NOTES ON DIFFERENTIAL FORMS

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ABSTRACT. These are notes on differential forms. They follow the last three weeks of a course given at the University of Chicago in Winter 2016.

CONTENTS

1. Differential forms 1
2. Integration 12
3. Chern classes 17
4. Acknowledgments 24
References 24

1. DIFFERENTIAL FORMS

1.1. **Exterior algebra.** Let V be a (real) vector space, and let V^* denote its dual. Then $(V^*)^{\otimes n} = (V^{\otimes n})^*$. For $u_i \in V^*$ and $v_i \in V$ the pairing is given by

$$u_1 \otimes u_2 \otimes \cdots \otimes u_n(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = u_1(v_1)u_2(v_2)\cdots u_n(v_n)$$

We denote by $T(V)$ the graded algebra $T(V) := \oplus_{n=0}^{\infty}(V^{\otimes n})$ and call it the *tensor algebra* of V. Let $I(V) \subset T(V)$ be the 2-sided ideal generated by elements of the form $v \otimes v$ for $v \in V$. This is a graded ideal and the quotient inherits a grading.

Definition 1.1 (Exterior Algebra)**.** The quotient $T(V)/I(V)$ is denoted $\Lambda(V)$. It is a graded algebra, and is called the *exterior algebra* of V.

The part of $\Lambda(V)$ in dimension j is denoted $\Lambda^j(V)$. If V is finite dimensional, and has a basis $v_1, \cdots, v_n$ then a basis for $\Lambda^j(V)$ is given by the image of j-fold "ordered" products $v_{i_1} \otimes \cdots \otimes v_{i_j}$ with $i_k < i_l$ for $k < l$. In particular, the dimension of $\Lambda^j(V)$ is $n!/j!(n-j)!$, and the total dimension of $\Lambda(V)$ is 2^n.

It turns out there is a natural isomorphism $(\Lambda(V))^* = \Lambda(V^*)$; equivalently, there is a nondegenerate pairing of $\Lambda(V^*)$ with $\Lambda(V)$. But in fact, there is more than one "natural" choice of pairing, so we must be more precise about which pairing we mean. If we think of $\Lambda(V)$ as a quotient of $T(V)$, then we could think of $\Lambda(V^*)$ as the *subgroup* of $T(V^*)$ consisting of tensors vanishing on the ideal $I(V)$. To pair $\alpha \in \Lambda(V^*) \subset T(V^*)$ with $\beta \in \Lambda(V) = T(V)/I(V)$ we choose any representative $\bar{\beta} \in T(V)$, and pair them by $\alpha(\beta) := \alpha(\bar{\beta})$ where the latter pairing is as above. Since (by definition) α vanishes on $I(V)$, this does not depend on the choice of $\bar{\beta}$.

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1.2. **Algebra structure.** As a quotient of $T(V^*)$ by an ideal, the exterior algebra inherits an algebra structure. But as a *subgroup* of $T(V^*)$ it is *not* a subalgebra, in the sense that the usual algebra product on $T(V^*)$ does not take $\Lambda(V^*)$ to itself.

Example 1.2. Observe that $\Lambda^1(V^*) = V^* = (\Lambda^1(V))^*$. But if $u \in V^*$ is nonzero, there is $v \in V$ with $u(v) = 1$, and then $u \otimes u(v \otimes v) = 1$ so that $u \otimes u$ is *not* in $\Lambda^2(V^*)$.

We now describe the algebra structure on $\Lambda(V^*)$ thought of as a subgroup of $T(V^*)$. Let S_j denote the group of permutations of the set $\{1, \cdots, j\}$. For $\sigma \in S_j$ the *sign* of σ, denoted $\mathrm{sgn}(\sigma)$, is 1 if σ is an even permutation (i.e. a product of an even number of transpositions), and -1 if σ is odd.

Definition 1.3 (Exterior product)**.** Let $u_1, \cdots, u_j \in V^*$. We define

$$u_1 \wedge u_2 \wedge \cdots \wedge u_j := \sum_{\sigma \in S_j} (-1)^{\mathrm{sgn}(\sigma)} u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes \cdots \otimes u_{\sigma(j)}$$

We call the result the *exterior product* (or sometimes *wedge product*).

It is straightforward to check that this element of $T(V^*)$ does indeed vanish on $I(V)$, and therefore lies in $\Lambda^j(V^*)$, and that every element of $\Lambda^j(V^*)$ is a finite linear combination of such products (which are called *pure* or *decomposable* forms). The space $\Lambda^j(V^*)$ pairs with $\Lambda^j(V)$ by

$$u_1 \wedge \cdots \wedge u_j(v_1 \otimes \cdots \otimes v_j) = \det(u_i(v_j))$$

on pure forms, and extended by linearity.

Let $S_{p,q}$ denote the set of p, q *shuffles*; i.e. the set of permutations σ of $\{1, \cdots, p+q\}$ with $\sigma(1) < \cdots < \sigma(p)$ and $\sigma(p+1) < \cdots < \sigma(p+q)$. It is a set of (canonical) coset representatives of the subgroup $S_p \times S_q$ in S_{p+q}. If $\alpha \in (\Lambda^p(V))^*$ and $\beta \in (\Lambda^q(V))^*$ then we can define $\alpha \wedge \beta \in (\Lambda^{p+q}(V))^*$ by

$$\alpha \wedge \beta(v_1 \otimes \cdots \otimes v_{p+q}) = \sum_{\sigma \in S_{p,q}} (-1)^{\mathrm{sgn}(\sigma)} \alpha(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(p)}) \beta(v_{\sigma(p+1)} \otimes \cdots \otimes v_{\sigma(p+q)})$$

One can check that this agrees with the notation above, so that $\Lambda(V^*)$ is an algebra with respect to exterior product, generated by $\Lambda^1(V^*)$. This product is associative and *skew-commutative*: i.e.

$$\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha$$

for $\alpha \in \Lambda^p(V^*)$ and $\beta \in \Lambda^q(V^*)$.

1.3. **Smooth manifolds and functions.** Let $U \subset \mathbb{R}^n$ be open. A map $\varphi : U \to \mathbb{R}^m$ is smooth if the coordinate functions are continuous and admit continuous mixed partial derivatives of all orders.

Definition 1.4 (Smooth Manifold)**.** An n-manifold is a (paracompact Hausdorff) topological space in which every point has a neighborhood homeomorphic to an open subset of $\mathbb{R}^n$. An n-manifold is *smooth* if it comes equipped with a family of open sets $U_\alpha \subset M$ (called *charts*) and maps $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$ which are homeomorphisms onto $\varphi_\alpha(U_\alpha) \subset \mathbb{R}^n$ open, so that for each pair α, β the *transition maps*

$$\varphi_\beta \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$$

are smooth, as maps between open subsets of $\mathbb{R}^n$.

Example 1.5. $\mathbb{R}^n$ is a smooth manifold. An open $U \subset \mathbb{R}^n$ is a smooth manifold. If M is smooth, an open $U \subset M$ is smooth.

Example 1.6. If M and N are smooth manifolds of dimensions m and n, then $M \times N$ is smooth of dimension $m+n$.

A function $f : M \to \mathbb{R}$ is *smooth* if it is smooth in local coordinates; i.e. if $f\varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha) \to \mathbb{R}$ is smooth for each α. Pointwise addition and multiplication make the smooth functions on M into a ring, denoted $C^\infty(M)$.

A map $\varphi : M \to N$ between smooth manifolds is smooth if it is smooth in local coordinates; i.e. if

$$\varphi_\beta\varphi\varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha) \to \varphi_\beta(V_\beta)$$

is smooth for each pair of charts U_α on M and V_β on N. Smooth functions *pull back*; i.e. there is a ring homomorphism $\varphi^* : C^\infty(N) \to C^\infty(M)$ given by $\varphi^*(f) = f\varphi$.

1.4. **Vectors and vector fields.** A map $\gamma : [0,\epsilon] \to M$ for some positive ϵ is smooth if its composition $\varphi_\alpha\gamma$ is smooth for each chart, where defined. We define an equivalence relation on the class of such maps, and say that γ and σ are *equivalent* if $\gamma(0) = \sigma(0)$, and if

$$\frac{d}{dt}\Big|_{t=0} \varphi_\alpha\gamma(t) = \frac{d}{dt}\Big|_{t=0} \varphi_\alpha\sigma(t)$$

where defined. Note that γ is equivalent to the restriction $\gamma|_{[0,\epsilon']}$ for any positive $\epsilon' \le \epsilon$; thus the equivalence class of γ depends only on the *germ* of γ at 0.

Definition 1.7 (Tangent space)**.** For $x \in M$, the set of equivalence classes of smooth maps $\gamma : [0,\epsilon] \to M$ for some positive ϵ with $\gamma(0) = x$ is called the *tangent space to M at x*, and is denoted T_xM.

The vector associated to the equivalence class of γ is usually denoted $\gamma'(0)$. By abuse of notation we denote by $\gamma'(t)$ the vector in $T_{\gamma(t)}M$ obtained by reparameterizing the domain of γ by a translation.

Definition 1.8 (Pushforward)**.** Vectors can be *pushed forward* by smooth maps. If $\varphi : M \to N$ is smooth, and $\gamma : [0,\epsilon] \to M$ is in the equivalence class of some vector at x, then $\varphi\gamma : [0,\epsilon] \to N$ is in the equivalence class of some vector at $\varphi(x)$. Thus there is an induced map $d\varphi_x : T_xM \to T_{\varphi(x)}N$.

Example 1.9. The identity map on $[0,\epsilon]$ determines the *coordinate* vector field ∂_t. Then $d\gamma_t(\partial_t) = \gamma'(t)$ for any $\gamma : [0,\epsilon] \to M$.

A vector $v \in T_xM$ defines a linear map $v : C^\infty(M) \to \mathbb{R}$ as follows. If f is a smooth (real-valued) function on M, and $\gamma : [0,\epsilon] \to M$ is a smooth map with $\gamma'(0) = v$, the composition $f\gamma$ is a smooth function on $[0,\epsilon]$, and we can define

$$v(f) := \frac{d}{dt}\Big|_{t=0} f\gamma(t)$$

By ordinary calculus, this satisfies the Leibniz rule $v(fg) = v(f)g(x) + f(x)v(g)$, and does not depend on the choice of representative γ.

