The double limit theorem

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1 Introduction and statement of theorem

The double limit theorem gives necessary conditions for a sequence of quasi-fuchsian manifolds $qf(X_i, Y_i)$ to have a limit, up to subsequence. It is the final step needed to finish the proof of the fibred case.

Let S be a complete hyperbolic surface, possibly with boundary. Then there is a compact surface $\bar{S}$ so that S is homeomorphic to $int(\bar{S})$. I will write $AH(S)$ for $AH(\bar{S} \times I, \partial\bar{S} \times I)$. When $\bar{S} = S$ they are, of course, the same.

Earlier Bers proved the following compactness result:

Theorem 1.1 *For any $Y \in Teich(S)$ the subset $B_Y = \{qf(X, Y) \mid X \in Teich(S)\}$ is pre-compact.*

The proof of this uses the fact that the family of conformal maps of the lower half-plane (suitably normalized) is a normal family. But morally it can be seen as an application of the following bound for the length of a geodesic in $N = qf(X, Y)$ in terms of the length of the poincare geodesics in X and Y.

Proposition 1.2

$$\frac{1}{l(\gamma)} \geq \frac{1}{2}\Big(\frac{1}{l_X(\gamma)} + \frac{1}{l_Y(\gamma)}\Big)$$

and in particular

$$l(\gamma) < 2 l_Y(\gamma).$$

Hence in a Bers slice B_Y the length of any set of curves representing a generating set for $\pi_1(S)$ is bounded away from infinity in any sequence $qf(X_i, Y)$; curves can shrink to length 0 - that is, cusp - in the limit, but cannot grow to have unbounded length. Bounding the lengths give a bound on the norms of the traces of the generators in $PSL_2(\mathbb{C})$ - so aside from some "twisting" and keeping track of the marking, the proposition above implies a sequence $qf(X_i, Y)$ lies in a compact subset of $AH(S)$, so that, up to subsequence, the sequence converges.

Since, in the situation we are in the conformal boundary disappears in the limit, a more internal formulation of this compactness is desireable.

Let $C(N)$ denote the smallest convex submanifold of N for which the inclusion is a homotopy equivalence. $C(N)$ is called the *convex core* of N. The convex core can be constructed by taking the quotient by $\pi_1(N)$ of hyperbolic convex hull of the limit set. Even more concretely, $C(N)$ has the following description.

Let $CH(\Lambda(\Gamma))$ denote the convex hull in $\mathbb{H}^3$ of the limit set $\Lambda(\Gamma)$, where Γ is the fundamental group of N. For any $y \in \mathbb{H}^3 - CH(\Lambda(\Gamma))$ let P_y be a hyperbolic plane containing the point $r(y)$, where r denotes the nearest point retraction map onto $CH(\Lambda(\Gamma))$ ($r(y)$ is unique by convexity). Moreover, let H_y denote the totally geodesic half-space bounded by P_y and containing y. Then by choosing a countable dense subset D of $\mathbb{H}^3 - CH(\Lambda(\Gamma))$ we find that

$$CH(\Lambda(\Gamma)) = \bigcap_{y \in D} (\mathbb{H}^3 - H_y)$$

is a countable intersection of closed half-spaces. Moreover it can be seen that

$$C(N) = CH(\Lambda(\Gamma)/\Gamma.$$

Hence, unless $\Lambda(\Gamma)$ is a circle, the boundary of $CH(\Lambda(\Gamma))$ is a collection of "pleated hyperbolic planes" - a union along a possibly uncountably collection of geodesics of a countable collection of subsets of hyperbolic planes. The boundary components of $C(N)$ are "pleated surfaces". We will discuss pleated planes and pleated surfaces in more detail in coming sections.

The following theorem links the geometry of the conformal boundary to that of the pleated surfaces in the boundary of $C(N)$.

Theorem 1.3 *(Sullivan)*

There is a constant $1 < K < \infty$ so that for every complete hyperbolic manifold N and any incompressible component S of $\partial C(N)$, there is a K-biLipshitz homeomorphism from S to the corresponding surface in the conformal boundary.

