**HOMEWORK 6 — HYPERBOLIC STRUCTURES ON 3–MANIFOLDS**

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*Problem* 1. The *Baumslag–Solitar* group $B(p,q)$ is defined to be the group with two generators $\alpha, \beta$ and the relation that $\beta$ conjugates the $p$th power of $\alpha$ to the $q$th power of $\alpha$. That is,

$$B(p,q) = \langle \alpha, \beta \mid \beta\alpha^p\beta^{-1} = \alpha^q \rangle$$

If $p,q$ are coprime integers $\geq 2$, show that $B(p,q)$ cannot be a subgroup of $\pi_1(M)$ for any hyperbolic 3–manifold $M$.

*Problem* 2. Let $M$ be a hyperbolic 3–manifold, and $\alpha, \beta \in \pi_1(M)$ two hyperbolic elements whose fixed point sets are disjoint. Show that for sufficiently large integers $n, m$, the group $G_n$ generated by $\alpha^n, \beta^m$ is isomorphic to $F_2$, the free group on two generators.

*Problem* 3 (Aitchison–Reeves). Let $L$ be the link in figure 1. Show that the complement admits a hyperbolic structure obtained from two copies of an ideal combinatorial polyhedron whose boundary is given combinatorially by the planar graph in the figure. Show that if the edges incident at 4–valent vertices have dihedral angle $\pi/2$ and other dihedral angles are $\pi/4$ this combinatorial ideal polyhedron can be realized as a geometric ideal polyhedron with regular faces.

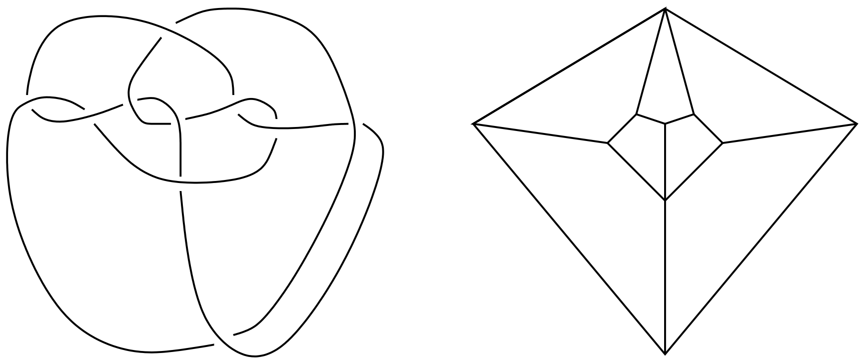

FIGURE 1. The planar graph obtained from the non–balanced link projection is realisable as an ideal hyperbolic polyhedron with regular faces, dihedral angles of $\pi/2$ along edges incident at 4–valent vertices, and $\pi/4$ along all other edges.

*Problem* 4. Let $\omega = \frac{1+\sqrt{3}i}{2}$, a sixth root of unity, and let $\mathbb{Z}[\omega]$ denote the ring of integers in the field $\mathbb{Q}(\omega)$. Let $\Gamma = \mathrm{PSL}(2, \mathbb{Z}[\omega])$, the quotient of the group of $2 \times 2$ matrices with entries in $\mathbb{Z}[\omega]$ and determinant 1 by the subgroup $\pm\mathrm{id}$. Show that the natural action of $\Gamma$ on $\mathbb{H}^3$ is properly discontinuous. If you can, find a fundamental domain for the action. (Warning: the action of $\Gamma$ is not free!)

*Problem* 5. Suppose $K$ is a knot in $S^3$ which admits an alternating projection with $n$ crossings. Show that there is a constant $c$ such that a hyperbolic structure on $S^3 - K$ has volume at most $nc$.

*Problem* 6. Let $L_1, L_2$ be the two links in $S^3$ depicted in figure 2. The second link is commonly known as the Borromean rings.

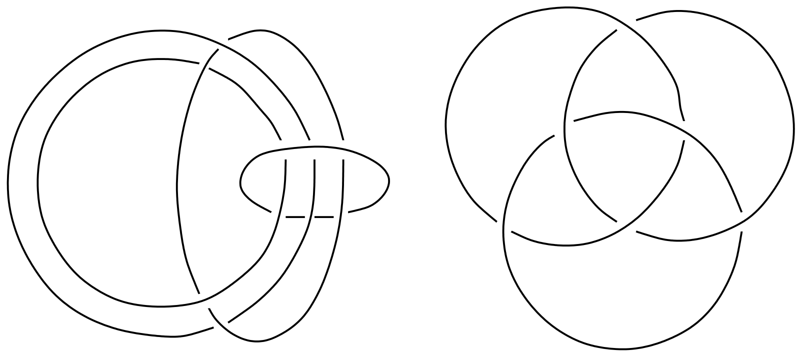

FIGURE 2. These two link complements in $S^3$ are *commensurable*.

1. Show that the regular ideal hyperbolic octahedra has dihedral angles $\pi/2$.
2. Show that the complement of either link in $S^3$ can be obtained by gluing two regular ideal octahedra along their faces. Give a combinatorial description of this gluing in either case.
3. Let $T$ be the tessellation of $\mathbb{H}^3$ by regular ideal octahedra, and $\Gamma$ the full group of symmetries of this tiling. Show that $\pi_1(S^3 - L_1)$ and $\pi_1(S^3 - L_2)$ can both be identified as finite index subgroups of $\Gamma$.
4. Conclude that there is a finite cover $M_1$ of $S^3 - L_1$ and a finite cover $M_2$ of $S^3 - L_2$ such that $M_1$ and $M_2$ are homeomorphic.

Two manifolds which admit homeomorphic finite covers are said to be *commensurable*.