Math151c, Spring 2008

1. (a) Show that S^4 admits infinitely many non-isomorphic $\mathbb{R}^3$ bundles, and classify them in terms of homotopy theoretic data.

   (b) Let T denote the tangent bundle to S^4. For each i let Λ^i denote the bundle over S^4 whose fiber over a point $p \in S^4$ is $\Lambda^i(T_p)$ (i.e. the antisymmetric part of $\otimes^i T_p$). Show that a Euclidean structure on T induces a Euclidean structure on Λ^i for all i, and that with respect to this Euclidean structure, there is an isomorphism (of Euclidean bundles) $\Lambda^4 \cong \epsilon^1$.

   (c) Define an isomorphism $* : \Lambda^i_p \to \Lambda^{4-i}_p$ for each point p by the formula

$$\alpha \wedge (*\beta) = \langle \alpha, \beta \rangle \mathbf{1}$$

   for any $\alpha, \beta \in \Lambda^i_p$, where $\wedge$ is the usual wedge product of alternating forms, $\langle \cdot, \cdot \rangle$ is the Euclidean inner product on the vector space Λ^i_p, and $\mathbf{1}$ is the vector in Λ^4_p corresponding to $1 \in \epsilon^1_p = \mathbb{R}$ under the isomorphism $\Lambda^4 \cong \epsilon^1$. Show that $* : \Lambda^2 \to \Lambda^2$ is a bundle map whose square is the identity, and let Λ^+, Λ^- be the bundles whose fibers are the ± 1 eigenspaces of $*$ in each Λ^2_p. Show that Λ^+ is a nontrivial $\mathbb{R}^3$ bundle over S^4, and work out which bundle it is, in terms of the classification in part (a).