The idea for this theorem is that bending a component of $C(N)$ along a closed curve has the effect in the conformal boundary of cutting along the bending curve and inserting (or "grafting") an annulus of Euclidean length equal to the amount of bending. Of course the resulting surface is no longer hyperbolic and one must re-uniformize, but one would expect that the length of the bending curve would decrease in the conformal boundary and stay the same in the convex core boundary. Curve transverse to the bending curve should have increased length in the conformal boundary and the same length in the convex core boundary. The amount of increase or decrease in length is bounded by the angle of bending. Hence the nearest point retraction from the complement of the convex core, extended to include the conformal boundary, will have a bi-Lipshitz constant bounded by an increasing function of the bending angles - since the convex core boundary is embedded, there is a bound on the bending angles and hence a bound on the biLipshitz constant. In general, all considerations are done in the universal cover, and bending can be done over curves that are not closed.

Hence in a Bers slice B_Y the length of the curves for any generating set for $\pi_1(N)$ is bounded by the length of their representatives on the component of $\partial C(N)$ facing Y. The boundary of the convex core is not a smooth surface in N, unless N is Fuchsian; it is an example of a pleated surface.

Definition 1.4 A *pleated surface* in a hyperbolic 3-manifold N is a complete hyperbolic surface S together with a continuous map $f : S \longrightarrow N$ which satisfies:

1. f takes every geodesic segment in S to a rectifiable arc in N with the same length - we say f is isometric;

2. for each $x \in S$ there is at least one open geodesic segment l_x through x which is mapped to a geodesic segment in N.

Note that f is generally not smooth and not even locally an embedding.

Proposition 1.5 *If $x \in S$ then either l_x is unique, or x has a neighbourhood which is mapped isometrically to a portion of a hyperbolic plane in N.*

Proof

Suppose that x lies on distinct geodesic segments l and m. The point x is contained in a quadrilateral with geodesic sides, two of which are contained in l and m. This quadrilateral is mapped by f to a quadrilateral with the same side lengths. The angles at $f(m)$ and $f(l)$ must be the same as the angles at m and l, as otherwise a curve joining l to m, near x, would get mapped to a shorter curve in N. Hence a neighbourhood of $f(x)$ is hyperbolically flat.

□

Definition 1.6 The set of points $x \in S$ for which l_x is unique is called the *pleating locus* $\Pi(f)$ for the surface $f : S \longrightarrow N$.

Proposition 1.7 *The pleating locus $\Pi(f)$ has the structure of a geodesic lamination.*

The Bers slice B_Y is pre-compact "because" for every sequence $N_i = qf(X_i, Y)$ there is a corresponding sequence of pleated surfaces $f_i : Y \longrightarrow N_i$ so that the length of any generating set for N_i is bounded by the length of the corresponding set of curves in Y. The approach to the double-limit theorem will be show that under certain conditions, a sequence $N_i = qf(X_i, Y_i)$ and a choice of generating set for $\pi_1(S)$, will have an associated sequence of pleated surfaces $f_i : S \longrightarrow N_i$ well-approximating N_i and contained in a compact subset of $Teich(S)$.

We are almost at the point where we can state the theorem. First a definition.

Definition 1.8 A pair of laminations (μ, ν) is *binding* if

1. they have no leaves in common;

2. for each component U of $S - \mu \cup \nu$, either the metric completion of U is a compact polygon, or the metric completion of U is a polygon with one ideal vertex which tends to a cusp, or the metric completion of U is a punctured polygon, where the puncture is a cusp on S.

Equivalently

Proposition 1.9 *A pair of laminations (μ, ν) is binding if and only if every simple geodesic on S has at least one transverse intersection with a leaf of μ or a leaf of ν.*

A pair of compactly supported measured laminations (μ, ν) is binding if and only if for every compactly supported measured lamination λ,

$$i(\lambda, \mu) + i(\lambda, \nu) > 0.$$

For example, if $[\mu_s]$ and $[\mu_u]$ are the stable and unstable projective measured laminations associated to a pseudo-Anosov mapping class then for any $\mu_s \in [\mu_s]$ and any $\mu_u \in [\mu_u]$ the pair (μ_s, μ_u) are binding.

Theorem 1.10 *Double Limit Theorem*

Let (μ, μ') bind S. Then for any sequence $(X_i, Y_i) \in Teich(S) \times Teich(S)$ converging to $([\mu], [\mu'])$, the sequence

$$\{qf(X_i, Y_i)\}$$

has a subsequence which converges algebraically.

