RIEMANNIAN GEOMETRY, SPRING 2019, HOMEWORK 1

DANNY CALEGARI

Homework is assigned on Thursdays; it is due at the start of class the week after it is assigned. So this homework is due April 11th.

Problem 1. If ∇ is a connection on a smooth bundle E, covariant differentiation $\nabla : \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$ is tensorial in the first term, but not in the second. However, if ∇_1, ∇_2 are two connections on E, show that the difference $\nabla_1 - \nabla_2$ *is* a tensor in the second term. Thus, if we choose a local trivialization of E, deduce that any covariant derivative on E can be expressed locally (in terms of this trivialization) as $d + \omega$, where ω is a matrix of 1-forms. I.e. if s_i are local sections of E, we can write

$$\nabla \left(\sum_i f_i s_i \right) = \sum_i df_i \otimes s_i + \sum_{i,j} f_i \omega_{ij} \otimes s_j$$

Problem 2. For the round unit sphere in $\mathbb{E}^3$ with its induced Riemannian metric, express the Levi-Civita connection on the sphere in terms of local polar coordinates.

Problem 3. Check that the definition of the Levi-Civita connection given (implicitly) by the formula

$$2\langle \nabla_X Y, Z\rangle = X\langle Y, Z\rangle + Y\langle X, Z\rangle - Z\langle X, Y\rangle + \langle [X,Y], Z\rangle - \langle [X,Z], Y\rangle - \langle [Y,Z], X\rangle$$

satisfies the properties of a connection.

Problem 4. Let ∇ be a connection on TM, which thereby induces a connection on T^*M in the usual way. Show that ∇ on TM is torsion-free if and only if the composition

$$\Gamma(T^*M) \xrightarrow{\nabla} \Gamma(T^*M \otimes T^*M) \xrightarrow{\pi} \Gamma(\Lambda^2 T^*M)$$

is equal to exterior d, where Γ denotes the space of smooth sections, and π is the antisymmetrizing map.

Problem 5. Show that a connection ∇ on TM preserves the metric if and only if the metric 2-tensor $g \in \Gamma(S^2 T^*M)$ is parallel; i.e. $\nabla g = 0$.

Problem 6 (harder). Let S be the surface in $\mathbb{R}^3$ defined by the equation $z = x^2 - y^2$ (this is called a *hyperbolic paraboloid*). Let $\gamma : S^1 \to S$ be the circle $\gamma(\theta) = (\cos(\theta), \sin(\theta), \cos^2(\theta) - \sin^2(\theta))$. Let $p = (1, 0, 1)$ be a point on γ. Compute the effect of parallel transport around γ on the tangent space T_pS. Remember that parallel transport for a surface in $\mathbb{R}^3$ is the effect of continuously translating a vector space in $\mathbb{R}^3$ and then orthogonally projecting it to TS as we move along the curve γ.

Department of Mathematics, University of Chicago, Chicago, Illinois, 60637
E-mail address: `dannyc@math.uchicago.edu`