Definition 1.10. A linear map $v : C^\infty(M) \to \mathbb{R}$ satisfying $v(fg) = v(f)g(x) + f(x)v(g)$ is called a *derivation* at x.

Every derivation at x arises from a unique vector at x, so the set of derivations is exactly T_xM. Note that this explains why this set has the structure of a vector space, of dimension n.

Each point $x \in M$ determines a unique maximal ideal $\mathfrak{m}_x$ in $C^\infty(M)$ consisting of the functions in M that vanish at x. This is exactly the kernel of the surjective homomorphism $C^\infty(M) \to \mathbb{R}$ given by $f \to f(x)$.

Proposition 1.11 (Pairing). *There is a natural pairing*

$$T_xM \otimes_{\mathbb{R}} \mathfrak{m}_x/\mathfrak{m}_x^2 \to \mathbb{R}$$

given by $v \otimes f \to v(f)$. This pairing is nondegenerate, so that $T_xM = (\mathfrak{m}_x/\mathfrak{m}_x^2)^*$.

Proof. If $f, g \in \mathfrak{m}_x$ then $v(fg) = v(f)g(x) + f(x)v(g) = 0$ so the pairing is well-defined.

Note that every derivation vanishes on the constants, so if $v \in T_xM$ is nonzero, it is nonzero on $\mathfrak{m}_x$. Thus $T_xM \to (\mathfrak{m}_x/\mathfrak{m}_x^2)^*$ is injective.

Conversely, any $\phi : \mathfrak{m}_x/\mathfrak{m}_x^2 \to \mathbb{R}$ extends to $\bar{\phi} : C^\infty(M) \to \mathbb{R}$ by $\bar{\phi}(f) := \phi(f - f(x))$, which is a derivation at x since

$$fg - f(x)g(x) = (f - f(x))(g - g(x)) + fg(x) + f(x)g - 2f(x)g(x)$$

so that

$$\phi(fg - f(x)g(x)) = \phi(f - f(x))g(x) + f(x)\phi(g - g(x))$$

$\square$

Derivations are natural: if $\varphi : M \to N$ and $v \in T_xM$ and $f \in C^\infty(N)$ then

$$v(\varphi^* f) = d\varphi_x(v)(f)$$

Definition 1.12 (Tangent bundle and vector fields). The collection of vector spaces T_xM for various $x \in M$ are the fibers of a smooth vector bundle TM called the *tangent bundle*. The fibers have dimension n, the same as the dimension of M. A smooth section of TM is called a *vector field*, and the space of all vector fields on M is denoted $\mathfrak{X}(M)$.

A smooth map $\varphi : M \to N$ induces a smooth map $d\varphi : TM \to TN$.

Example 1.13. If $\varphi : M \to N$ is a diffeomorphism, $d\varphi$ is a bundle isomorphism.

Vector fields do not push forward under maps in general, but when $\varphi : M \to N$ is a diffeomorphism and $X \in \mathfrak{X}(M)$ then it makes sense to define $d\varphi(X) \in \mathfrak{X}(N)$.

If f is a smooth function and X is a vector field, we can apply X as a derivation pointwise to obtain a new smooth function $X(f)$. This defines a pairing $\mathfrak{X}(M) \otimes_{\mathbb{R}} C^\infty(M) \to C^\infty(M)$. This pairing is *not* a pairing of $C^\infty(M)$ modules. It satisfies the Leibniz rule $X(fg) = X(f)g + fX(g)$.

1.5. **Local coordinates.** If $x_1, \cdots, x_n$ are smooth coordinates on M near a point x, the partial differential operators

$$\partial_i|_x := \frac{\partial}{\partial x_i}\Big|_x$$

are derivations at x, and thereby can be thought of as elements of T_xM (beware that the notation ∂_i ignores the dependence on the choice of local coordinates $x_1, \cdots, x_n$).

To translate this into geometric language, the subset of M near x where $x_j = x_j(x)$ for $j \neq i$ is a smooth 1-manifold, parameterized locally by x_i. Thus there is a smooth map $\gamma : [0, \epsilon] \to M$ uniquely defined (for sufficiently small ϵ) by $\gamma(0) = x$, by $x_j(\gamma(t)) = x_j(x)$ for $j \neq i$, and $x_i(\gamma(t)) = x_i(x) + t$. Then as vectors, $\partial_i|_x = \gamma'(0)$. Note that we need *all* n coordinates x_j to define any operator $\partial_i|_x$.

The (local) sections ∂_i defined in a coordinate patch U span $\mathfrak{X}(M)$ (locally) as a free $C^\infty(U)$ module; i.e. any vector field may be expressed (throughout the coordinate patch) uniquely in the form

$$X := \sum X_i \partial_i$$

1.6. **Lie bracket.** If $X, Y \in \mathfrak{X}(M)$ the operator $[X, Y] := XY - YX$ on smooth functions, *a priori* of second order, turns out to be first order and to satisfy the Leibniz rule; i.e. it defines a vector field, called the *Lie bracket* of X and Y. The reason is that partial differentiation commutes (for functions which are at least C^2), so when we antisymmetrize, the second order terms cancel.

To see this, first observe that the claim is local, so we can work in a coordinate patch. Then take $X = \sum X_i \partial_i$ and $Y = \sum Y_i \partial_i$, and let f be a smooth function. Then

$$\begin{aligned}
[X, Y]f = X(Y(f)) - Y(X(f)) &= \sum_i X_i \partial_i \big(\sum_j Y_j \partial_j(f) \big) - \sum_i Y_i \partial_i \big(\sum_j X_j \partial_j(f) \big) \\
&= \sum_j \big(\sum_i (X_i \partial_i(Y_j) - Y_i \partial_i(X_j)) \big) \partial_j(f)
\end{aligned}$$

Proposition 1.14 (Lie algebra)**.** *Lie bracket is antisymmetric in X and Y, linear in each variable, and satisfies the* Jacobi identity

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$$

for any three vector fields $X, Y, Z \in \mathfrak{X}(M)$.

The proof is by calculation. One says that bracket makes $\mathfrak{X}(M)$ into a *Lie algebra*.

Proposition 1.15 (Naturality)**.** *Lie bracket is natural; i.e. if $\varphi : M \to N$ is a diffeomorphism, and $X, Y \in \mathfrak{X}(M)$ then $d\varphi([X, Y]) = [d\varphi(X), d\varphi(Y)]$*

Since Lie bracket is defined without reference to coordinates, naturality under diffeomorphism follows.

Example 1.16 (Local coordinates)*.* If $x_1, \cdots, x_n$ are local coordinates with associated vector fields $\partial_1, \cdots, \partial_n$ then $[\partial_i, \partial_j] = 0$ for all i, j (this is equivalent to the usual statement that “partial derivatives commute”).

1.7. **Flow of a vector field.** Let X be a vector field on M. If we think of a vector as a(n equivalence class of) smooth map from $[0,\epsilon]$ to M, it is natural to ask for a smooth map $\phi : M \times [0,\epsilon] \to M$ so that the vector $X(x)$ agrees with the equivalence class of $\phi(x,*) : [0,\epsilon] \to M$. In fact, there is a *canonical* choice of the *germ* of such a ϕ along $M \times 0$, which furthermore satisfies $\phi(x, s+t) = \phi(\phi(x,s),t)$ for all sufficiently small (positive) s,t (depending on x).

If we denote $\phi(x,t)$ by $\phi_t(x)$ we can think of ϕ_t as a 1-parameter family of diffeomorphisms whose orbits are *tangent* to the vector field X. The existence and uniqueness of ϕ (for small enough positive ϵ on compact subsets of M) is just the fundamental theorem of ODE. We say that the family ϕ_t of diffeomorphisms is obtained by *integrating* X. The orbits of ϕ_t are the *integral curves* of the vector field X.

Formally we write ϕ_{-t} for t small and positive as the (time t) flow associated to the vector field $-X$. Thus we have ϕ_t defined for all $t \in [-\epsilon, \epsilon]$. Notice that $\phi_{s+t} = \phi_s \phi_t$ for any real s,t where defined.

If X and Y are vector fields, and ϕ_t is obtained by integrating X, we may form the family of vector fields $d\phi_{-t}(Y)$.

Definition 1.17 (Lie Derivative of vector fields)**.** The *Lie derivative* $\mathfrak{L}_X : \mathfrak{X}(M) \to \mathfrak{X}(M)$ is defined by

$$\mathfrak{L}_X(Y) := \lim_{t \to 0} \frac{d\phi_{-t}(Y) - Y}{t}$$

Said another way, we think of Y as a vector field along the integral curves of X. The flow ϕ_t gives us a way to identify the vector spaces at different points along an integral curve so that we can "differentiate" Y.

Proposition 1.18 (Derivative is bracket)**.** *For $X, Y \in \mathfrak{X}(M)$ we have $\mathfrak{L}_X(Y) = [X,Y]$.*

The proof is by calculation.

Example 1.19 (Jacobi identity). Thinking of Lie bracket as a Lie derivative gives us another way to think about the Jacobi identity. For vector fields X, Y, Z the Jacobi identity is equivalent to the statement that

$$\mathfrak{L}_X([Y,Z]) = [\mathfrak{L}_X(Y), Z] + [Y, \mathfrak{L}_X(Z)]$$

in other words, $\mathfrak{L}_X$ acts as a "derivation" with respect to the (Lie) algebra structure on $\mathfrak{X}(M)$. One way to see this is to differentiate the identity

$$d\phi_{-t}([Y,Z]) = [d\phi_{-t}(Y), d\phi_{-t}(Z)]$$

at $t = 0$ (this latter identity is just naturality of Lie bracket).