Remark The Bers used his bound on the lengths of curves in a Bers slice to show the existence of manifolds in the closure of a Bers slice that were cusps - that is, components of the conformal boundary had punctures. In fact, using the length bound he showed that any collection of disjoint, simple closed curves could be made to cusp in the limit. Analogously, the double limit theorem allow the construction of manifolds which have prescribed ending laminations, provided those ending laminations bind.

2 Pleated Surfaces

Before beginning the proof of the double limit theorem we need some understanding of pleated surfaces.

A simple, closed geodesic $g : S^1 \longrightarrow N$ lifts to an embedding $\bar{g} : S^1 \longrightarrow P(N)$, where $P(N)$ is the projectivized tangent bundle to N. Since laminations are generalizations of simple closed curves we expect similar behaviour.

Theorem 2.1 *Let $f : S \longrightarrow N$ be an incompressible pleated surface in $N \simeq int(S \times I)$. Then the map $f : \Pi(f) \longrightarrow N$ lifts to an embedding $\bar{f} : \Pi(f) \longrightarrow P(N)$.*

Note that this is a special case of a more general theorem. For an arbitrary hyperbolic 3-manifold N the lift may not be an embedding - it may factor through a cover. Topological restrictions placed on f ensure that the lift is an embedding. These topological restrictions are immediate for $N = int(S \times I)$.

Definition 2.2 A map $f : S \longrightarrow N$ is *doubly incompressible* provided

1. f is incompressible;

2. homotopy classes of maps $(I, \partial I) \longrightarrow (S, cusps(S))$ are mapped injectively by f to homotopy classes of maps $(I, \partial I) \longrightarrow (N, cusps(N))$;

3. For every essential cyclinder $c : (S^1 \times I, \partial(S^1 \times I)) \longrightarrow (N, f(S))$, so that $\partial c = c|_{\partial(S^1 \times I)}$ factors as $f \circ c_0$, the image of $\pi_1(c_0)$ consists of parabolic elements of $\pi_1(S)$, or extends to a map of $S^1 \times I$ into S;

4. every maximal abelian subgroup of $\pi_1(S)$ is mapped to a maximal abelian subgroup of $\pi_1(N)$.

A map $f : S \longrightarrow N$ is *weakly doubly incompressible* provided properties 1, 2 and 3 hold, but instead of 4 we will require the weaker condition that every maximal cyclic subgroup of $\pi_1(S)$ is mapped to a maximal cyclic subgroup of $\pi_1(N)$.

Theorem 2.3 *If $f : S \longrightarrow N$ is weakly doubly incompressible and if λ is any geodesic lamination so that f takes every leaf of λ to a geodesic in N by a local homeomorphism, then*

$$\bar{f} : \lambda \longrightarrow P(N)$$

is an embedding.

Proposition 2.4 *Let $f_i : (S_i, x_i) \longrightarrow (N_i, y_i)$ be a sequence of doubly incompressible pleated surfaces, with $inj_{S_i}(x_i) \geq \epsilon$, converging geometrically to*

$$f : S \longrightarrow N.$$

Then f is weakly doubly incompressible.

$\square$

The next theorem gives us some control over how the image of the thick part of S sits in N - it cannot double-over too much.

Theorem 2.5 *(Uniform Injectivity)*

Let $\epsilon_0 >$ and $A > 0$ be given. Among all doubly incompressible

$$f : S \longrightarrow N$$

of area not greater than A the maps

$$\bar{f} : \Pi(f) \longrightarrow P(N)$$

are uniformly injective on the ϵ_0-thick part of S.
That is, for every $\epsilon > 0$ there is a $\delta > 0$ so that for any such S, N and f and for any two points $x, y \in \Pi(f)$ whose injectivity radii are greater than ϵ_0, if $d(x, y) \geq \epsilon$ then $d(\bar{f}(x), \bar{f}(y)) \geq \delta$.

Proof

The space of surfaces with base-frame (S, x) such that S has area bounded by A and $inj_S(x) \geq \epsilon_0$ is compact in the geometric topology.

There is a constant C such that for any surface S there exists two curves γ_1, γ_2 through x whose lengths are bounded above by C, and $< \gamma_1, \gamma_2 >$ is a free group on two generators. This is seen by observing that there is always a geodesic of length less than or equal to $2\cosh^{-1}\left(\frac{A}{2\pi}+1\right)$. If the constant does not exist then there is a sequence of surfaces (S_i, x_i) so that in the geometric limit all curves crossing this bounded length curve have infinite length, an impossibility for a finite area surface. Hence two curves of bounded length always exist and by the Margulis lemma generate a free group.

