MATH 139 FINAL EXAM

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Answer all problems. These problems are all of equal weight. If you have questions or comments, contact me by email at `dannyc@math.harvard.edu`. Solutions to this exam are due in my mailbox at 3pm on Tuesday, May 15th.

Problem 1. How would you define the Dehn invariant for three dimensional hyperbolic polyhedra? What about for finite volume polyhedra possibly with ideal vertices? Find an infinite family of three dimensional hyperbolic polyhedra with volume uniformly bounded above and Dehn invariant 0.

Problem 2. Fix a point $p \in \mathbb{H}^3$ and consider a sequence of points $q_i, r_i \in \mathbb{H}^3$ such that the distance $d(p, q_i) \to \infty$ and $d(p, r_i) \to \infty$. Show that the angles at p of the hyperbolic triangles $q_i p r_i$ converge to 0 if and only if the distances

$$d(p, q_i) + d(p, r_i) - d(q_i, r_i) \to \infty$$

Problem 3. This problem involves the calculation of a volume of a certain hyperbolic tetrahedron.

(1) Let
$$A(x) = \int_0^x \frac{1}{1+t^2} \log \frac{4}{1+t^2} dt$$
for $t \in \mathbb{R}$. Show that $A(x) = 2\Lambda(\cot^{-1} x)$ where Λ is Lobachevsky's function.

(2) Let $G_{\alpha,\gamma}$ be the hyperbolic tetrahedron with one ideal vertex defined as follows: let the vertices of T be p, q, r, s with p an ideal point. Suppose that the triangle pqr and pqs have right angles at q. Suppose that the dihedral angle along edge pq is α and along rs is γ, and the dihedral angles along qr,qs and pr are $\pi/2$. Show that $G_{\alpha,\gamma}$ is uniquely defined up to isometry by these conditions.

(3) Show that the volume of $G_{\alpha,\gamma}$ is given by
$$\text{vol}(G_{\alpha,\gamma}) = \frac{1}{8}\Big(A\Big(\frac{1-ac}{a+c}\Big) + A\Big(\frac{1+ac}{a-c}\Big) + 2A(a)\Big)$$
where $a = \tan(\alpha)$ and $c = \tan(\gamma)$.

Problem 4. We say a knot K in $\mathbb{S}^3$ has *tunnel number* 1 if there is an arc α in $\mathbb{S}^3 - K$ with both endpoints on K, such that $\mathbb{S}^3 - N(K \cup \alpha)$ is a genus 2 handlebody (here N denotes a small thickened neighborhood of the graph $K \cup \alpha$).

(1) Show that if K has tunnel number 1, there is a presentation for $\pi_1(\mathbb{S}^3 - K)$ with two generators and one relation.

(2) Show that the figure eight knot has tunnel number 1.

Problem 5. Give three examples of knots whose complements do not admit complete finite volume hyperbolic structures, for three different reasons.

Give an example of a knot — *other than the figure eight knot!* — whose complement in $\mathbb{S}^3$ admits a complete finite volume hyperbolic structure. Construct such a structure by finding a decomposition of the complement into ideal hyperbolic tetrahedra, and find a solution to the edge and completeness equations with all simplices positively oriented.