Example 1.20 (Coordinate vector fields). If $\partial_1, \cdots, \partial_n$ are the vector fields associated to local coordinates $x_1, \cdots, x_n$ then the flow ϕ_t associated to ∂_1 is characterized by the properties $x_i(\phi_t(x)) = x_i(x)$ for $i > 1$ and $x_1(\phi_t(x)) = x_1(x) + t$. In particular, the flows associated to the different ∂_i, ∂_j *commute*. Differentiating this fact recovers the identity $[\partial_i, \partial_j] = 0$.

In fact, if ϕ_t, ψ_s are the flows obtained by integrating vector fields X, Y then ϕ_t and ψ_s commute for all (small) t, s if and only if $[X,Y] = 0$. To see this, differentiate $\phi_t \psi_s = \psi_s \phi_t$ with respect to s and t. This explains the geometric "meaning" of the Lie bracket.

1.8. **Frobenius' Theorem.** How can we recognize families of vector fields $X_1, \cdots, X_n$ which are of the form $\partial_1, \cdots, \partial_n$ for some local coordinates? A necessary and sufficient condition is that the X_i span locally, and satisfy $[X_i, X_j] = 0$. For, in this case, the flows ϕ^i_t generated by the X_i commute, and define the desired local coordinates (unique up to constants).

More generally, if $X_1, \cdots, X_p$ for $p \le n$ are independent and satisfy $[X_i, X_j] = 0$ throughout U, then through any point in U we can find a smooth p-dimensional submanifold swept out by the orbits of the commuting flows ϕ^j_t, and these submanifolds decompose U locally into a product (i.e. they are the leaves of a *foliation*).

Frobenius' theorem gives necessary and sufficient conditions under which we can find such vector fields.

Theorem 1.21 (Frobenius)**.** *Let ξ be a p-dimensional sub-bundle of the tangent bundle TM over an open set U. Then ξ is tangent to the leaves of a foliation if and only if the sections of ξ are closed under Lie bracket (thought of as vector fields on M).*

Proof. One direction is easy. Locally, the leaves of a foliation are obtained by setting some subset $x_{p+1}, \cdots, x_n$ of a system of local coordinates to a constant; thus sections of ξ are spanned by ∂_i for $i \le p$, and are therefore closed under Lie bracket.

Conversely, choose p independent sections $X_1, \cdots X_p$ which span ξ locally. There is a sort of "Gram-Schmidt" process which replaces these sections with commuting ones, while staying linearly independent and living in ξ.

We have

$$[X_1, X_j] = \sum_{k=1}^{p} c^k_{1j} X_k$$

Fix a point $x \in U$. We would like to replace X_2 by $\hat{X}_2 := X_2 + \sum f_i X_i$ for suitable smooth functions f_i so that $[X_1, X_2] = 0$. If the f_i vanish at x, then $\hat{X}_2$ and the other X_i will still span near x. Now,

$$[X_1, X_2 + \sum f_i X_i] = \sum_k (c^k_{12} + X_1(f_k) + \sum_i f_i c^k_{1i}) X_k$$

so we would like to solve the system of first order linear ODEs

$$c^k_{12} + X_1(f_k) + \sum_i f_i c^k_{1i} = 0$$

for the functions f_i. There is a unique solution along each integral curve of X_1 if we specify the values of the f_i at a point. So choose a transversal to the integral curve of X_1 through x, and let the f_is be equal to zero on this transversal.

Doing this inductively, we obtain p commuting independent vector fields (near x) in ξ, whose associated flows sweep out the leaves of the desired foliation.

Alternate proof. Choose local coordinates $x_1, \cdots, x_n$ and express each X_i as $X_i := \sum X^j_i \partial_j$. Reorder coordinates if necessary so that the matrix $[X^j_i]_{i,j \le p}$ is nonsingular near x. Then if we define $Y_i := \bar{X}^i_j X_i$, where the matrix $[\bar{X}^i_j]_{i,j \le p}$ is the inverse of $[X^j_i]_{i,j \le p}$ pointwise, we have $Y_i = \partial_i + \sum_{j>p} Y^j_i \partial_j$. Then the Y_i are linearly independent, and $[Y_i, Y_j]$ is in the linear span of the ∂_k for $k > p$. On the other hand, it is in the linear span of the Y_l for $l \le p$, so it is identically zero. $\square$

1.9. **1-forms.** For each $x \in M$ denote the dual space $(T_xM)^*$ by T_x^*M. The collection of T_x^*M for various x are the fibers of a smooth vector bundle T^*M whose sections are called *1-forms*, and denoted $\Omega^1(M)$. In a local coordinate patch, $\Omega^1(U)$ is spanned freely (as a $C^\infty(U)$-module) by sections $dx_1, \cdots, dx_n$ defined at each $x \in U$ by the condition

$$dx_i|_x(\partial_j|_x) = \delta_{ij}$$

There is thus for every open $U \subset M$ a pairing of $C^\infty(U)$ modules

$$\Omega^1(U) \otimes_{C^\infty(U)} \mathfrak{X}(U) \to C^\infty(U)$$

whose restriction to each coordinate patch U is nondegenerate.

A smooth map $\varphi : M \to N$ *pulls back* 1-forms $\varphi^* : \Omega^1(N) \to \Omega^1(M)$ by the defining property

$$\varphi^*(\alpha)|_x(X|_x) = \alpha|_{\varphi(x)}(d\varphi_x(X|_x))$$

for each vector field X.

Now, for each smooth function f we can define a 1-form df to be the *unique* 1-form with the property that for all smooth vector fields X, we have

$$df(X) = X(f)$$

The pairing in each coordinate patch defines df there uniquely (just take X to be the ∂_i), and the definitions agree in the overlaps, so this is well-defined. In each local coordinate patch, we can compute

$$df = \sum_i \partial_i(f) dx_i$$

Notice that with this definition, the exterior derivatives of the coordinate functions $d(x_i)$ are precisely equal to the 1-forms dx_i, so our notation is consistent. Thus the 1-form dx_i can be defined *without* specifying the other coordinate functions (unlike the operators ∂_i).

Since the definition is natural (i.e. does not depend on a choice of coordinates) it respects pullback. That is, $d(\varphi^* f) = \varphi^* df$ for any smooth $\varphi : M \to N$.

Example 1.22. Recall that $\mathfrak{m}_x$ is the maximal ideal in $C^\infty(M)$ consisting of functions that vanish at x. Observe that $d : \mathfrak{m}_x/\mathfrak{m}_x^2 \to T_x^*M$ defined by $d(f) = df|_x$ is an isomorphism. The pairing of T_xM with $\mathfrak{m}_x/\mathfrak{m}_x^2$ defined in Proposition 1.11 agrees with the pairing of vector fields and 1-forms pointwise.

1.10. **Differential forms.** Let $\Lambda^j T^*M$ denote the vector bundle whose fiber over each point x is $\Lambda^j(T_x^*M)$ which we identify with a subgroup of the tensor algebra of T_x^*M as in § 1.1 and § 1.2, made into an algebra by exterior product. Denote by $\Omega^j(M)$ the smooth sections of $\Lambda^j T^*M$. Its elements are called *j-forms*. Denote $\oplus_j \Omega^j(M)$ by $\Omega^*(M)$. Its elements are *forms*. Exterior product fiberwise gives $\Omega^*(M)$ the structure of a graded (associative, skew-commutative) ring. If $\alpha \in \Omega^p$ and $\beta \in \Omega^q$ then $\alpha \wedge \beta \in \Omega^{p+q}$ and satisfies

$$\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha$$

Note that $\Lambda^0(T_x^*M) = \mathbb{R}$ canonically for each x, so that $\Omega^0(M) = C^\infty(M)$.

Differential forms pull back under smooth maps. By naturality, this pullback respects exterior product:

$$\varphi^*(\alpha) \wedge \varphi^*(\beta) = \varphi^*(\alpha \wedge \beta)$$

Example 1.23 (Volume forms). If M is n-dimensional, $\Lambda^n T^* M$ is a line bundle, which is trivial if and only if M is orientable. Nowhere zero sections of $\Lambda^n T^* M$ are called *volume forms*. In every local coordinate patch $x_1, \cdots, x_n$ a volume form can be expressed uniquely as $f dx_1 \wedge \cdots \wedge dx_n$ where f is nowhere zero.

If $y_1, \cdots, y_n$ are another system of local coordinates, we have

$$dy_1 \wedge \cdots \wedge dy_n = \det(\partial y_i / \partial x_j) dx_1 \wedge \cdots \wedge dx_n$$

The matrix $(\partial y_i / \partial x_j)$ is called the *Jacobian* of the coordinate change.

1.11. **Exterior derivative.** We already saw the existence of a natural differential operator

$$d : C^\infty(M) \to \Omega^1(M)$$

defined by $f \to df$. We would like to extend this operator to all of $\Omega^*(M)$ in such a way that it satisfies the Leibniz rule

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$$

for $\alpha \in \Omega^p(M)$ and $\beta \in \Omega^q(M)$.

It is tricky to give a coordinate-free definition of exterior d in general (although it is possible). If we choose local coordinates $x_1, \cdots, x_n$ every p-form α may be expressed locally in a unique way as a sum

$$\alpha = \sum_I f_I dx_{i_1} \wedge \cdots \wedge dx_{i_p}$$

where the sum is over multi-indices $I := i_1 < i_2 < \cdots < i_p$.

Definition 1.24 (Exterior derivative)**.** The exterior derivative of a p-form α is given in local coordinates by

$$d\alpha = \sum_I \sum_i \partial_i(f_I) dx_i \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p}$$

Proposition 1.25 (Properties of d)**.** *Exterior d satisfies $dd = 0$ and the Leibniz rule*

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$$

for $\alpha \in \Omega^p(M)$, and furthermore it is uniquely characterized by these two properties.

Proof. By definition,

$$d(d\alpha) = \sum_I \sum_i \sum_j \frac{\partial^2 f_I}{\partial x_i \partial x_j} dx_i \wedge dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_p}$$

But $dx_i \wedge dx_j = -dx_j \wedge dx_i$ for all pairs i, j so everything cancels. This shows $dd = 0$.