Let γ be a closed geodesic through $f(x)$. Then one of $< \gamma, f(\gamma_1) >$ or $< \gamma, f(\gamma_2) >$ is non-abelian, since $< \gamma_1, \gamma_2 >$ is a free group of rank 2. Without loss of generality, suppose $< \gamma, f(\gamma_1) >$ is non-abelian. Since $< \gamma, f(\gamma_1) >$ is a free group of rank 2, there is a lower bound on $length(\gamma)$ since if $length(\gamma)$ is too small, the length $length_N(f(\gamma_1))$ will be forced to exceed C, an impossibility. Hence the injectivity radius of N at $f(x)$ is bounded away from 0.

The space of hyperbolic manifolds with base-frame (N, y) with $inj_N(y) \geq \epsilon_1$ is compact.

Hence given a sequence $f_i : S_i \longrightarrow N_i$, we can extract a limit $f : S \longrightarrow N$. By general principles f is 1-Lipshitz. Since the laminations $\Pi(f_i)$ converge we find that f is a pleated surface.

Suppose that there is a sequence of pleated surfaces

$$f_i : S_i \longrightarrow N_i$$

and points $x_i, y_i \in \Pi(f_i)$ so that $d(\bar{f}_i(x_i), \bar{f}_i(y_i)) \to 0$. We want to show that $d(x_i, y_i) \to 0$.

By the above considerations there is a limit surface $f : S \longrightarrow N$, and points $x, y \in \Pi(f)$ such that $x_i \to x$, $y_i \to y$.

By an earlier proposition we find that f is doubly incompressible, and hence $barf$ is an embeddiing, implying that $x = y$, as required.

$\square$

We now begin to understand the intrinsic geometry of the pleated surface is related to that of the 3-manifold N.

Definition 2.6 Recall that a lamination λ is *maximal* provided that each component of the complement $S - \lambda$ is isometric to the interior of an ideal triangle.

Let λ be a finite maximal lamination and μ a measured lamination with compact support. Define the *alternation number*

$$a(\lambda, \mu)$$

as follows. Using that fact weighted simple closed curves are dense in $ML(S)$ we may assume that μ is a simple closed curve.

If μ is a leaf of λ then $a(\lambda, \mu) = 0$. Otherwise there is a countable set of intersections of μ and λ. If x and y are intersection points with no intervening intersections along μ, then the leaves of λ through x and y are two sides of an ideal triangle. These leaves are therefore asymptotic on either one side of μ or the other. If x, y and z are three successive intersection points along μ, and if the leaves of λ through x and y are asymptotic on the opposite side of μ to the leaves of λ through y and z, then call y a *boundary intersection.*

It may be that a point y in the intersection is on an accumulation point of leaves of λ. The leaf of λ through y will be a closed leaf (since λ is finite), and the leaves of λ will spiral to the closed leaf on both sides. If the directions of spiraling are different then call y a *boundary intersection.*

Note that boundary intersections are isolated and their number is finite. Let $a(\lambda, \mu)$ be the total number of boundary intersections.

For a general lamination μ, compute $a(\lambda, \mu)$ as the total μ-transverse measure of the set of boundary intersections.

Theorem 2.7 *(Continuity of alternation number) For a fixed maximal finite lamination λ, the alternation number $a(\lambda, \mu)$ is a finite-valued, continuous function of the measured lamination μ.*

Proof Every isolated leaf l of λ, lifted to the universal cover, separates two

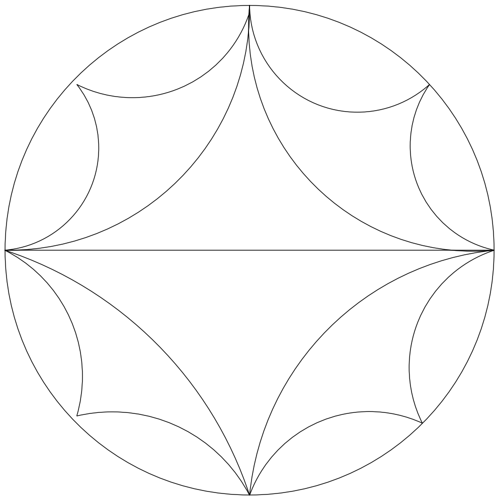

Figure 1:

ideal triangles, whose edges are lifts of leaves of λ. The union of two triangles forms an ideal quadrilateral Q_l. If a geodesic m crosses l it is a boundary intersection if and only if m passes through opposite sides of the quadrilateral.

