HOMEWORK 1 — SCISSORS CONGRUENCE AND EQUIDECOMPOSABILITY IN EUCLIDEAN SPACE

DANNY CALEGARI

Homework is assigned on Tuesdays; it is due at the start of class two weeks after it is assigned. Problems marked "Hard" are extra credit; in other words, doing all the ordinary problems correctly will earn full credit. But by doing hard problems, it is possible to make mistakes on, or fail to complete, all the ordinary problems, and still earn full credit.

Problem 1 (Tensor products). Identify the more common names of the following tensor products of $\mathbb{Z}$–modules (remember that $\mathbb{Z}$–modules are just Abelian groups)

(1) $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$ for n a positive integer
(2) $\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z}$ for n, m positive integers
(3) $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$
(4) $\mathbb{Z}^n \otimes_{\mathbb{Z}} \mathbb{Q}$

Problem 2. Find a *bounded* subset of the plane which is congruent to a proper subset of itself.

Problem 3. Calculate the Dehn invariants of the regular platonic solids of side length 1. What is the relation between the Dehn invariant of the octahedron and the tetrahedron? Show that an octahedron and two tetrahedra of the same edge length can be glued up to make a parallelepiped which tiles Euclidean space.

Problem 4. Let R and S be two squares in the plane of equal side length. Find a decomposition of R into pieces which can be moved by *translation alone* to give a decomposition of S. Try to find a decomposition with as few pieces as possible.

Problem 5. The *rhombic triacontahedron* is obtained in the following way from a regular dodecahedron: erect a little five–sided pyramid on each of the twelve faces of the dodecahedron. For a certain choice of pyramid, each of the five sides of the pyramid on one face of the dodecahedron joins up with a side of the pyramid on a neighboring face of the dodecahedron to make a rhombus. Explicitly, if F_1, F_2 are two faces which share an edge a, b, and if c_1, c_2 are the apex of the pyramid erected on F_1, F_2 respectively, then if the heights of the pyramids are chosen to be a certain (equal) number, the points c_1, c_2, a, b are in the same plane and span a rhombus. Thus this solid has thirty congruent rhombic faces. Draw a good picture of this solid. Show that the diagonals of the rhombus faces have a ratio of $\frac{1+\sqrt{5}}{2}$ — that is, the golden ratio. Show that the dihedral angles of the faces are *rational* multiples of π, and therefore that the Dehn invariant of this solid is zero.

Remark 1. The rhombic triacontahedron can be decomposed into parallelepipeds (which already shows it has Dehn invariant zero). These parallelepipeds tile three dimensional space in a non–periodic fashion, which is a three–dimensional version of Penrose tiles.

Problem 6. Let D be the unit disk. Show that there is no partition of D into two disjoint sets $D = X \coprod Y$ such that X and Y are congruent. Hint: where does the center go?

Problem 7. Using the fact that the group of orientation–preserving isometries of the plane is isomorphic to a semi–direct product of $\mathbb{R}^2$ and S^1, show that this group does not contain a subgroup isomorphic to F_2, the free group on two generators. Hint: show that $\text{Isom}^+(\mathbb{E}^2)$ has a homomorphism to an abelian group with abelian kernel, and that F_2 does not.

Problem 8 (Zylev's theorem; Hard). Show that if two polyhedra are stably scissors congruent, then they are scissors congruent. That is, if $P \coprod A \sim Q \coprod A$ then $P \sim Q$. (Note: the notation $X \coprod Y$ means the disjoint union of X and Y.)

DEPARTMENT OF MATHEMATICS, HARVARD, CAMBRIDGE, MA 02138
E-mail address: dannyc@math.harvard.edu