The Leibniz rule follows from the product rule for partial derivatives. Now apply associativity of wedge product to see that d is determined on all forms by its values on 1-forms. But $d(dx_i) = 0$ for all i and we are done. ∎

The Leibniz rule makes no reference to local coordinates, and nor does the property $dd = 0$. Thus the definition of exterior d given above is independent of local coordinates, and is well-defined on $\Omega^*(M)$.

The next proposition allows us to define d in a manifestly coordinate-free way by induction:

Proposition 1.26 (Inductive formula)**.** *For any $\alpha \in \Omega^p(M)$ and any $X_0, X_1, \cdots, X_p \in \mathfrak{X}(M)$ there is a formula*

$$d\alpha(X_0, \cdots, X_p) = \sum_i (-1)^i X_i(\alpha(X_0, \cdots, \hat{X}_i, \cdots, X_p)) + \sum_{i<j} (-1)^{i+j} \alpha([X_i, X_j], X_0, \cdots, \hat{X}_i, \cdots, \hat{X}_j, \cdots, X_p)$$

where the "hat" means omission.

Proposition 1.26 can be proved by induction, although we won't do it here. But note that in the special case that the $X_0, \cdots, X_p$ are commuting vector fields (e.g. if they are equal to some subset of the coordinate vector fields ∂_i) this is essentially equivalent to the formula in Definition 1.24.

Example 1.27. If α is a 1-form, and $X, Y \in \mathfrak{X}(M)$ then

$$d\alpha(X, Y) = X(\alpha(Y)) - Y(\alpha(X)) - \alpha([X, Y])$$

Example 1.28. If α is a 1-form, and $X, Y, Z \in \mathfrak{X}(M)$ then

$$\begin{aligned}
0 &= d(d\alpha)(X, Y, Z) \\
&= X(d\alpha(Y, Z)) - Y(d\alpha(X, Z)) + Z(d\alpha(X, Y)) - d\alpha([X, Y], Z) + d\alpha([X, Z], Y) - d\alpha([Y, Z], X) \\
&= X(Y(\alpha(Z))) - X(Z(\alpha(Y))) - X(\alpha([Y, Z])) - Y(X(\alpha(Z))) + Y(Z(\alpha(X))) + Y(\alpha([X, Z])) \\
&+ Z(X(\alpha(Y))) - Z(Y(\alpha(X))) - Z(\alpha([X, Y])) - [X, Y](\alpha(Z)) + Z(\alpha([X, Y])) + \alpha([[X, Y], Z]) \\
&+ [X, Z](\alpha(Y)) - Y(\alpha([X, Z]) - \alpha([[X, Z], Y]) - [Y, Z](\alpha(X)) + X(\alpha([Y, Z])) + \alpha([[Y, Z], X]) \\
&= \alpha([[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y])
\end{aligned}$$

Since α is arbitrary, we deduce that $dd = 0$ for 1-forms is *equivalent* to the Jacobi identity for vector fields.

1.12. **Interior product and Cartan's formula.**

If X is a vector field we define the *interior product*

$$\iota_X : \Omega^*(M) \to \Omega^{*-1}(M)$$

for each p by contaction of tensors. In other words, if $\omega \in \Omega^p$ then

$$(\iota_X \omega)(X_1, \cdots, X_{p-1}) = \omega(X, X_1, \cdots, X_{p-1})$$

for any $X_1, \cdots, X_{p-1} \in \mathfrak{X}(M)$. On 1-forms this reduces to $\iota_X(\alpha) = \alpha(X)$.

Proposition 1.29 (Properties of ι)**.** *Interior product satisfies $\iota_X \iota_Y \omega = -\iota_Y \iota_X \omega$ and the Leibniz rule*

$$\iota_X(\alpha \wedge \beta) = (\iota_X \alpha) \wedge \beta + (-1)^p \alpha \wedge (\iota_X \beta)$$

whenever $\alpha \in \Omega^p$.

The proofs are immediate.

Now, let X be a vector field, generating a flow ϕ_t of diffeomorphisms. If α is a p-form we may form the family of p-forms $\phi_t^*(\alpha)$.

Definition 1.30 (Lie Derivative of forms). The *Lie derivative* $\mathcal{L}_X : \Omega^*(M) \to \Omega^*(M)$ is defined by

$$\mathcal{L}_X(\alpha) := \lim_{t\to 0} \frac{\phi_t^*(\alpha) - \alpha}{t}$$

The Lie derivative of forms can be expressed in terms of exterior derivative and interior product by Cartan's "magic formula":

Proposition 1.31 (Cartan's Magic Formula). *For any $X \in \mathfrak{X}(M)$ and $\alpha \in \Omega^p(M)$ there is an identity*

$$\mathcal{L}_X(\alpha) = \iota_X(d\alpha) + d(\iota_X(\alpha))$$

In other words, as operators on forms,

$$\mathcal{L}_X = \iota_X d + d\iota_X$$

The proof is by calculation.

Example 1.32 (Lie derivative of functions). Since $\iota_X f = 0$ for a function f we have $\mathcal{L}_X f = \iota_X df = df(X) = X(f)$.

Example 1.33 (Leibniz formula for forms and vector fields). If $X_0, \cdots, X_p$ are vector fields and α is a p-form, there is a "Leibniz formula" for $\mathcal{L}_{X_0}$:

$$\mathcal{L}_{X_0}(\alpha(X_1, \cdots, X_p)) = (\mathcal{L}_{X_0}(\alpha))(X_1, \cdots, X_p) + \sum \alpha(X_0, \cdots, \mathcal{L}_{X_0}(X_i), \cdots, X_p)$$

To see this, we compute

$$\begin{aligned}
(\mathcal{L}_{X_0}(\alpha))(X_1, \cdots, X_p) &= d\alpha(X_0, \cdots, X_p) + d(\iota_{X_0}(\alpha))(X_1, \cdots, X_p) \\
&= \sum_i (-1)^i X_i(\alpha(\cdots \hat{X}_i \cdots)) + \sum_{i<j} (-1)^{i+j} \alpha([X_i, X_j], \cdots \hat{X}_i \cdots \hat{X}_j \cdots) \\
&+ \sum_{0<i} (-1)^{i-1} X_i(\alpha(\cdots \hat{X}_i \cdots)) + \sum_{0<i<j} (-1)^{i+j-2} \alpha(X_0, [X_i, X_j], \cdots \hat{X}_i \cdots \hat{X}_j \cdots) \\
&= X_0(\alpha(X_1, \cdots, X_p)) + \sum_j (-1)^j \alpha([X_0, X_j], X_1 \cdots \hat{X}_j \cdots)
\end{aligned}$$

In the special case $p = 1$ this reduces to

$$X(\alpha(Y)) = d\alpha(X, Y) + Y(\alpha(X)) + \alpha([X, Y])$$

1.13. **de Rham cohomology.** Since $dd = 0$ the groups $\Omega^*(M)$ form a complex. The cohomology of this complex is the *de Rham cohomology* of M. That is,

$$H^j_{dR}(M) := \{\alpha \in \Omega^j(M) \text{ with } d\alpha = 0\}/d\Omega^{j-1}(M)$$

A smooth map $\varphi : M \to N$ induces pullback $\varphi^* : \Omega^*(N) \to \Omega^*(M)$. Since d is natural, this is a chain map, and we get induced homomorphisms

$$\varphi^* : H^*_{dR}(N) \to H^*_{dR}(M)$$

Thus de Rham cohomology is an invariant of the diffeomorphism type of M.

de Rham cohomology actually forms a graded (skew-commutative) *ring*. If $[\alpha] \in H^p_{dR}$ and $[\beta] \in H^q_{dR}$ are represented by forms α, β with $d\alpha = d\beta = 0$ then $d(\alpha \wedge \beta) = 0$ by the Leibniz rule, so there is a class $[\alpha \wedge \beta] \in H^{p+q}_{dR}$.

Proposition 1.34. *The class of $[\alpha \wedge \beta]$ is well-defined, and thus there is an associative and skew-commutative multiplication on H^*_{dR}.*

Proof. If we replace α by $\alpha + d\gamma$ with $\gamma \in \Omega^{p-1}$ then

$$\begin{aligned}(\alpha + d\gamma) \wedge \beta &= \alpha \wedge \beta + d\gamma \wedge \beta \\ &= \alpha \wedge \beta + d(\gamma \wedge \beta)\end{aligned}$$

since $d\beta = 0$. □

A form α with $d\alpha = 0$ is *closed*. The forms $d\beta$ for some β are *exact*. Thus de Rham cohomology measures "closed forms modulo exact forms".

The de Rham Theorem says the following:

Theorem 1.35 (de Rham Theorem)**.** *There is a natural isomorphism*

$$H^*_{dR}(M) = H^*(M; \mathbb{R})$$

between de Rham cohomology and (ordinary) singular cohomology with real coefficients.

We will prove the de Rham Theorem in § 2.5.

2. Integration

2.1. **Integration in $\mathbb{R}^p$.** Let $K \subset U \subset \mathbb{R}^p$ where K is a compact polyhedron and U is open. Let $\alpha \in \Omega^p(U)$. Then there is a unique smooth function f_α on U so that $\alpha = f_\alpha dx_1 \wedge \cdots \wedge dx_p$.

Let μ denote the restriction of (p-dimensional) Lesbesgue measure on $\mathbb{R}^p$ to K, and define

$$\int_K \alpha := \int_K f_\alpha d\mu$$

Note that this is zero if $\mu(K) = 0$.

We can likewise define $\int_U \alpha = \int_U f_\alpha d\mu$ whenever f_α is in $L^1(U)$.

2.2. **Smooth singular chains.** For all p we identify the standard p-simplex Δ^p with the simplex in $\mathbb{R}^p$ with vertices at 0 and at the coordinate vectors e_i. By abuse of notation we write $e_0 = 0$ and refer to each e_i as the "ith vertex". As a subset of $\mathbb{R}^p$ it is determined by the inequalities $x_i \geq 0$ and $\sum x_i \leq 1$.

