HOMEWORK 6 — PROPERLY DISCONTINUOUS GROUPS

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This homework is due Friday December 8th at the start of class.

Problem 1. Suppose Γ is an infinite properly discontinuous group in $\text{Isom}^+(\mathbb{E}^2)$ which contains some elements of order 3. Show that the set P of fixed points of elements of order 3 form the vertices of a tessellation by equilateral triangles or the vertices of a tessellation by equilateral hexagons. Deduce *Napoleon's theorem*: if T is a triangle, and three equilateral triangles T_1, T_2, T_3 are constructed on the three sides of T, then the centers of the triangles T_i themselves are the vertices of an equilateral triangle.

Hint: find a tessellation of $\mathbb{E}^2$ by copies of T and the T_i and let Γ be its group of symmetries.

Problem 2. If $\Gamma \subset \text{Isom}^+(\mathbb{E}^2)$ is a properly discontinuous group of symmetries, then we have seen that Γ can be written as a semi–direct product

$$\Gamma = T \rtimes R$$

where T is a group of translations, and R is the subgroup of rotations fixing some point. Find an example of a properly discontinuous group of symmetries $\Gamma \subset \text{Isom}(\mathbb{E}^2)$ which cannot be written as a semi–direct product, even though there is a short exact sequence

$$1 \to T \to \Gamma \to S \to 1$$

where S is a finite subgroup of $O(2, \mathbb{R})$.

Remark. Every finite subgroup of $SO(2, \mathbb{R})$ is *cyclic* — that is, generated by a single element. Thus a short exact sequence

$$1 \to T \to \Gamma \to S \to 1$$

where S is a finite subgroup of $SO(2, \mathbb{R})$, *splits* — that is, Γ is a semi–direct product. But if S is not cyclic, this sequence does not necessarily split. This is the explanation for the phenomenon in the previous problem.

Problem 3. What is the $(2, 2, \infty)$–spherical orbifold? Show that it admits a flat (i.e. Euclidean) structure, and identify its orbifold fundamental group as the infinite dihedral group. How does this group act in an orientation preserving way on $\mathbb{E}^2$?

Problem 4. The group S_4 acts on $\mathbb{R}^4$ by permuting the co–ordinates. This action is linear and fixes the origin; that is, it is a homomorphism of the group into $O(4, \mathbb{R})$. This action fixes the vector $(1, 1, 1, 1)$, and therefore it fixes the three dimensional subspace $\pi \subset \mathbb{R}^4$ of vectors whose co–ordinates sum to 0. Show that the restriction $S_4|_\pi$ is the group of (not necessarily orientation–preserving) symmetries of the regular tetrahedron centered at the origin in π whose vertices are

$$(3, -1, -1, -1), (-1, 3, -1, -1), (-1, -1, 3, -1), (-1, -1, -1, 3)$$

Problem 5. Let T be an arbitrary Euclidean triangle. Show that there is a tetrahedron Δ whose four sides are congruent to T which has a $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ group of symmetries. This group is also known as the *Klein* 4*–group*. Here a *tetrahedron* is just a polygonal sphere with four faces which are flat Euclidean triangles. It is not assumed that the tetrahedron is the boundary of a solid object in $\mathbb{R}^3$ with flat faces. Show the spherical polyhedron can be realized as the boundary of a solid tetrahedron (in the usual sense) in a way which preserves the group of symmetries, if T is *acute*.

Problem 6. Let S be a genus 2 surface. Show that S can be built from regular right angled pentagons with four pentagons around every vertex. If γ is a geodesic (a straight closed curve) in the 1–skeleton of the surface so obtained, we can *twist* the surface along this curve, and reglue the pentagons on either side in a new configuration. Enumerate the combinatorially distinct decompositions of S into right angled pentagons, and show how they are related by twisting in a curve.

Problem 7. Let Σ be a hyperbolic orbifold (possibly obtained as a quotient of $\mathbb{H}^2$ by a group containing some orientation–reversing elements). Using the Gauss–Bonnet theorem for orbifolds, show that the area of Σ is at least $\frac{\pi}{42}$. (Hint: show Σ has a double cover which is an orientable orbifold and therefore has at worst point singularities.) Deduce that if S is any hyperbolic surface of genus g, the group G of orientation–preserving symmetries of S has at most $84(g-1)$ elements.

Problem 8. Let $p > 3$ be a prime number and let $\Gamma(p)$ denote the kernel of the obvious homomorphism

$$\phi_p : PSL(2,\mathbb{Z}) \to PSL(2,\mathbb{Z}/p\mathbb{Z})$$

- What is the order of the group $PSL(2,\mathbb{Z}/p\mathbb{Z})$?
- Show that $\Gamma(p)$ acts on $\mathbb{H}^2$ freely. What is the hyperbolic area of the quotient surface?
- For $p = 7$ identify the quotient $\mathbb{H}^2/\Gamma(7)$ explicitly, and find a decomposition of it into ideal triangles.

Problem 9. Recall that a closed curve on a surface Σ is *simple* if it is embedded, and *essential* if it does not bound a disk in Σ. If γ, δ are two essential simple closed curves on a hyperbolic surface which do not intersect, show that their geodesic representatives do not intersect. Deduce that a combinatorial pair–of–pants decomposition of a hyperbolic surface can be realized by cutting along $3g-3$ geodesics.

Problem 10 (hard). Let $\alpha_1, \alpha_2, \beta_1, \beta_2$ be four disjoint infinite geodesics in $\mathbb{H}^2$ which are not "nested", in the sense that there is a point p which can be joined by arcs to any one of the α_i, β_i without crossing any others. Let γ, δ be two hyperbolic translations, where γ takes α_1 to α_2 and δ takes β_1 to β_2.

- Show that $\Gamma = \langle \gamma, \delta \rangle$, the group generated by γ and δ, is a free group on two elements $\mathbb{Z} * \mathbb{Z}$.
- Suppose $\alpha_1, \beta_1, \alpha_2, \beta_2$ are the sides (in order) of an ideal quadrilateral. Show that the quotient $\mathbb{H}^2/\Gamma$ has finite area if and only if the commutator $[\gamma, \delta] = \gamma\delta\gamma^{-1}\delta^{-1}$ is parabolic.

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