HOMEWORK 3 — HYPERBOLIC GEOMETRY CONTINUED

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Problem 1. Recall the definition of the Bernoulli numbers: they are given by the Taylor series expansion

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n$$

If $\Lambda(\theta)$ denotes the Lobachevsky function, prove the identity

$$\Lambda(\theta) = \theta\left(1 - \log|2\theta| + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{B_{2n}}{2n} \frac{(2\theta)^{2n}}{(2n+1)!}\right)$$

Hint: integrate $\cot(z)$ twice.

Use this formula to find an approximation to the volume of a regular ideal tetrahedron. How many terms do you need to get 4 decimal places?

Remark 1. With notation as above, $B_n = 0$ for n odd and > 2, so some authors use the notation B_n to denote the $2n$–th Bernoulli numbers, with the notation above.

Problem 2. Using the notation

$$D(z) = \text{Im}(\text{Li}_2(z)) + \log|z| \arg(1 - z)$$

and its interpretation as the volume of an ideal hyperbolic simplex with parameter z, prove the *five term identity*:

$$D(x) - D(y) + D\Big(\frac{y}{x}\Big) - D\Big(\frac{1 - x^{-1}}{1 - y^{-1}}\Big) + D\Big(\frac{1 - x}{1 - y}\Big) = 0$$

Hint: what are the cross–ratios of the 4–tuples from the set $(0, 1, x, y, \infty)$?

Problem 3. Let H denote the quaternions, numbers of the form

$$h = a + bi + cj + dk$$

with $a, b, c, d \in \mathbb{R}$. For h a quaternion, denote

$$h' = a - bi - cj + dk, \overline{h} = a - bi - cj - dk$$

Define $SB(2, H)$ to be the group of 2×2 matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with entries in H such that $d = a', b = c'$ and $a\overline{a} - c\overline{c} = 1$.

Consider the unit ball

$$B = \{u = u_0 + u_1 i + u_2 j \in H \mid u_0^2 + u_1^2 + u_2^2 < 1\}$$

- Show that the natural "fractional linear" action of $SB(2, H)$ on B has image exactly the group of orientation–preserving conformal isomorphisms of the 3–ball.
- Show that this action of $SB(2, H)$ has kernel exactly $\pm$Id and therefore that there is an isomorphism $SB(2, H)/\pm \text{Id} \cong PSL(2, \mathbb{C})$.

Problem 4. The group $SU(2) \subset SL(2,\mathbb{C})$ is the group of 2×2 matrices such that $\overline{A}^t A = \mathrm{Id}$ and $\det(A) = 1$. Define $PSU(2) = SU(2)/\pm\mathrm{Id}$ which can be thought of as a subgroup of $PSL(2,\mathbb{C})$. Show $PSU(2)$ can be identified with the stabilizer of a point in $\mathbb{H}^3$. Deduce that $PSU(2) \cong SO(3,\mathbb{R})$. Give an explicit isomorphism $PSU(2) \to SO(3,\mathbb{R})$.

The group $SO(3,\mathbb{C})$ is the group of 3×3 matrices A such that $A^t A = \mathrm{Id}$ and $\det(A) = 1$. Show that $PSL(2,\mathbb{C}) \cong SO(3,\mathbb{C})$, and give an explicit isomorphism between these groups in such a way that the restriction to $PSU(2)$ induces the isomorphism in the first part of the problem.

Problem 5 (Thurston). Descriptions of spaces and geometries are often implicit. Manifolds often arise in practice by parameterizing other geometric objects. The following pretty example is due to Bill Thurston.

Consider the space of (not necessarily simple) pentagons in the plane having $108°$ angles and sides parallel to the corresponding sides of a model regular pentagon, and parameterize this space by the (signed) side lengths $s_1, \dots, s_5$.

- Show that the s_i are subject to a linear relation that confines them to a three dimensional linear subspace V of $\mathbb{R}^5$.
- Show that the area enclosed by a pentagon is a quadratic form on V which is isometric to $E^{2,1}$. How do you measure area for a pentagon whose boundary is not topologically a circle?
- Describe a model for the hyperbolic plane in terms of the subset of V consisting of pentagons of unit area. Does your model have a single component?
- Show that the space of simple pentagons of unit area is a right–angled pentagon in the hyperbolic plane.
- There is an area–preserving “butterfly operation” on pentagons that replaces a side of length s by a side of length $-s$, changing the lengths of the two neighboring sides to make it fit. Interpret this operation in hyperbolic geometry.
- Given a non–simple pentagon in V, when is it possible to modify it by a sequence of butterfly moves until it is simple?

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