The ith face of Δ^p is the simplex of dimension $p-1$ spanned by all but the ith vertex of Δ^p. For each i there is a unique affine map $d_i : \Delta^{p-1} \to \Delta^p$ called the ith *face map* which takes the ordered vertices of Δ^{p-1} to the ordered vertices of the ith face of Δ^p.

A *smooth singular p-simplex* in M is a smooth map $\sigma : \Delta^p \to M$. Each smooth singular p-simplex σ determines $p+1$ smooth singular $(p-1)$-simplices by composition σd_i. The smooth singular p-simplices generate a free abelian group of *smooth singular p-chains* and these chain groups form a complex under ∂ defined by

$$\partial \sigma = \sum_{i=0}^{p} (-1)^i \sigma d_i$$

and the homology of this complex is the *smooth singular cohomology* $H^*(M;\mathbb{R})$ of M. It is isomorphic to ordinary singular cohomology (in which we do not insist that the maps σ are smooth). This is more easy to see for homology: cycles can be approximated by smooth cycles, and homologies between them can be approximated by smooth homologies. Then apply the universal coefficient theorem.

If σ is a smooth singular p-simplex, and $\alpha \in \Omega^p(M)$ then $\sigma^*(\alpha)$ is a p-form on Δ^p and we can define

$$\int_\sigma \alpha := \int_{\Delta^p} \sigma^*\alpha$$

2.3. **Stokes' Theorem.** The "simplest" version of Stokes' Theorem is the following:

Theorem 2.1 (Stokes' Theorem)**.** *Let $\sigma : \Delta^p \to M$ be a smooth singular p-simplex and let $\alpha \in \Omega^{p-1}(M)$ be a $(p-1)$-form. Then*

$$\int_{\partial\sigma} \alpha := \sum_{i=0}^{p} (-1)^i \int_{\sigma d_i} \alpha = \int_\sigma d\alpha$$

Proof. Let $\sigma^*\alpha = \sum f_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_p$ as a smooth $(p-1)$-form on $\Delta^p \subset \mathbb{R}^p$. Denote the individual terms by β_i. Note that $\sigma^*(d\alpha) = d\sigma^*\alpha$. Since d and $\int$ are linear, it suffices to prove the theorem when $\sigma^*\alpha = \beta_i$.

Evidently $d_j^*\beta_i = 0$ when $j > 0$ and $j \neq i$, and $d_i^*\beta_i = (f_i d_i) dx_1 \wedge \cdots \wedge dx_{p-1}$. Similarly, we calculate $d_0^*\beta_i = (-1)^i (f_i d_0) dx_1 \wedge \cdots \wedge dx_{p-1}$. We assume $\alpha = \beta_p$ for simplicity (the other cases are similar). We need to show

$$\int_{\Delta^p} \partial_p(f_p) dx_1 \wedge \cdots \wedge dx_p = \int_{\Delta^{p-1}} (f_p d_0) dx_1 \wedge \cdots \wedge dx_{p-1} - \int_{\Delta^{p-1}} f_p dx_1 \wedge \cdots \wedge dx_{p-1}$$

(we have cancelled factors of $(-1)^{p-1}$ that should appear on both sides). This reduces to the fundamental theorem of Calculus on each integral curve of ∂_p, plus Fubini's theorem. □

Stokes' Theorem says that the map from differential forms to smooth singular (real-valued) cochains is a *chain map*. It therefore induces a map on cohomology. Consequently there is a pairing $H^p_{dR}(M) \times H_p(M;\mathbb{R}) \to \mathbb{R}$ defined by taking a p-form α representing a de Rham cohomology class $[\alpha]$, and a smooth p-cycle $\sum t_i\sigma_i$ representing a homology class A, and defining

$$[\alpha](A) := \sum t_i \int_{\sigma_i} \alpha$$

By Stokes' Theorem and elementary homological algebra, this is independent of the choices involved.

Example 2.2. Let N be a compact smooth oriented p-manifold, possibly with boundary, let $\varphi : N \to M$ be a smooth map, and let α be a p-form on M. Then $\varphi^*\alpha$ is a p-form on N which is automatically closed, and we can pair it with the fundamental class of N in homology to obtain a number. We call this the result of integrating α over N, and denote it by $\int_\varphi \alpha$ or $\int_N \alpha$ if φ is understood. The most typical case will be that N is a smooth submanifold of M, and φ is inclusion.

Example 2.3. Let $\varphi : N^p \to M$ be as above, and let α be a $(p-1)$-form. Then α pulls back to a $(p-1)$-form on ∂N, and $d\alpha$ pulls back to a p-form on N, and Stokes' Theorem gives us

$$\int_{\partial N} \alpha = \int_N d\alpha$$

This is the more "usual" statement of Stokes' Theorem.

Stokes' Theorem can also be used (together with the Poincaré Lemma) to prove the de Rham Theorem. We will carry out the proof over the next few sections, but for now we explain the strategy. Choose a smooth triangulation of M and use this to identify the (ordinary) singular cohomology groups $H^*(M;\mathbb{R})$ with the *simplicial* cohomology groups $H^*_\Delta(M;\mathbb{R})$ associated to the triangulation. If we choose smooth characteristic maps for each simplex, we get natural maps $\Omega^*(M) \to C^*_\Delta(M;\mathbb{R})$ obtained by integrating p-forms over the smooth singular characteristic maps associated to the p-simplices of the triangulation.

Stokes' Theorem says this is a map of chain complexes, so there is an induced map on cohomology $H^*_{dR}(M) \to H^*_\Delta(M;\mathbb{R})$. It is easy to construct p-forms with compact support integrating to any desired value on a p-simplex, and forms defined locally can be smoothly extended throughout the manifold, so the map of chain complexes is surjective.

We need to show the induced map on cohomology is an isomorphism.

2.4. **Poincaré Lemma.** First we augment the complex Ω^* by $\epsilon : \mathbb{R} \to \Omega^0$ whose image is the constant functions. The comhomology of this augmented complex is the *reduced* de Rham cohomology, and denoted $\tilde{H}_{dR}(M)$.

Theorem 2.4 (Poincaré Lemma). $\tilde{H}_{dR}(\mathbb{R}^n) = 0$. *That is, any closed p-form on $\mathbb{R}^n$ is exact if $p > 0$ or a constant if $p = 0$.*

Proof. Define the *radial vector field* $X := \sum x_i \partial_i$. This defines a flow ϕ_t on $\mathbb{R}^n$ defined for all time. For $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}^n$ let $\lambda \cdot x \in \mathbb{R}^n$ denote the point whose coordinates are obtained from those of x by multiplying them by λ. Then $\phi_t(x) = e^t \cdot x$. In other words, ϕ_t is the dilation centered at 0 that scales everything by e^t.

Now, we have

$$\phi_t^*(f_I(x)dx_{i_1} \wedge \cdots \wedge dx_{i_p}) = e^{kt} f_I(e^t \cdot x) dx_{i_1} \wedge \cdots \wedge dx_{i_p}$$

for all multi-indices I with $|I| = k$. This converges (pointwise) to 0 as $t \to -\infty$ providing $k > 0$ (this makes sense: if we "zoom in" near 0 any smooth form of positive dimension pairs less and less with the "unit" vectors).

Now, for *any* vector field X with associated flow ϕ_t on any manifold, and for any form α,

$$\int_{t=a}^b \phi_t^*(\mathcal{L}_X(\alpha))dt = \int_{t=a}^b \phi_t^* \left(\lim_{s\to 0} \frac{\phi_s^*\alpha - \alpha}{s} \right) dt$$
$$= \lim_{s \to 0} \frac{1}{s} \left(\int_{t=a+s}^{b+s} \phi_t^* \alpha \, dt - \int_{t=a}^b \phi_t^* \alpha \, dt \right) = \phi_b^*\alpha - \phi_a^*\alpha$$

and therefore for the radial vector field on $\mathbb{R}^n$, we have $\int_{-\infty}^0 \phi_t^*(\mathcal{L}_X(\alpha))dt = \alpha$ for any p-form with $p > 0$, or $= \alpha - \alpha(0)$ if $p = 0$ (i.e. if α is a function).

Define the operator $I : \Omega^*(\mathbb{R}^n) \to \Omega^*(\mathbb{R}^n)$ by

$$I(\alpha) := \int_{-\infty}^{0} \phi_t^*(\alpha) dt$$

Notice that I commutes with d, by linearity of integration, since ϕ_t^* does for any t.

We have shown $I\mathcal{L}_X\alpha = \alpha$ for p-forms with $p > 0$, and $I\mathcal{L}_X f = f - f(0)$ on functions. But then if α is a *closed* p-form,

$$\alpha = I\mathcal{L}_X\alpha = I(\iota_X d\alpha + d\iota_X\alpha) = dI\iota_X\alpha$$

which exhibits α as an exact form (or a constant if $p = 0$). □

With Poincaré under our belt we can compute the (reduced) cohomology of several other spaces.

Proposition 2.5. *Let β be a closed q-form on $S^p \times \mathbb{R}^{n-p}$. If $p \neq q$ then β is exact. Otherwise β is exact (in reduced cohomology) if and only if $\int_{S^p \times 0} \beta = 0$.*

Proof. We let $U_\pm$ be open collar neighborhoods of the upper and lower hemispheres $D^p_\pm$ in S^p. Notice that each $U_\pm \times \mathbb{R}^{n-p}$ is diffeomorphic to $\mathbb{R}^n$. The restriction of β to each piece is closed, and therefore exact by the Poincaré Lemma. Thus there are $(q-1)$-forms $\alpha_\pm$ on the two pieces with $d\alpha_\pm = \beta$ where defined.

The intersection of the two pieces is diffeomorphic to $S^{p-1} \times \mathbb{R}^{n-p+1}$ and $\alpha_+ - \alpha_-$ is closed there.

If $p = q$, by Stokes' Theorem,

$$\int_{S^{p-1} \times 0} \alpha_+ - \alpha_- = \int_{D^p_+ \cup D^p_- \times 0} \beta = 0$$

Therefore by induction $\alpha_+ - \alpha_-$ is exact. If $p \neq q$ then $\alpha_+ - \alpha_-$ is exact by induction unconditionally.

