POINT SET TOPOLOGY, WINTER 2015, FINAL

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This final exam was given out on Thursday, March 19.

Problem 1. Show that the map $f : [-1,1] \to [-1,1]$ defined by $f(x) = 2x^2 - 1$ is continuous (e.g. by showing that the preimage of each open/closed set is open/closed), both in the "usual" topology, and the cofinite topology.

Problem 2. Let $[-1,1]^{\mathbb{N}}$ have the product topology, where each interval has the usual topology. Points of $[-1,1]^{\mathbb{N}}$ are sequences $(x_1, x_2, \cdots)$ of real numbers, where each $x_i \in [-1,1]$. Let X be the subset of $[-1,1]^{\mathbb{N}}$ consisting of sequences $(x_1, x_2, \cdots)$ for which $x_i = 2x_{i+1}^2 - 1$ for all $i \in \mathbb{N}$. Show that X is compact and Hausdorff, and the map $\sigma : X \to X$ defined by $\sigma(x_1, x_2, \cdots) = (x_2, x_3 \cdots)$ is a homeomorphism from X to itself. (Bonus question: draw a picture of X and explain what σ does to X).

Problem 3. Let U be an open subset of $\mathbb{R}^2$. Show that U is locally path connected, and therefore deduce that the components and the path components of U are the same. In this (or any other) way conclude that U has only countably many components.

Problem 4. Write down definitions of the terms

(1) locally compact;
(2) Hausdorff;
(3) completely regular.

Show that every locally compact Hausdorff space is completely regular. (Yes, I know this was a homework problem. You should be thanking me!)

Problem 5. Show that the Stone-Čech compactification $\beta\mathbb{N}$ of the integers $\mathbb{N}$ with the discrete topology has uncountably many points (hint: construct a bounded sequence of real numbers which contains uncountably many subsequences converging to different numbers).

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