HOMEWORK 5 — FUNDAMENTAL GROUPS AND COVERING SPACES

DANNY CALEGARI

Problem 1. Let X be path connected. That is, given any two points $x, y \in X$ there is a path $\alpha : I \to X$ with $\alpha(0) = x$ and $\alpha(1) = y$. Show that the following are equivalent:

(1) $\pi_1(X, x)$ is trivial for all x (we also say in this case that X is *simply connected*)
(2) Every map $\beta : S^1 \to X$ extends to a map of the closed unit disk D into X.
(3) If σ, τ are paths in X with the same initial points and the same terminal points then $\sigma \simeq \tau$ rel. endpoints.

Problem 2. Let $f, g : \mathbb{S}^n \to \mathbb{S}^n$ be maps such that for all $x \in \mathbb{S}^n$, the points $f(x)$ and $g(x)$ are not antipodal. Show $f \simeq g$. If in addition there is $x_0 \in X$ such that $f(x_0) = g(x_0)$, show $f \simeq g$ rel. x_0.

Problem 3 (Brouwer's fixed point theorem). In this problem we prove the 2–dimensional version of Brouwer's theorem, and derive a theorem of linear algebra as a consequence.

- Show that the circle is not a retract of the closed unit disk. That is, show there is no map $r : D^2 \to S^1$ where $S^1 = \partial D^2$ such that $r|_{S^1} = \text{id}$. In general for $A \subset X$ a map $X \to A$ whose restriction to A is the identity is called a *retraction* from X to A.
- As a consequence, prove Brouwer's theorem: a continuous map $f : D^2 \to D^2$ has a fixed point. Hint: suppose a fixed point free map f exists. Then we can join x to $f(x)$ by a line, and move along this line from $f(x)$ to x and beyond until we reach a point $r(x) \in S^1$, thus defining a retraction $D^2 \to S^1$.
- Show that every 3×3 matrix with positive real entries has an eigenvector with positive eigenvalues. Hint: consider the triangle $T = \{x + y + z = 1; x, y, z \geq 0\}$ and the self–map of T obtained by composing the matrix transformation with radial projection.

Problem 4. Suppose $\phi : F \to E$ is a covering space, and suppose E is path–connected. Show that all the points in E have the same number of preimages.

Problem 5. There is a natural projection $SO(3, \mathbb{R}) \to \mathbb{S}^2$ which is the projection to the coset space of a subgroup stab(p) for any $p \in S^2$ under the standard action of the group. The preimage of any point in S^2 is a circle. Show that $SO(3, \mathbb{R})$ is not homotopy equivalent to $S^2 \times S^1$.

Problem 6. Let G be the topological space obtained from two circles by gluing them together at a point (so that G is homeomorphic to the figure "8"). Draw all the connected coverings of G of order 3. Which coverings are regular?

Problem 7. Find a presentation for $\pi_1(\mathbb{S}^3 - K)$ for K the figure 8 knot.