If $(q-1) = 0$ this means $\alpha_+ - \alpha_-$ is constant, so we can adjust α_- by a constant on one piece to get a new function with $d\alpha_\pm = \beta$ and $\alpha_+ = \alpha_-$ on the overlaps. These glue together to give α with $d\alpha = \beta$ as desired.

If $(q-1) > 0$ then $\alpha_+ - \alpha_- = d\gamma$ for some $(q-2)$-form γ. We extend γ smoothly over one of the pieces and substitute $\alpha_- \to \alpha_- + d\gamma$. Then $d\alpha_\pm = \beta$ still, and $\alpha_+ = \alpha_-$ on the overlaps, so we can glue together to get α with $d\alpha = \beta$ everywhere. □

2.5. Proof of de Rham's Theorem.

We now prove de Rham's Theorem. We have chosen a smooth triangulation τ, and defined a map $\int : \Omega^*(M) \to C^*_\Delta(M;\mathbb{R})$ by integration. Stokes' Theorem implies that this is a chain map, so that we have $H^*_{dR}(M) \to H^*_\Delta(M;\mathbb{R})$.

Lemma 2.6. *The map $H^*_{dR}(M) \to H^*_\Delta(M;\mathbb{R})$ is injective.*

Proof. Let α be a closed p-form whose image in $H^*_\Delta(M;\mathbb{R})$ is trivial. Since the map on chain complexes is surjective, we can adjust α by an exact p-form so that it maps to the 0-cochain; i.e. its integral over each p-simplex in the triangulation is zero. We now show, by induction on the skeleton, that we can adjust α by an exact form to make it vanish identically.

Suppose α vanishes in a neighborhood of the i-skeleton of the triangulation. Choose an $(i+1)$-simplex, and let $\varphi : \mathbb{R}^n \to M$ map diffeomorphically onto a thickened neighborhood of the interior of the simplex. We pull back α to $\varphi^*\alpha$ which is supported in $D^{i+1} \times \mathbb{R}^{n-i-1}$. The pullback is closed, and therefore exact on $\mathbb{R}^n$ by the Poincaré Lemma, and is therefore equal to $d\beta$ for some $(p-1)$-form β. Notice that $\mathbb{R}^n - (D^{i+1} \times \mathbb{R}^{n-i-1})$ is diffeomorphic to $S^i \times \mathbb{R}^{n-i}$ and $d\beta = \varphi^*\alpha = 0$ there, so that β is closed there.

If $p = (i+1)$ then our hypothesis on α gives $\int_{S^i \times 0} \beta = 0$ by Stokes' Theorem, so whether or not $p = (i+1)$, Proposition 2.5 says that $\beta = d\gamma$ for some $(p-2)$-form γ on $\mathbb{R}^n - (D^{i+1} \times \mathbb{R}^{n-i-1})$. Extend γ smoothly throughout $\mathbb{R}^n$ and substitute $\beta \to \beta - d\gamma$. Then $d\beta = \alpha$, and β vanishes where α does. Now multiply β by a bump function ϕ equal to 1 in a sufficiently big subset of $\mathbb{R}^n$, and substitute $\alpha \to \alpha - d(\phi\beta)$. The new α is smooth, cohomologous to the old, and vanishes in a neighborhood of the $(i+1)$-skeleton.

This shows that the map $H^*_{dR}(M) \to H^*_\Delta(M; \mathbb{R})$ is injective. □

Lemma 2.7. *The map $H^*_{dR}(M) \to H^*_\Delta(M; \mathbb{R})$ is surjective.*

Proof. This is proved by a direct and local construction, but the details are fiddly.

We claim that we can construct a map $\alpha : C^*_\Delta \to \Omega^*$ satisfying

(1) α is a chain map; i.e. $d(\alpha(\phi)) = \alpha(\delta\phi)$ for all simplicial cochains ϕ; and
(2) α inverts $\int$; i.e. $\int_\sigma \alpha(\phi) = \phi(\sigma)$ for all simplices σ of τ.

This will prove the lemma. For, if ϕ is a p-cocycle, then $\alpha(\phi)$ is a closed p-form, and the cohomology class $[\alpha(\phi)]$ maps to the cohomology class $[\phi]$. It remains to construct α.

Assume for the moment that M is compact, so that the triangulation makes it into a finite simplicial complex. If we label the vertices from 0 to n then we can identify these vertices with the coordinate vertices $e_0 := (1, 0, \cdots, 0)$ through $e_n := (0, \cdots, 0, 1)$ in $\mathbb{R}^{n+1}$ with coordinates $x_0, \cdots, x_n$, and we can identify each p-simplex of τ with vertices $i_0, \cdots, i_p$ with the simplex $0 \le x_{i_j} \le 1$, $\sum_{j=0}^p x_{i_j} = 1$ in the subspace spanned by the x_{i_j}. Call $K \subset \mathbb{R}^{n+1}$ the union of these simplices; thus K is (in a natural way) a polyhedron on $\mathbb{R}^{n+1}$.

There is a *smooth homeomorphism* $\varphi : M \to K$ which takes each simplex of τ to the corresponding simplex of K. This might seem strange, but it is not hard to construct skeleton by skeleton: just make sure that the derivatives of φ vanish to all orders at the "corners" so that the map is smooth there.

For each p-simplex σ of τ, we let $\varphi(\sigma)$ denote the corresponding simplex of K. Let ϕ_σ denote the p-cochain taking the value 1 on σ and 0 everywhere else. We will define a form $\beta(\sigma)$ on $\mathbb{R}^{n+1}$ and then define $\alpha(\phi_\sigma) = \varphi^*\beta(\sigma)$ on M.

Suppose $\varphi(\sigma)$ is the simplex spanned by $e_0, \cdots, e_p$ for convenience. In our previous notation, $\varphi(\sigma) = d_0(\Delta^{p+1})$, the 0th face of the "standard" $(p+1)$-simplex. Then we define

$$\beta(\sigma) = p! \sum (-1)^i x_i dx_0 \wedge \cdots \widehat{dx_i} \cdots \wedge dx_p$$

as a form on all of $\mathbb{R}^{n+1}$.

Claim. *α is a chain map.*

Proof. It suffices to check on $\alpha(\phi_\sigma)$. We calculate

$$d\beta(\sigma) = (p+1)!dx_0 \wedge \cdots \wedge dx_p$$

i.e. it is equal to $(p+1)!$ times the pullback of the volume form on $\mathbb{R}^{p+1}$ under the map $\mathbb{R}^{n+1} \to \mathbb{R}^{p+1}$ which projects out the other coordinates. We need to check that $d\beta(\sigma) = \sum_{\sigma'} \pm\beta(\sigma')$ on K, where the sum is taken over $(p+1)$-simplices σ' with σ as a face, and the sign comes from the difference between the orientations of σ and $\partial\sigma'$. To check this, we check it on each $(p+q)$ simplex.

Let σ'' be a $(p+q)$-simplex of τ. If σ is not a face of σ'' then $d\beta(\sigma)$ and $\sum_{\sigma'} \pm\beta(\sigma')$ are both identically zero on $\varphi(\sigma'')$. Likewise, even if σ is a face of σ'', then $\beta(\sigma')$ is zero on $\varphi(\sigma'')$ unless σ' is a face of σ''.

So let σ'' be a $(p+q)$-simplex with σ as a face. Without loss of generality, we can suppose $\varphi(\sigma'') = d_0(\Delta^{p+q+1})$; i.e. it is the simplex where $\sum_{i=0}^{p+q} x_i = 1$. Thus $\sum_{j=1}^{q} x_{p+j} = 1 - \sum_{i\le p} x_i$ and $\sum_{j=1}^{q} dx_{p+j} = -\sum_{i \le p} dx_i$ on $\varphi(\sigma'')$. But then

$$\begin{aligned}
d\beta(\sigma) &= (p+1)!\Big(\sum_{j=1}^{q} x_{p+j}dx_0 \wedge \cdots \wedge dx_p + \sum_{i\le p}(-1)^i x_i dx_i \wedge dx_0 \wedge \cdots \widehat{dx_i} \cdots \wedge dx_p\Big) \\
&= (p+1)!\Big(\sum_{j=1}^{q} x_{p+j}dx_0 \wedge \cdots \wedge dx_p + \sum_{j=1}^{q}\sum_{i\le p}(-1)^{i+1} x_i dx_{p+j} \wedge dx_0 \wedge \cdots \widehat{dx_i} \cdots \wedge dx_p\Big) \\
&= \sum_{\sigma'} \pm\beta(\sigma')
\end{aligned}$$

on the simplex $\varphi(\sigma'')$.

It follows by linearity that $\alpha(\delta\phi) = d\alpha(\phi)$ for all cochains ϕ; i.e. that α is a chain map. $\square$

Claim. *α inverts $\int$.*

Proof. It suffices to check on $\alpha(\phi_\sigma)$. Evidently $\beta(\sigma)$ is zero on every p-simplex of K except $\varphi(\sigma)$. So it suffices to show $\int_{\varphi(\sigma)} \beta(\sigma) = 1$. If we identify $\varphi(\sigma) = d_0(\Delta^{p+1})$ then observe that $\beta(\sigma)$ vanishes on all the other faces of Δ^{p+1}. So by Stokes' Theorem,

$$\int_{\varphi(\sigma)} \beta(\sigma) = \int_{\partial\Delta^{p+1}} \beta(\sigma) = \int_{\Delta^{p+1}} (p+1)!dx_0 \wedge \cdots \wedge dx_p = 1$$

$\square$

This completes the proof of the lemma when M is compact. But actually, we did not use anywhere the compactness of M. If M is noncompact so that K is infinite and lives in $\mathbb{R}^\infty$ it is nevertheless true that every construction or calculation above takes place in a finite dimensional subspace, and makes perfect sense there. So we are done in general. $\square$

This completes the proof of the de Rham Theorem.

3. Chern classes

3.1. **Connections.** Let $\pi : E \to M$ be a smooth n-dimensional vector bundle over M with fiber E_x over x (E could be real or complex). Denote smooth sections of E by $\Gamma(E)$. This is a $C^\infty(M)$-module (real or complex).

