LINEAR ALGEBRA

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Abstract. These notes are based on an undergraduate course taught at the University of Chicago in Winter 2019.

Contents

1. Vector spaces

1.1. What is Linear Algebra? [1]

1.1.1. *Icecream.* When I was seven years old, I had a funny dream that I lived on an alien planet. I'm not sure exactly what I was doing there; maybe I was visiting relatives? In my dream the aliens looked and talked and acted exactly like middle-class Melbournians from the 1970s, and in fact were completely indistinguishable from all the ordinary people around me every day in every respect but one: they ate icecream for dinner, and then had vegetables for dessert afterwards. In the dream I ate my 'dinner' — scoop after scoop of vanilla straight from the blue Peters' tub. All the while the aliens smiled and nodded, 'what a good little boy he is,' they said, 'eating up all his icecream without complaining!'

And they were even more impressed at the end of the meal when I was too full for peas and carrots. Truly, I was the cat who got the cream.[2]

That's Linear Algebra: it's good for you, and every bite is heaven. If you don't feel like you're getting away with something, you're doing it wrong.

1.1.2. *Machine code.* If you've ever done any programming, you know that machine code is very important. It gives unparalleled precision and control, and allows you direct access to the CPU. But machine code is not the ideal medium for human-to-human communication, even when both of the humans in question are you.

Machine code — like a lot of the 'code' in which written mathematics is carried out, especially when the emphasis is calculation — is largely algebraic. But a great deal of the specialized hardware of the human brain is optimized to do geometry: to interpret

Date: January 9, 2019.

[1]Why have I chosen to make this first section so silly? See David Foster Wallace, 'This is Water': https://youtu.be/8CrOL-ydFMI

[2]kitschy Australian cultural reference

visual scenes in terms of a geometric configuration of objects in space, and to perform mental manipulations of such configurations in order to hit a ball, or to stick a landing on a tumbling pass. The internal language of this hardware *is* Linear Algebra. The human condition — or at least the 70% of it that concerns our figuring out our relation to the physical world — **is** Linear Algebra. Icecream

All the pain of an undergraduate Linear Algebra course comes from the necessity to mediate our relationship to the subject through clumsy machine code like this:

$$\left|\sum_{i=1}^{n} u_i \cdot v_i\right|^2 \leq \left(\sum_{j=1}^{n} |u_j|^2\right) \cdot \left(\sum_{k=1}^{n} |v_k|^2\right)$$

instead of by direct apprehension like this:

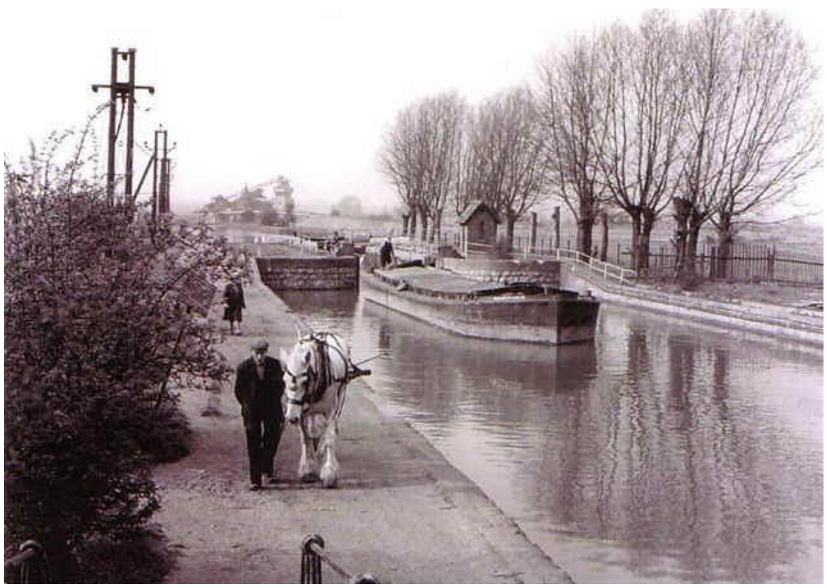


FIGURE 1. A horse doing Linear Algebra

1.1.3. *Why PAY-go is wrong.* By which I mean 'Prove As You Go'. Here's why it's wrong: because the purpose of a course in Linear Algebra isn't to verify that the assertions of the subject are true (they are, by the way), but to *communicate* what Linear Algebra really is, and to give the people taking the class the tools to use it, and the perspective to see where and why they can and should.

So: I'm not going to prove each assertion immediately at the time I make it. Even more shockingly, I'm not going to give precise definitions of mathematical objects before talking about them and using them.

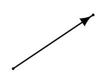
I am going to give precise definitions and rigorous proofs eventually though, and I'm even going to do it as early in the process as is pedagogically reasonable. So don't freak out just yet!

