 2

DANNY CALEGARI

Homework is assigned on Thursdays; it is due at the start of class the week after it is assigned. So this homework is due April 18th.

Problem 1. Let C be the cone $x^2+y^2=z^2$ in Euclidean $\mathbb{E}^3$, which is smooth away from the point $(0,0,0)$. Determine the geodesics on this cone (as smooth curves in $\mathbb{E}^3$) by directly solving the geodesic equations. Now slit the cone open along the ray $x=z$ and lay it flat in the plane (by "unrolling" it); what do the geodesics look like when the cone is laid flat in the plane?

Problem 2. If Σ is a smooth (2-dimensional) oriented surface in $\mathbb{E}^3$, the *Gauss map* is a map $G:\Sigma\to S^2$ (the unit sphere in $\mathbb{E}^3$) defined uniquely by the property that $T_p\Sigma$ and $T_{G(p)}S^2$ are parallel (in $\mathbb{E}^3$) as *oriented* planes. Show that $T\Sigma$ can be naturally identified with the pullback G^*TS^2. Derive the following consequence: the pullback of the Gauss map commutes with parallel transport; i.e. if $\gamma:[0,1]\to\Sigma$ is a smooth curve, and $V\in\Gamma(TS^2)$ is parallel along $G\circ\gamma$ (i.e. $\nabla_{(G\circ\gamma)'}(V)=0$) then the pullback G^*V (as a section of $T\Sigma$) is parallel along γ.

Problem 3. If the Riemannian metric is expressed locally in coordinates x_i in the form $g:=\sum g_{ij}dx_idx_j$, derive a formula for the Christoffel symbols Γ_{ij}^k (in the same coordinates) in terms of the g_{ij}.

Problem 4. Let x_i be geodesic normal coordinates centered at a point p (i.e. obtained from the exponential map by exponentiating orthonormal linear coordinates on T_pM). Show that the metric $g_{ij}dx_idx_j$ in these coordinates satisfies

$$g_{ij}(p)=\begin{cases}1 \text{ if } i=j\\ 0 \text{ otherwise}\end{cases}$$

and $(\partial_k g_{ij})(p)=0$ for all i,j,k. In other words, the metric "osculates" the Euclidean metric to first order (at p).

Problem 5. Let S be the flat torus $\mathbb{R}^2/\mathbb{Z}^2$ with its Euclidean metric minus a point. Compute the maximal domain of definition of the exponential map $\exp_p$ at each point $p\in S$, and describe how this domain varies with p (note: we may use the flat structure to canonically identify the tangent space at each point with $\mathbb{R}^2$).

Department of Mathematics, University of Chicago, Chicago, Illinois, 60637
E-mail address: dannyc@math.uchicago.edu