There is no canonical way to identify the fibers of E over different points. A *connection* on E is a *choice* of such an identification, at least "infinitesimally".

Definition 3.1. Let E be a smooth real vector bundle over M. A *connection* ∇ is a linear map (*not* a $C^\infty(M)$-module homomorphism)

$$\nabla : \Gamma(E) \to \Omega^1(M) \otimes \Gamma(E)$$

satisfying the Leibniz rule

$$\nabla(fs) = df \otimes s + f\nabla(s)$$

for $f \in C^\infty(M)$ and $s \in \Gamma(E)$.

Such an operator ∇ is also (more usually) called a *covariant derivative*. We denote $\nabla(s)(X) = \nabla_X(s)$ for $X \in \mathfrak{X}(M)$. Note that $\nabla_{fX}(s) = f\nabla_X s$ for a smooth function f.

Example 3.2. The trivial bundle $E = M \times \mathbb{R}^n$ admits the "trivial connection" $\nabla(s) = ds$ where we identify sections of E with n-tuples of smooth functions by using the trivialization.

We sometimes use the notation $\Omega^p(M; E) := \Omega^p(M) \otimes \Gamma(E)$ i.e. for the space of sections of $\Lambda^p T^*M \otimes E$. We can extend ∇ to operators

$$\nabla : \Omega^p(M; E) \to \Omega^{p+1}(M; E)$$

by

$$\nabla(\alpha \otimes s) = d\alpha \otimes s + (-1)^p \alpha \wedge \nabla(s)$$

It is *not* typically true that $\nabla^2 = 0$.

Proposition 3.3. *Any smooth real vector bundle admits a connection. The space of connections on E is an* affine space *for* $\Omega^1(M; \mathrm{End}(E))$.

Proof. The first claim follows from the second by using partitions of unity to take convex combinations of connections defined locally e.g. from "trivial" connections with respect to local trivializations of the bundle.

To prove the second claim, let $\nabla, \tilde{\nabla}$ be two connections. Then

$$(\nabla - \tilde{\nabla})(fs) = f(\nabla - \tilde{\nabla})(s)$$

which shows that $(\nabla - \tilde{\nabla})$ is a $C^\infty(M)$-module homomorphism from $\Gamma(E)$ to $\Omega^1(M) \otimes \Gamma(E)$; i.e. an element of $\Omega^1(M; \mathrm{End}(E))$. □

3.2. **Curvature and parallel transport.**

Definition 3.4. A section s is *parallel* along a path $\gamma : [0,1] \to M$ if $\nabla_{\gamma'(t)}(s) = 0$ throughout $[0,1]$.

By the fundamental theorem of ODEs there is a unique parallel section over any path with a prescribed initial value. Thus a path $\gamma : [0,1] \to M$ determines an isomorphism $E_{\gamma(0)} \to E_{\gamma(1)}$ called the result of *parallel transport along* γ.

By abuse of notation we can think of the total space of E as a smooth manifold. A connection gives a unique way to lift a path in M to a parallel path in E. Taking derivatives, we get a map $\mathfrak{X}(M) \to \mathfrak{X}(E)$ which we call "tilde" and denote $X \to \tilde{X}$. The image consists of *parallel vector fields*.

The parallel vector fields span pointwise a distribution of n-plane fields ξ on E called the *horizontal distribution*.

For X a vector field on M, the lift $\tilde{X}$ generates a flow of bundle automorphisms $\tilde{\phi}_t$ on E lifting the flow of diffeomorphisms ϕ_t of M generated by X. We can define

$$\mathcal{L}_{\tilde{X}} s = \lim_{t \to 0} \frac{\tilde{\phi}_{-t}(s) - s}{t}$$

The following is immediate from the definitions:

Proposition 3.5. *With notation as above* $\mathcal{L}_{\tilde{X}} s = \nabla_X s$.

If $X, Y \in \mathfrak{X}(M)$ then we can form $R(X,Y) := [\tilde{X}, \tilde{Y}] - \widetilde{[X,Y]} \in \mathfrak{X}(E)$. This is a *vertical* vector field (i.e. it is tangent to the fibers of E).

If V is a vector space, there is a canonical identification $T_v V = V$ at every $v \in V$. Thus a vector field on V is the same thing as a smooth map $V \to V$. Since the parallel vector fields integrate to flows by bundle automorphisms, R respects the vector space structure in each fiber. So $R(X,Y)$ determines a *linear* endomorphism of each fiber E_x; i.e. $R(X,Y)|_{E_x} \in \mathrm{End}(E_x)$ so that $R(X,Y) \in \Gamma(\mathrm{End}(E))$.

Proposition 3.6.

$$R(X,Y)(s) = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s$$

for any $s \in \Gamma(E)$.

This is a calculation.

Definition 3.7. R is called the *curvature* of the connection. A connection is *flat* if R is identically zero.

If ∇ is flat, the parallel vector fields are closed under Lie bracket, and the horizontal distribution integrates to a foliation (by Frobenius' Theorem) whose leaves are everywhere parallel.

Proposition 3.8. $R \in \Omega^2(M; \mathrm{End}(E))$.

Proof. The content of this proposition is that R is $C^\infty(M)$-linear in all three entries. We check

$$\begin{aligned} R(fX,Y)(s) &= f\nabla_X \nabla_Y s - \nabla_Y f \nabla_X s - \nabla_{[fX,Y]} s \\ &= f\nabla_X \nabla_Y s - Y(f)\nabla_X s - f\nabla_Y \nabla_X s - f\nabla_{[X,Y]} s + Y(f)\nabla_X s \\ &= fR(X,Y)(s) \end{aligned}$$

Since R is antisymmetric in X and Y, we get $C^\infty(M)$-linearity for Y too. Finally,

$$\begin{aligned} R(X,Y)(fs) &= f\nabla_X \nabla_Y s + X(f)\nabla_Y s + Y(f)\nabla_X s + X(Y(f))s \\ &\quad - f\nabla_Y \nabla_X s - Y(f)\nabla_X s - X(f)\nabla_Y s - Y(X(f))s \\ &\quad - [X,Y](f)s - f\nabla_{[X,Y]} = fR(X,Y)(s) \end{aligned}$$

and we are done. $\square$

3.3. **Flat connections.**

Proposition 3.9. *There is an identity* $(\nabla(\nabla(s)))(X,Y) = R(X,Y)(s)$. *Thus* $\Omega^*(M;E)$ *is a complex with respect to* ∇ *if and only if* ∇ *is flat.*

Proof. By $C^\infty(M)$-linearity to prove the first claim it suffices to check this identity on coordinate vector fields ∂_i, ∂_j, where it is immediate.

In general we can compute

$$\nabla(\nabla(\alpha \otimes s)) = \alpha \otimes \nabla(\nabla(s))$$

so $\nabla^2 = 0$ on $\Omega^*(M;E)$ if and only if it is zero on $\Omega^0(M;E) = \Gamma(E)$, and this is exactly the condition that $R = 0$. □

If ∇ is flat, parallel transport is *homotopy invariant*. That is, if $\gamma_1, \gamma_2 : [0,1] \to M$ are homotopic rel. endpoints, parallel transport along γ_1 and along γ_2 define the *same* isomorphism from $E_{\gamma_i(0)}$ to $E_{\gamma_i(1)}$. For, a homotopy between them lifts to a parallel homotopy of sections in the integral manifold of the horizontal distribution.

Thus a flat connection determines a *holonomy representation* $\rho : \pi_1(M,x) \to \mathrm{Aut}(E_x)$. Since ∇ is flat, the groups $\Omega(M;E)$ form a complex with respect to ∇, and we denote the cohomology of this complex by $H^*_{dR}(M;E)$.

The analog of the de Rham Theorem in this context is an isomorphism

$$H^*_{dR}(M;E) = H^*(M;\rho)$$

where the right hand side denotes (singular) cohomology with coefficients in the local system determined by ρ.

3.4. **Invariant polynomials and Chern classes.**

Now let E be a real or complex n-dimensional bundle over M.

If we trivialize E locally over $U \subset M$ as $U \times \mathbb{R}^n$ or $U \times \mathbb{C}^n$ we can identify $\mathrm{End}(E)$ with a bundle of $n \times n$ matrices. Thus, locally any $\alpha \in \Omega^p(M;\mathrm{End}(E))$ can be expressed as a *matrix of p-forms.*

Example 3.10 (Change of trivialization). Let ∇ be a connection. If we trivialize E locally, we can express the covariant derivative (relative to this trivialization) as

$$\nabla(s) = ds + \omega \otimes s$$

for some $\omega \in \Omega^1(M;\mathrm{End}(E))$ which we think of as a matrix of 1-forms. If we change the local trivialization by a section $h \in \Gamma(\mathrm{Aut}(E))$ then

$$\nabla(hs) = dh \otimes s + hds + h\omega \otimes s$$

so ω transforms by $\omega \to dh \cdot h^{-1} + h\omega h^{-1}$.

"Matrix multiplication" defines a product on $\Omega^*(M;\mathrm{End}(E))$:

Definition 3.11 (Wedge product)**.** There is a product on $\Omega^*(M;\mathrm{End}(E))$ that we denote by $\wedge$, which is wedge product on forms, and fiberwise composition of endomorphisms. On decomposable vectors

$$(\alpha \otimes A) \wedge (\beta \otimes B) = (\alpha \wedge \beta) \otimes AB$$

where $\alpha, \beta \in \Omega^*(M)$ and $A, B \in \Gamma(\mathrm{End}(E))$.