For each quadrilateral and geodesic m passing through opposite sides of the quadrilateral let $\phi_{l,m}$ be a continuous function on $Q_l \cap m$ with integral 1 and value 0 on the end-points. Lift $\phi_{l,m}$ to $\bar{\phi}_l : P(\mathbb{H}^2) \longrightarrow \mathbb{R}$ so that $\bar{\phi}_l(x, m) = \phi_{l,m}(x)$.

Let $C : P(\mathbb{H}^2) \longrightarrow \mathbb{R}$ be

$$C(x, m) = \sum_l \phi_l(x, m).$$

The function C is well-defined since for any (x, m) there are only 3 non-zero terms in the sum (since any $x \in \mathbb{H}^2$ lies in at most 3 quadrilaterals). It is continuous, because when a sequence of geodesics m which cross opposite sides tends to one that doesn't, the length goes to infinity, so that the function tends to 0. Since the construction is invariant under deck transformations, we obtain a continuous function, also called C on $P(S)$.

Similarly, if α is a closed geodesic for λ which is spiraling in opposite directions on its two sides, add a contribution for each geodesic m which crosses α, supported on the intersection of m with an ϵ-neighbourhood of α. The result we call $C(\lambda)$.

If μ is a measured lamination there is an associated measure $M(\mu)$ on $P(S)$ the projectivized tangent bundle of S, defined as the product of the transverse measure with Lebesgue measure.

Clearly

$$a(\lambda, \mu) = \int_{P(S)} C(\lambda) dM(\mu),$$

implying that $a(\lambda, \cdot)$ is continuous.

□

We can now state and prove the main theorem needed for the proof of the double limit theorem.

Theorem 2.8 *(Efficiency of Pleated surfaces)*

For any $\epsilon > 0$ there is a constant $C < \infty$ such that the following holds:

Let λ be any finite maximal lamination on S and N any element of $AH(S)$. Suppose that no closed leaf of λ has length less than ϵ in N, so that in particular no such leaf is parabolic, and a surface $f_\lambda : P_\lambda \longrightarrow N$ exists with $\Pi(f_\lambda) = \lambda$.

Let μ be any compactly supported measured lamination which is realizable in N. Then

$$length_N(\mu) < length_{P_\lambda}(\mu) < length_N(\mu) + Ca(\lambda, \mu).$$

Proof

By continuity of the alternation number we may assume that μ is a weighted simple closed geodesic. Moreover, since the inequality is homogeneous in μ, we may assume that μ has weight 1.

First we deal with the special cases:

1. μ is a leaf of λ. Then $f : P \longrightarrow N$ maps μ to a geodesic in N and we are trivially done;

2. If μ is a geodesic shorter than ϵ on S, then it is not a leaf of λ and $a(\lambda, \mu) \geq 2$. It follows then that

$$length_{P_\lambda}(\mu) \geq length_N(\mu) + Ca(\lambda, \mu) \geq length_N(\mu) + 2C$$

   implies

$$length_N(\mu) \leq length_{P_\lambda}(\mu) - 2C$$

   which is false for C sufficiently large, depending on ϵ.

Hence we may assume that $a(\lambda, \mu) > 0$ and there is a chain of length $a(\lambda, \mu)$ consisting of the leaves of λ on which there are boundary intersections with μ.

Lift μ to $\tilde{\mu}$ in $\mathbb{H}^2$. Let R be the $\frac{1}{2}$-neighbourhood of $\tilde{\mu}$. Let $g_1, g_2, \ldots, g_n, \ldots$ be the lifts of the leaves of λ which intersect μ in the chain of leaves which have boundary intersections with μ.