1.1.4. *What's linear about Linear Algebra?* In an object-oriented language like `C++` the emphasis isn't on things, it's on *classes*. A class is when you think of a thing not in isolation, but together with the kinds of functions you're allowed to apply to the thing.

In Linear Algebra the 'things' are *vector spaces* and the 'functions' are *linear transformations*. The main thing about a vector space is that it's chock full of things called *lines*, and the linear transformations take lines to lines in a simple way. That's the 'linear' in Linear Algebra.

Actually, there's another thing. A vector space has a very special point in it, called **zero**, which people usually write as 0 to save chalk. Every line goes through zero. Every linear transformation takes zero to zero. Zero is (also) the 'linear' in Linear Algebra.

1.2. **Vector spaces.** OK, what's a vector space? It's a space — whatever that is; anyway, a kind of set — whose elements are *vectors*. What's a vector? It's just a word for an element of a vector space. This sounds circular, and it is. So that our brains have something to latch onto, let's represent a vector by a little arrow, like this:

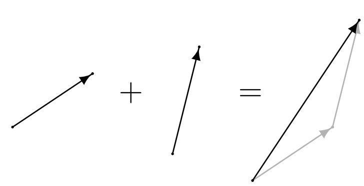


It goes from the 'tail' to the 'head'. If I draw two arrows on the page (or on a blackboard or whatever), and I can get one of the arrows from the other by translating it, then I want to think of them as representing the 'same' vector. This sounds complicated but it isn't, it's exactly like the principle that this 6 represents the same number as this 6 here, even though they're in different places.

You have to be a bit careful though: rotating an arrow can definitely change the vector it represents. Just like 6 and 9 represent different numbers.

There's a special vector — the *zero* vector — whose length is zero; if we have to draw it, we'll draw it by a dot. All dots are the same, so there's only one zero.

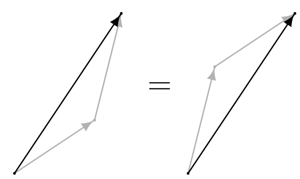
1.2.1. *Adding vectors.* Vectors can be *added*. In our representation, we add vectors (arrows) by concatenating them head to tail, and then drawing a new arrow (the result of the addition) from the tail of the first to the head of the second:



Evidently, adding zero does nothing; for any vector v, we have $0 + v = v + 0 = v$. This justifies the term 'zero vector', I guess.

1.2.2. *Addition of vectors is commutative.* In a vector space the order of addition doesn't matter — addition is *commutative*. In other words, we should get the same result if we interchange the order of concatenation:

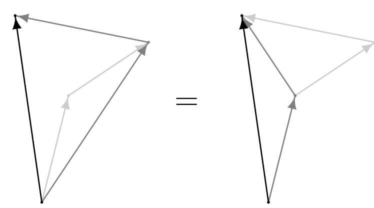
Addition of arrows is commutative: the sum of two vectors is the diagonal of the parallelogram with the given two vectors as sides; commutativity is equivalent to the fact that



such a parallelogram exists at all. For arrows, this is an assertion that needs proof.[3] But for vector spaces, commutativity is something we put in as part of the definition, and then we look for interesting examples where it holds.

1.2.3. *Addition of vectors is associative.* Now suppose we have three vectors. We can add the first two together and then add the third to the result. Or we can add the first to the result of adding the last two together. In other words, we can produce $(u + v) + w$ or $u + (v + w)$. We want these two expressions to be equal; if they are, we say addition of vectors is *associative*. Again, this is a requirement that we put in by fiat; having done that, the onus is on us to come up with examples that satisfy it.

Addition of arrows is associative:



Here's why: because concatenation is associative. If we get a bunch of arrows and lay them head to tail, the end result doesn't depend on which order we lay them out. It's a bit like the fact that the end result of putting together a jigsaw puzzle doesn't depend on the order in which you put the pieces in.

Because addition of vectors is both commutative and associative, it makes sense to do things like: take a bunch of vectors and add them all together. I don't need to say which order to lay out the vectors, or which order to do the adding.

1.2.4. *Where are the lines?* OK, I said that vector spaces are full of lines. Where are they? The lines aren't the *elements* of the vector space, they're certain kinds of *subsets*.

1.2.5. *Scaling vectors.*

1.2.6. *Scaling vectors is distributive.*

2. Acknowledgments

Danny Calegari was supported by NSF grant DMS 1405466.

[3] In principle; actually it's equivalent to the 'parallel postulate' in Euclidean geometry, usually taken as an axiom. You need to start from somewhere.

References

[1] Sergei Treil, *Linear Algebra done Wrong*, online textbook, https://www.math.brown.edu/~treil/papers/LADW/LADW_2017-09-04.pdf

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