Definition 3.12 (Lie bracket)**.** For any vector space V, we can make $\mathrm{End}(V)$ into a Lie algebra by the bracket $[A, B] := AB - BA$. Doing this fiberwise defines a bracket on $\Gamma(\mathrm{End}(E))$, which extends to $\Omega^*(M; \mathrm{End}(E))$ as follows: if $\alpha \in \Omega^p(M; \mathrm{End}(E))$ and $\beta \in \Omega^q(M; \mathrm{End}(E))$ then

$$[\alpha, \beta] = \alpha \wedge \beta - (-1)^{pq} \beta \wedge \alpha$$

Example 3.13. Let $\alpha \in \Omega^1(M; \mathrm{End}(E))$. Then

$$\alpha \wedge \alpha = \frac{1}{2}[\alpha, \alpha]$$

Now, let ∇ be a connection. Trivialize E locally, and express the covariant derivative (relative to this trivialization) as

$$\nabla(s) = ds + \omega \otimes s$$

for some $\omega \in \Omega^1(M; \mathrm{End}(E))$. We compute

$$\nabla(\nabla(s)) = \nabla(ds + \omega \otimes s) = \omega \wedge ds + d\omega \otimes s - \omega \wedge ds - \omega \wedge \omega \otimes s$$

so that

$$R = d\omega - \omega \wedge \omega = d\omega - \frac{1}{2}[\omega, \omega]$$

as an honest identity in $\Omega^2(M; \mathrm{End}(E))$.

Example 3.14 (Bianchi Identity). With notation as above,

$$dR = [\omega, R]$$

For,

$$dR = -d\omega \wedge \omega + \omega \wedge d\omega = [\omega, d\omega] = [\omega, R]$$

because $[\omega, \omega \wedge \omega] = 0$.

Definition 3.15 (Invariant polynomial)**.** Let V be a vector space (real or complex). An *invariant polynomial of degree* p is a function P from $\mathrm{End}(V)$ to the scalars, satisfying

1. **(degree p):** $P(\lambda A) = \lambda^p P(A)$ for all scalars λ;
2. **(invariance):** $P(gAg^{-1}) = P(A)$ for all $g \in \mathrm{Aut}(V)$.

If E is a bundle over M whose fibers are isomorphic to V, then any invariant polynomial P defines a map which we likewise denote P:

$$P : \Omega^q(M; \mathrm{End}(E)) \to \Omega^{pq}(M)$$

by applying P fiberwise in any local trivialization. Invariance means that the result does not depend on the choice.

Definition 3.16 (Polarization)**.** Suppose P is an invariant polynomial on $\mathrm{End}(V)$ of degree p. The *polarization* of P is a multilinear map from $\mathrm{End}(V)^p$ to the scalars (which, by abuse of notation, we also denote by P) which satisfies

1. **(specialization):** $P(A, \cdots, A) = P(A)$;
2. **(symmetry):** P is invariant under permutations of the entries; and
3. **(invariance):** $P(gA_1g^{-1}, \cdots, gA_pg^{-1}) = P(A_1, \cdots, A_p)$ for all $g \in \mathrm{Aut}(V)$ and $A_i \in \mathrm{End}(V)$.

If P is an invariant polynomial, its polarization can be defined by setting $P(A_1, \cdots, A_p)$ equal to the coefficient of $t_1 t_2 \cdots t_p$ in $P(t_1 A_1 + t_2 A_2 + \cdots + t_p A_p)/p!$

Differentiating the invariance property shows that

$$\sum_i P(A_1, \cdots, [B, A_i] \cdots, A_p) = 0$$

for any invariant polarization, whenever $B, A_i \in \mathrm{End}(V)$.

Proposition 3.17. *If $\alpha_i \in \Omega^{m_i}(M; \mathrm{End}(E))$, and $\beta \in \Omega^1(M; \mathrm{End}(E))$ then*

$$\sum_i (-1)^{m_1 + \cdots + m_{i-1}} P(\alpha_1, \cdots, [\beta, \alpha_i], \cdots, \alpha_p) = 0$$

Proof. This is just the invariance property, and follows by differentiation (the sign comes from moving the 1-form β past each m_j-form α_j.) □

We explain how to apply invariant polynomials to curvature forms to obtain invariants of bundles.

Proposition 3.18. *Let E be a real or complex vector bundle with fibers isomorphic to V, and let P be an invariant polynomial of degree p on $\mathrm{End}(V)$. Pick a connection ∇ on E with curvature $R \in \Omega^2(M; \mathrm{End}(E))$. Then $P(R) \in \Omega^{2p}(M)$ is closed, and its cohomology class does not depend on the choice of connection.*

Proof. Let ∇_0, ∇_1 be two connections. Their difference $\nabla_1 - \nabla_0 = \alpha$ is in $\Omega^1(M; \mathrm{End}(E))$. Define a family of connections $\nabla_t := \nabla_0 + t\alpha$. Locally, in terms of a trivialization, we can write $\nabla_t = d + \omega_t$ where $\omega_t := \omega_0 + t\alpha$ for some $\omega_0 \in \Omega^1(M; \mathrm{End}(E))$ (depending on the trivialization), and then

$$R_t = d\omega_t - \frac{1}{2}[\omega_t, \omega_t] = R_0 + t(d\alpha - [\omega_0, \alpha]) - \frac{1}{2}t^2[\alpha, \alpha]$$

Thus

$$\frac{1}{p}\frac{d}{dt}P(R_t) = P(d\alpha - [\omega_t, \alpha], R_t, \cdots, R_t)$$

Now, the Bianchi identity gives

$$dP(\alpha, R_t, \cdots, R_t) = P(d\alpha, R_t, \cdots, R_t) - (p-1)P(\alpha, [\omega_t, R_t], R_t, \cdots, R_t)$$

but invariance of P gives (by Proposition 3.17)

$$P([\omega_t, \alpha], R_t, \cdots, R_t) - (p-1)P(\alpha, [\omega_t, R_t], R_t, \cdots, R_t) = 0$$

So $d/dtP(R_t)$ is exact, and therefore (by integration) so is $P(R_1) - P(R_0)$.

Locally we can always choose a connection whose curvature vanishes pointwise. So any $P(R)$ is locally exact, which is to say it is closed. □

We can now define Chern forms of connections on complex vector bundles:

Definition 3.19 (Chern classes)**.** Let E be a complex vector bundle, and choose a connection. The *Chern forms* of the connection $c_j \in \Omega^{2j}(M; \mathbb{C})$ are the coefficients of the "characteristic polynomial" of $R/2\pi i$. That is,

$$\det\left(\mathrm{Id} - t\frac{R}{2\pi i}\right) = \sum c_i t^i$$

Likewise we can define Pontriagin forms of connections on real vector bundles:

Definition 3.20 (Pontriagin classes)**.** Let E be a real vector bundle, and choose a connection. The *Pontriagin forms* of the connection $p_j \in \Omega^{4j}(M;\mathbb{R})$ are the coefficients of the "characteristic polynomial" of $-R/2\pi$. That is,

$$\det\left(\mathrm{Id} + t\frac{R}{2\pi}\right) = \sum p_j t^{2j}$$

By Proposition 3.18 the forms c_j and p_j are closed, and give rise to well-defined cohomology classes which are invariants of the underlying bundles.

Theorem 3.21. *With notation as above, $[c_j]$ and $[p_j]$ are the usual Chern and Pontriagin classes, and therefore lie in $H^{2j}(M;\mathbb{Z})$ and $H^{4j}(M;\mathbb{Z})$ respectively.*

Proof. If E is a real vector bundle, $p_j(E) = (-1)^j c_{2j}(E_{\mathbb{C}})$ so it suffices to prove this theorem for the Chern classes. This can be done axiomatically.

Connections can be pulled back along with bundles, so the classes as defined above are certainly natural. If ∇_1, ∇_2 are connections on bundles E_1, E_2 then $\nabla_1 \oplus \nabla_2$ is a connection on $E_1 \oplus E_2$ with curvature $R_1 \oplus R_2$. Thus the Whitney product formula follows.

This shows that the $[c_j]$ as defined above agree with the usual Chern classes up to a multiplicative constant. We compute on an example (this is by far the hardest part of the proof!) For $\mathbb{CP}^1$ we should have $c_1(T\mathbb{CP}^1) = \chi(\mathbb{CP}^1) = 2$ for the right normalization, so to prove the theorem it suffices to check that

$$\int_{\mathbb{CP}^1} \frac{-R}{2\pi i} = 2$$

where R is the curvature of a connection on the tangent bundle (which is an honest 2-form, since $\mathrm{End}(E)$ is the trivial line bundle for any complex line bundle E).

We cover $\mathbb{CP}^1$ with two coordinate charts z and w, and have $w = z^{-1}$ on the overlap. Any vector field is of the form $f(z)\partial_z$ (on the z-coordinate chart).

The vector fields ∂_z and ∂_w give trivializations of the bundle on the two charts. We define a connection form as follows. Where z is finite, we can express the connection as

$$\nabla\partial_z = \alpha \otimes \partial_z$$

We choose the connection

$$\alpha := \frac{-2\bar{z}\,dz}{|z|^2+1}$$

Where w is finite, we can express the connection as

$$\nabla\partial_w = \beta \otimes \partial_w$$

Having the connections agree on the overlap will determine β, and we claim that β (defined implicitly as above) extends smoothly over $w = 0$.

On the overlap we have $\partial_w = -z^2\partial_z$, so by the formula in Example 3.10 it follows that

$$\beta = \alpha + \frac{d(-z^2)}{-z^2} = \left(\frac{-2\bar{z}}{|z|^2+1} + \frac{2}{z}\right)dz = \left(\frac{2\bar{w}^{-1}}{|w|^{-2}+1} - 2w\right)\frac{dw}{w^2} = \frac{-2\bar{w}\,dw}{|w|^2+1}$$

which extends smoothly over 0 (and explains the choice of α).

Then

$$R = d\alpha = \partial_{\bar{z}}(\alpha) d\bar{z} \wedge dz = \frac{-4i}{(x^2+y^2+1)^2} dx \wedge dy$$

where we have used $z = x + iy$ so that $dz = dx + idy$ and $d\bar{z} = dx - idy$. Finally, we compute

$$\int_{\mathbb{CP}^1} \frac{-R}{2\pi i} = \frac{4}{2\pi} \int_{\theta=0}^{2\pi} \int_0^\infty \frac{r}{(1+r^2)^2} dr d\theta = -\frac{4}{2(1+r^2)}\Big|_0^\infty = 2$$

□

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