Choosing some arbitrary orientation on μ, let a_i be the leftmost point of $R \cap g_i$ and b_i the rightmost endpoint. Let A_i be the perpendicular projections of a_i and b_i to $\tilde{\mu}$. Then $A_i < A_{i+1}$, $B_i < B_{i+1}$, and $|A_{i+1} - B_i|$ is bounded by $C_0 = log(\frac{1}{sin(\theta)})$ where θ is the angle such that

$$\frac{1}{2} = \int_{\frac{\pi}{2}}^{\theta + \frac{\pi}{2}} \frac{\phi}{sin(\phi)} d\phi.$$

If $B_i > A_{i+1}$ then let x_{i+1} be the point of g_{i+1} whose projection to $\tilde{\mu}$ is B_i; otherwise let $x_{i+1} = a_{i+1}$. The choice is made so that when we form our polygonal approximation to $\tilde{\mu}$ the approximating path will make "forward progress".

Define a polygonal path m in $\mathbb{H}^2$ to consist of the geodesic segments

$$\cdots \overline{x_i b_i}\ \overline{b_i x_{i+1}}\ \overline{x_{i+1} b_{i+1}} \cdots$$

Given the bound on $|A_{i+1} - B_i|$, one shows using hyperbolic trigonometry that the length of the projection to P_λ of m exceeds that of μ by a constant times $a(\lambda, \mu)$. The "jump" segment $\overline{b_{i-1}x_i}$ has length at most $1 + C_0$, so that the lenght of the segment $\overline{b_{i-1}x_i}\ \overline{x_ib_i}$ is at most $2 + C_0$ more than the distance from B_i to B_{i+1}. Hence m lies on the leaves of λ except for a portion of its length which is a constant times $a(\lambda, \mu)$.

$$length_{P_\lambda}(\mu) \leq length_{P_\lambda}(m) \leq length_{P_\lambda}(\mu) + C_1 a(\lambda, mu)$$

where $C_1 = 2 + C_0$.

We will now alter m so that none of the jumps between the leaves of λ occur in the ϵ-thin part of P_λ. Observe that all the leaves of λ which enter the thin part remain close together and are nearly parallel for the entire intersection of the thin part.

DRAW PICTURE OF HOW THIS IS DONE.

Because the leaves of λ are nearly parallel, and everthing is happening in the 2ϵ thin part we obtain that the new curve, n, has length exceeding $length(m)$ by at most a constant times $a(\lambda, \mu)$. More formally, if K is the maximal length of any boundary component of $P_{\lambda Thin(\epsilon_0)}$, then

$$length_{P_\lambda}(\mu) \leq length_N(n) \leq length_{P_\lambda}(\mu) + C_2 a(\lambda, \mu)$$

where $C_2 = 2 + C_0 + K$.

Moreover, each jump is in the thick part of P_λ and has length bounded by $max(1 + C_0, K)$.

Let $f : P_\lambda \longrightarrow N$ be our pleated surface. The image $p = f(n)$ of n is homotopic to a polygonal path p by homotoping each jump segment to the geodesic segment in N with the same endpoints. By the fellow-traveller property the length of p differs from the length of μ by at most a constant times $a(\lambda, \mu)$. The polygonal path p has $2a(\lambda, \mu)$ sides.

Let μ' be the geodesic in N homotopic to $f(\mu)$. There is a pleated annulus A representing the homotopy between p and μ'. It has area at most $2\pi a(\lambda, \mu)$.

Say that $x \in n$ lies in n_0 if it lies within a distance $max(1+C_0, K)/2$ from on of the endpoints of the geodesic segment on which it lies. Then n_0 contains all the jump segments of n, and there are at most $a(\lambda, \mu)$ components to n_0. Let p_0 be the corresponding portion of p. Then

$$length_{P_\lambda}(n_0) \leq 2Ka(\lambda, \mu).$$

Let p_1 consist of the points x in $p - p_0$ for which the geodesic segment p_x beginning at x perpendicular to p and terminating at ∂A has length at least

ϵ. That is, p_1 consists of those points of $p - p_0$ for which p_1 has a regular neighbourhood in A of width ϵ. Let R_1 consist of geodesics p_x. Observe that

$$area(R_2) \geq length_N(p_1)\frac{\epsilon}{2}.$$

Hence, since the area of A is a bounded by $2\pi a(\lambda, \mu)$ we find that

$$length_{P_\lambda}(n_1) \leq \frac{2area(A)}{\epsilon} \leq \frac{4\pi}{\epsilon}a(\lambda, mu).$$

The remainder $p_2 = p - (p_1 \cup p_0)$ runs very close to portions of the boundary of A. It consists of segments of leaves of λ passing very near to other boundary of A, and hence are nearly parallel to other portions of the boundary of A.

