HOMEWORK 4 — HEEGAARD DECOMPOSITIONS AND LINK SURGERY

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Problem 1. If M_1 has a Heegaard decomposition of genus g_1 and M_2 has a Heegard decomposition of genus g_2, show that the connect sum $M_1 \# M_2$ has a Heegaard decomposition of genus $g_1 + g_2$.

Problem 2. Suppose $M = H_1 \cup H_2$ is a Heegaard decomposition of M into two handlebodies of genus g. Suppose there are embedded loops

$$\gamma_1, \gamma_2 \subset \Sigma$$

where Σ is the splitting surface $\Sigma = H_1 = H_2$, such that γ_i bounds a disk in H_i for each i, and such that γ_1 intersects γ_2 transversely in a single point. Show that M admits a Heegaard decomposition into two handlebodies of genus $g - 1$.

Problem 3. Generalize Alexander's theorem to show that every tame torus T^2 in $\mathbb{S}^3$ bounds a solid torus on at least one side. What about a surface of genus 2? Higher genus?

Problem 4. Two standard descriptions of the Lens space $L(p, q)$ are

- $\frac{p}{q}$ surgery on the unknot in $\mathbb{S}^3$
- The quotient space $\mathbb{S}^3/\langle\tau\rangle$ where $\tau : \mathbb{S}^3 \to \mathbb{S}^3$ is the transformation given by
$$(z_1, z_2) \to (z_1\omega, z_2\omega^q)$$
where we think of $\mathbb{S}^3 \subset \mathbb{C}^2$ as the set of vectors (z_1, z_2) with $\|z_1\|^2 + \|z_2\|^2 = 1$, and $\omega = e^{\frac{2\pi i}{p}}$.

Show that these two descriptions give topologically equivalent manifolds.

Problem 5. $\mathbb{RP}^3$ is the topological 3–manifold whose points are the vectors of length 1 tangent to the unit sphere S. Identify $\mathbb{RP}^3$ with the points in the group $SO(3, \mathbb{R})$. Let Γ be the group of orientation–preserving symmetries of the regular dodecahedron, thought of as a subgroup of $SO(3, \mathbb{R})$. Show that the coset space $SO(3, \mathbb{R})/\Gamma$ is a 3–manifold P. Give some explicit combinatorial description of this manifold; e.g. show how to obtain it by glueing faces of polyhedra, or give a surgery description or Heegaard decomposition.

Problem 6. Suppose we have a link L consisting of a collection of components K_i, and suppose that K_1 is unknotted and links some strands of the other components in the manner indicated in figure 1. Suppose we do 1 surgery on K_1. Show that the result of this is given by "twisting" the strands that pass through K_1, as indicated in the figure. How does the framing change on the twisted strands?

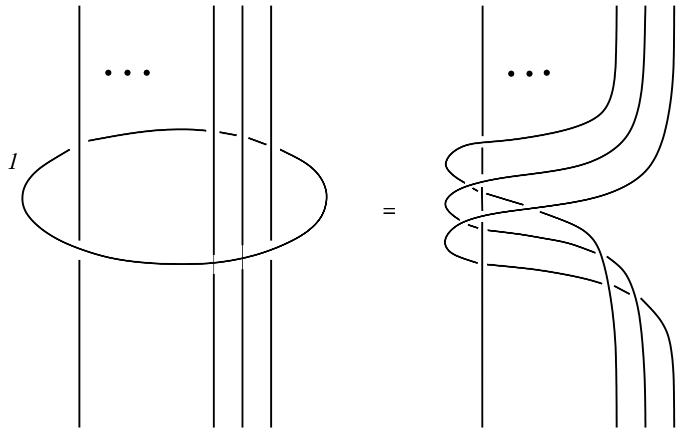

FIGURE 1. 1 surgery on an unknot induces a "twist" on the strands passing through a disk spanned by the knot.

Problem 7. Show that the surgery diagram in figure 2 describes the 3–manifold $\Sigma_g \times S^1$ where Σ_g is the oriented surface of genus g, and there are g "clasps" in the figure. If $g = 1$ this is 0 surgery on each component of the Borromean rings, and the result of surgery is the 3–torus $S^1 \times S^1 \times S^1$.

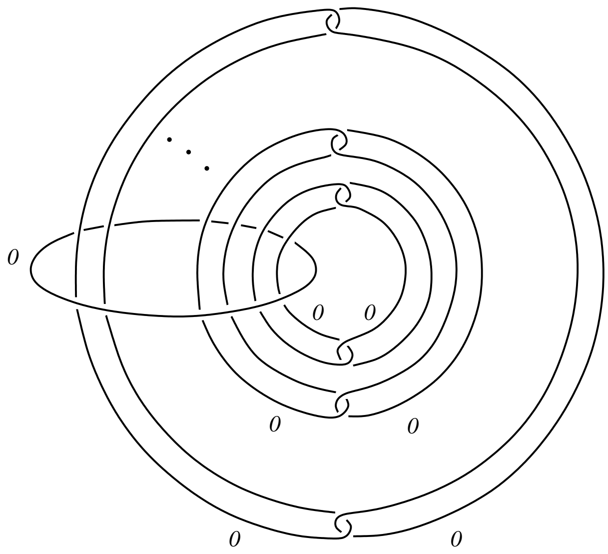

FIGURE 2. If there are g clasps in the figure, this is a surgery diagram for $\Sigma_g \times S^1$.