For any portion of the remainder of p in the thick part of P_λ, by the uniform injectivity theorem, we find that the portion of ∂A ϵ-close to this portion of p is contained in μ', or there is a way to shorten n. Consider a segment α of p_2 for which the projection of α to ∂A is a segment β of p_2. The map $f : P_\lambda \longrightarrow N$ maps the set of thin components of P_λ injectively into the set of thin components of N, so the preimages of both α and β are contained in the thick part of P_λ. If x_1 and x_2 are the end points of α, and y_1 and y_2 are the end points of β, labeled to that $d(x_i, y_i) \leq \epsilon$, then there are paths γ_i between x_i and y_i of length bounded by ϵ, and portions r_i of n so that $r_i \cup \gamma_i$ bounds a disk on P_λ for each i. We may assume that r_1 contains both α and β. Then γ_1 may be used to shorten the length of n by at least $2length(\alpha) - \epsilon$. The number of such shortenings is bounded by a constant times $a(\lambda, \mu)$.

Let I be an interval in n with endpoints in the thin part which is homotopic rel endpoints into the thin part. f is injective between the thin parts of P_λ and the thin parts of N. Hence the segments of leaves of λ running through the thin part cannot be close in A, because then two leaves would be close a short distance away in the thick part. We conclude that the remainder of p is ϵ close to μ', and hence has length bounded away from the length of its neighbour in μ'.

Hence we find that

$$length_{P_\lambda}(\mu) \leq length_{P_\lambda}(n) \leq length_N(\mu') + Ca(\lambda, \mu),$$

as required.

$\square$

The following proposition is a useful criterion for compactness in $Teich(S)$.

Proposition 2.9 *(Binding confinement)*

Let μ_1 and μ_2 bind S.

Then for any constant $C > 0$, the set of hyperbolic structures g on S for which

$$length_g(\mu_1) \leq C$$

and

$$length_g(\mu_2) \leq C$$

is compact.

$\square$

3 Proof of the double limit theorem

If $\bar{S}$ has boundary, choose a lamination λ on S with one ideal vertex and no closed leaves which is the 1-skeleton of an ideal triangulation.

Represent μ and μ' by measured laminations with length 1, also called μ and μ', respectively.

By the compactification of Teichmuller space by projective measured laminations there are measured laminations $\mu_i \to \mu$ so that for all compactly supported λ intersecting μ

$$|length_{X_i}(\lambda) - i(\lambda, \mu_i)| \to 0,$$

so that

$$length_{X_i}(\mu_i) \to 0.$$

Similarly there are $\mu_i' \to \mu'$ so that

$$length_{Y_i}(\mu_i') \to 0.$$

Set $N_i = qf(X_i, Y_i)$. By Bers' result or Sullivan's, it follows that the lengths

$$length_{N_i}(\mu_i) \to 0$$

and

$$length_{N_i}(\mu_i') \to 0.$$

Let $f_i : P_{\lambda,i} \longrightarrow N_i$ be a pleated surface in N_i with pleating locus λ. Using efficiency of pleated surfaces we find that since

$$lengthN_i(\mu_i) < length_{P_{\lambda,i}}(\mu_i) < length_{N_i}(\mu_i) + Ca(\lambda, \mu_i),$$

the length of μ in $P_{\lambda,i}$ is bounded for all i.

And similarly for μ'.

By the proposition about binding confinement, we find that the structures $P_{\lambda,i}$ lie in a bounded subset of $Teich(S)$. Therefore generators of the group $qf(X_i, Y_i)$ remain bounded, since they move a base-point of $\tilde{P_{\lambda,i}}$ not further in $\mathbb{H}^3$ than in $\tilde{P_{\lambda,i}}$.

To prove the case in which $\bar{S}$ has no boundary, we need to be able to choose our lamination λ so that throughout the sequence $qf(X_i, Y_i)$, the closed geodesics in λ are not short. We will then be able to apply efficiency of pleated surfaces and finish the proof as above.

Since we need but one closed geodesic to build λ it suffices to show that for any sequence $qf(X_i, Y_i)$ there is a curve that is not short infinitely often.

Proposition 3.1 *four not small*

Corollary 3.2 *four curves not all short*